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Explicit solution for freezing of humid porous half-space with a heat flux condition

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Abstract

Explicit solutions for temperature and moisture distribution in a porous half-space with a heat flux condition at $x = 0$ of the type $q_0/\sqrt{t}$ are obtained. An inequality for the coefficient q_0 is necessary and sufficient in order to obtain that exact solution. An equivalence between this problem and the analogous corresponding to a phase change problem with a temperature condition is also obtained.

Keywords: Phase change; Porous medium; Freezing; Similarity method

1. Introduction

Heat and mass transfer with phase change problems, taking place in a porous medium, such as evaporation, condensation, freezing, melting, sublimation and desublimation, have wide application in separation processes, food technology, heat and mixture migration in soils and grounds, etc. Due to the non-linearity of the problem, solutions usually involve mathematical difficulties. Only a few exact solutions have been found for idealized cases (see [1,5,10,11], for example). A large bibliography on free and moving boundary problems for the heat-diffusion equation was given in [13].

Mathematical formulation of the heat and mass transfer in capillary porous bodies has been established by Luikov [6–9]. Two different models was presented by Mikhailov [10] for solving the problem of evaporation of liquid moisture from a porous medium. For the problem of freezing

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Nomenclature

$A_i, B_i,\ i = 1,2,3$	constants of integration
$a_i,\ i = 1,2$	thermal diffusivity of the i region phase
a_{12}	ratio of thermal diffusivities from region 1 to 2
$a_{\rm m}$	moisture diffusivity
F_0	Fourier number
$k_i,\ i = 1,2$	thermal conductivity of the i region phase
k_{12}	ratio of thermal conductivity from region 1 to 2
$K_0 = ru_0/c_2\Delta t$	Kossovitch number
l_0	the characteristic length
$\mathscr{L}_u = a_{\rm m}/a_2$	Luikov number
$\mathscr{P}_n = \delta\Delta t/u_0$	Posnov number
q_0	coefficient that characterizes the heat flux at $x = 0$
r	latent heat
$s(\tau)$	position of the evaporation front
$t_i(x,\tau),\ i = 1,2$	temperature in the region i
t_0	initial temperature
$t_{\rm s}$	temperature at surface $x = 0$
$t_{\rm v}$	temperature at vaporization state
$T_i,\ i = 1,2$	non-dimensional temperature
u	mass-transfer potential
x	length coordinate
X	non-dimensional length
Z	potential defined by Eq. (21)
Greek symbols	
ε	phase change criterion
δ	thermal gradient coefficient
ρ_2	density of the porous body in the region 2
λ	non-dimensional constant
τ	time
Θ	non-dimensional mass transfer potential

(desublimation) of humid porous half-space, Mikhailov also presented an exact solution [11]. Other problems in this direction are in Refs. [2–4].

In the following, freezing (desublimation) of moisture in a porous medium with heat flux condition at $x = 0$ of the type $q_0/\sqrt{t}$, with $q_0 > 0$, will be studied. An analytical model of the process is defined and exact solutions for temperature and moisture distributions are obtained. An inequality for the coefficient q_0 is necessary and sufficient in order to obtain the corresponding explicit solution. Finally, an equivalence between a phase change problem with a temperature condition and a phase change problem with a heat flux condition of the type $q_0/\sqrt{t}$ on the surface is also obtained.

Let us consider the flow of heat and moisture through a porous half-space during freezing. The position of phase change front at time τ is given by $x = s(\tau)$. It divides the porous body into two regions. In the freezing region, $0 < x < s(\tau)$, there is no moisture movement and the temperature distribution is described by the heat equation

$$\frac{\partial t_1}{\partial \tau}(x,\tau) = a_1 \frac{\partial^2 t_1}{\partial x^2}(x,\tau), \quad 0 < x < s(\tau),\ \tau > 0. \tag{1}$$

The region $s(\tau) < x < +\infty$ is humid capillary porous body in which there are coupled heat and moisture flows. The process is described by the well-known Luikov's system [10] for the case $\varepsilon = 0$ (ε is the phase conversion factor of liquid into vapor):

$$\frac{\partial t_2}{\partial \tau}(x,\tau) = a_2 \frac{\partial^2 t_2}{\partial x^2}(x,\tau), \quad x > s(\tau),\ \tau > 0, \tag{2}$$

$$\frac{\partial u}{\partial \tau}(x,\tau) = a_{\mathrm{m}} \frac{\partial^2 u}{\partial x^2}(x,\tau) + a_{\mathrm{m}}\delta \frac{\partial^2 t_2}{\partial x^2}(x,\tau), \quad x > s(\tau),\ \tau > 0. \tag{3}$$

The initial distributions of temperature and moisture are uniform

$$t_2(x,0) = t_2(+\infty,\tau) = t_0, \quad u(x,0) = u(+\infty,\tau) = u_0. \tag{4}$$

It is assumed that on the surface of the half-space the heat flux depends on the time in the following way, like in [12]:

$$k_1 \frac{\partial t_1}{\partial x}(0,\tau) = q_0/\sqrt{\tau}, \tag{5}$$

where $q_0 > 0$ is a coefficient which characterizes the heat flux at the fixed face $x = 0$. On the freezing front, there exists an equality between the temperatures:

$$t_1(s(\tau),\tau) = t_2(s(\tau),\tau) = t_{\mathrm{v}}, \quad \tau > 0, \tag{6}$$

where $t_{\mathrm{v}} < t_0$.

Heat and moisture balance at the freezing front yields

$$k_1 \frac{\partial t_1}{\partial x}(s(\tau),\tau) - k_2 \frac{\partial t_2}{\partial x}(s(\tau),\tau) = u(s(\tau),\tau)\rho_2 r \frac{\mathrm{d}s}{\mathrm{d}\tau}(\tau), \quad \tau > 0, \tag{7}$$

$$\frac{\partial u}{\partial x}(s(\tau),\tau) + \delta \frac{\partial t_2}{\partial x}(s(\tau),\tau) = 0, \quad \tau > 0. \tag{8}$$

The set of equations and conditions (1)–(8) is called problem P.

In Section 2 we obtain an exact solution for the problem P when an inequality on q_0 holds. Next, in Section 3 we introduce the problem $\widetilde{P}$, which is the problem P changing condition (5) by a temperature condition in $x = 0$, and we study the behavior of the solution of this problem considering when the parameter latent heat r tends to infinity. In Section 4 we establish a relationship between problem P (heat flux condition at $x = 0$) and $\widetilde{P}$ (temperature condition at $x = 0$), and we obtain an inequality that the coefficient which characterizes the free boundary verifies. Finally in Section 5 we give some illustrative results in order to study the effects of the parameter q_0 on our phase change process. In particular, we give the behavior of λ (which characterizes the free boundary (22)) as a function of q_0 and the behavior of the temperature distributions as a function of the position x for a given time $F_0 = 1$ for different values of q_0.

2. Solution of the problem

If we consider the next transformations

$$X = x/l_0, \quad F_0 = a_2\tau/l_0^2, \quad S(F_0) = s(\tau)/l_0,$$

$$T_i(X, F_0) = (t_i(x,\tau) - t_v)/(t_0 - t_v), \quad i = 1, 2,$$

$$\Theta(X, F_0) = (u_0 - u(x,\tau))/u_0,$$

then the set of equations and conditions (1)–(8) can be put in a dimensionless form as follows:

$$\frac{\partial T_1}{\partial F_0}(X, F_0) = a_{12}\frac{\partial^2 T_1}{\partial X^2}(X, F_0), \quad 0 < X < S(F_0), \ F_0 > 0, \tag{9}$$

$$\frac{\partial T_2}{\partial F_0}(X, F_0) = \frac{\partial^2 T_2}{\partial X^2}(X, F_0), \quad X > S(F_0), \ F_0 > 0, \tag{10}$$

$$\frac{\partial \Theta}{\partial F_0}(X, F_0) = \mathscr{L}_u\frac{\partial^2 \Theta}{\partial X^2}(x,\tau) + \mathscr{L}_u\mathscr{P}_n\frac{\partial^2 T_2}{\partial X^2}(x,\tau), \quad X > S(F_0), \ F_0 > 0, \tag{11}$$

$$T_2(X, 0) = T_2(+\infty, F_0) = 1, \quad \Theta(X, 0) = \Theta(+\infty, F_0) = 0, \tag{12}$$

$$\frac{\partial T_1}{\partial X}(0, F_0) = q_0\sqrt{a_2}\Big/\left[k_1(t_0 - t_v)\sqrt{F_0}\right], \tag{13}$$

$$T_1(S(F_0), F_0) = T_2(S(F_0), F_0) = 0, \quad F_0 > 0, \tag{14}$$

$$\frac{\partial \Theta}{\partial X}(S(F_0), F_0) + \mathscr{P}_n\frac{\partial T_2}{\partial X}(S(F_0), F_0) = 0, \quad F_0 > 0, \tag{15}$$

$$k_{12}\frac{\partial T_1}{\partial X}(S(F_0),F_0)-\frac{\partial T_2}{\partial X}(S(F_0),F_0)=K_0(1-\Theta(S(F_0),F_0))\frac{\partial S}{\partial F_0}(F_0),\quad F_0>0. \tag{16}$$

For convenience in the derivation of the solution, we introduce now a new unknown function, which couples T_2 and Θ, i.e.

$$Z(X,F_0)=T_2(X,F_0)+[(1-\mathscr{L}_u)/(\mathscr{L}_u\mathscr{P}_n)]\Theta(X,F_0),\quad X>S(F_0),\ F_0>0. \tag{17}$$

After some elementary computations we obtain

$$\frac{\partial Z}{\partial F_0}(X,F_0)=\mathscr{L}_u\frac{\partial^2 Z}{\partial X^2}(X,F_0),\quad X>S(F_0),\ F_0>0. \tag{18}$$

Then Eqs. (9), (10) and (18) have the following solutions:

$$T_1(X,F_0)=A_1+B_1\,\mathrm{erf}\left(X/2\sqrt{a_{12}F_0}\right),\quad 0<X<S(F_0),\ F_0>0, \tag{19}$$

$$T_2(X,F_0)=A_2+B_2\,\mathrm{erf}\left(X/2\sqrt{F_0}\right),\quad X>S(F_0),\ F_0>0, \tag{20}$$

$$Z(X,F_0)=A_3+B_3\,\mathrm{erf}\left(X/2\sqrt{\mathscr{L}_uF_0}\right),\quad X>S(F_0),\ F_0>0, \tag{21}$$

$$S(F_0)=2\lambda\sqrt{F_0},\quad F_0>0, \tag{22}$$

where the constants λ, A_i and B_i, $i=1,2,3$, must to be chosen so that they satisfy the seven conditions corresponding to the initial and boundary conditions (12)–(16).

From the initial condition (12) we obtain a system of two algebraic equations and its solution yields

$$A_2=1-B_2,\quad A_3=1-B_3. \tag{23}$$

Eq. (13) leads to

$$B_1=\sqrt{\pi a_1}q_0/[k_1(t_0-t_v)]. \tag{24}$$

From condition (14), we obtain

$$A_1=-[\sqrt{\pi a_1}q_0/(k_1(t_0-t_v))]\,\mathrm{erf}(\lambda/\sqrt{a_{12}}), \tag{25}$$

$$B_2=(1-\mathrm{erf}\,\lambda)^{-1}. \tag{26}$$

It follows from Eq. (15) that

$$B_3 = \exp\left(\lambda^2/\mathscr{L}_u\right)\Big/\left[\sqrt{\mathscr{L}_u}\exp\left(\lambda^2\right)\right](1-\operatorname{erf}\lambda). \tag{27}$$

After determining the constants A_1, A_2, A_3, B_1, B_2 and B_3 from Eqs. (23)–(27) the solutions (19), (20) and (22) can be written as

$$T_1(X,F_0) = [\sqrt{\pi a_1}q_0/(k_1(t_0-t_v))]\left(\operatorname{erf}\left(X/2\sqrt{a_{12}F_0}\right) - \operatorname{erf}(\lambda/\sqrt{a_{12}})\right), \tag{28}$$

$$T_2(X,F_0) = (1-\operatorname{erf}\lambda)^{-1}\left(\operatorname{erf}\left(X/2\sqrt{F_0}\right) - \operatorname{erf}\lambda\right), \tag{29}$$

$$\Theta(X,F_0) = 1 - \left\{\exp\left(\lambda^2/\mathscr{L}_u\right)\Big/\left\{\left[\sqrt{\mathscr{L}_u}\exp\left(\lambda^2\right)\right](1-\operatorname{erf}\lambda)\right\}\right\}\left(1-\operatorname{erf}\left(X/2\sqrt{\mathscr{L}_uF_0}\right)\right). \tag{30}$$

All three functions T_1, T_2 and Θ are explicited as a function of the parameter λ which must be determined by condition (16) which gives us the following trascendental equation:

$$\begin{aligned}&\{\sqrt{\pi a_2}q_0/[k_2(t_0-t_v)]\}\exp\left(-\lambda^2/a_{12}\right) - F_1(\lambda)\\ &\quad = \sqrt{\pi}K_0\lambda(1-[(1-\mathscr{L}_u)/(\mathscr{L}_u\mathscr{P}_n)](1-H(\lambda))), \quad \lambda > 0,\end{aligned} \tag{31}$$

where

$$\begin{aligned}&F_1(x) = \exp\left(-x^2\right)\cdot(1-\operatorname{erf}x)^{-1},\\ &Q(x) = \sqrt{\pi}x\exp\left(-x^2\right)[1-\operatorname{erf}x],\\ &H(x) = Q\left(x/\sqrt{\mathscr{L}_u}\right)/Q(x)\end{aligned} \tag{32}$$

are real functions defined for $x > 0$.

For convenience in the notation, we define the following real functions:

$$\mu(x) = \{\sqrt{\pi a_2}q_0/[k_2(t_0-t_v)]\}\exp\left(-x^2/a_{12}\right) - F_1(x), \quad x > 0, \tag{33}$$

$$\beta(x) = \sqrt{\pi}K_0x(1-[(1-\mathscr{L}_u)/(\mathscr{L}_u\mathscr{P}_n)](1-H(x))), \quad x > 0. \tag{34}$$

Then Eq. (31) can be written saying that λ must be the solution of the equation

$$\mu(x) = \beta(x), \quad x > 0. \tag{35}$$

then we can enunciate and demonstrate the following property:

Theorem 1. *If*

$$q_0 > k_2(t_0 - t_v)/\sqrt{\pi a_2}, \tag{36}$$

then there exists one and only one solution $\lambda > 0$ *of Eq.* (35).

If $q_0 \leqslant k_2(t_0 - t_v)/\sqrt{\pi a_2}$, *then there is no solution of problem* (1)–(8) *as a phase change problem*; *it is only a heat conduction problem for the initial phase.*

Proof. In order to solve Eq. (35) we shall study the behavior of the functions μ and β. In [12] were studied the properties of $F_1(x)$ and its derivative. We know that

$$F_1(0) = 1, \quad F_1(+\infty) = +\infty \quad \text{and} \quad F_1'(x) > 0 \ \forall x > 0.$$

By the way, the function Q has the following properties:

$$Q(0) = 0, \quad Q(+\infty) = 1 \quad \text{and} \quad Q'(x) > 0 \ \forall x > 0.$$

Then, we see the properties of the function $H(x)$:

$$H(0) = \lim_{x\to 0} Q\left(x/\sqrt{\mathscr{L}_u}\right)/Q(x) = \mathscr{L}_u^{-1/2}, \quad H(+\infty) = 1,$$

$$H'(x) = 2(\pi\mathscr{L}_u)^{-1/2}\left(\frac{Q(x) - 1}{F_1\left(x/\sqrt{\mathscr{L}_u}\right)} - \frac{Q(x/\sqrt{\mathscr{L}_u}) - 1}{\sqrt{\mathscr{L}_u}F_1(x)}\right)[F_1(x)]^2.$$

Then we get

$$H'(x) > 0 \quad \text{if } \mathscr{L}_u < 1,$$
$$H'(x) < 0 \quad \text{if } \mathscr{L}_u > 1,$$
$$H'(x) = 0 \quad \text{if } \mathscr{L}_u = 1.$$

Therefore, H is a strictly increasing function if $\mathscr{L}_u < 1$, and is a strictly decreasing function if $\mathscr{L}_u > 1$.

Now, we can obtain the properties of the function $\beta(x)$, that is

$$\beta(0) = 0, \quad \beta(+\infty) = +\infty,$$

$$\beta'(x) = \sqrt{\pi}K_0(1 - [(1 - \mathscr{L}_u)/(\mathscr{L}_u\mathscr{P}_n)](1 - H(x))) + \sqrt{\pi}K_0 x\left([\mathscr{L}_u\mathscr{P}_n/(1 - \mathscr{L}_u)]H'(x)\right). \tag{37}$$

Note that if $\mathscr{L}_u < 1$, we have $[\mathscr{L}_u\mathscr{P}_n/(1 - \mathscr{L}_u)] > 0$ and $H'(x) > 0$, and then

$$\sqrt{\pi}K_0 x\left([\mathscr{L}_u\mathscr{P}_n/(1 - \mathscr{L}_u)]H'(x)\right) > 0, \tag{38}$$

Similarly, if $\mathscr{L}_u > 1$, $[\mathscr{L}_u \mathscr{P}_n/(1-\mathscr{L}_u)] < 0$ and $H'(x) < 0$, and again (38) is valid. Therefore, in any case we have $\beta'(x) > 0\ \forall x > 0$.

Finally, we deduce the properties of the function $\mu(x)$:

$$\mu(0) = \{\sqrt{\pi a_2} q_0/[k_2(t_0 - t_{\mathrm{v}})]\} - 1, \quad \mu(+\infty) = -\infty,$$
$$\mu'(x) = -\left\{(\{\sqrt{\pi a_2} q_0/[k_2(t_0 - t_{\mathrm{v}})]\}(2x/a_{12}))\exp\left(-x^2/a_{12}\right) + F_1'(x)\right\} < 0 \quad \forall x > 0. \tag{39}$$

Therefore, $\mu(x)$ is strictly decreasing for $x > 0$. Then, we obtain one and only one solution of Eq. (35) in the case $\{\sqrt{\pi a_2} q_0/[k_2(t_0 - t_{\mathrm{v}})]\} - 1 > 0$, that is, q_0 satisfies inequality (36). In the case $q_0 \leqslant k_2(t_0 - t_{\mathrm{v}})/\sqrt{\pi a_2}$, there is no solution of Eq. (35) and then there is no solution of problem (1)–(8) as a phase change problem. So, if $q_0 \leqslant k_2(t_0 - t_{\mathrm{v}})/\sqrt{\pi a_2}$ there exists only a heat conduction problem for the initial phase. □

As we said above, if $q_0 \leqslant k_2(t_0 - t_{\mathrm{v}})/\sqrt{\pi a_2}$ there exists only a heat conduction problem for the initial phase. Let us consider this problem. In this case we have the following.

There is no phase change front, so the porous body consists of only one region. The region $0 < x < +\infty$ is humid capillary porous body in which there are coupled heat and moisture flows. The process is described by the well-known Luikov's system [10] for the case $\varepsilon = 0$ (ε is the phase conversion factor of liquid into vapor):

$$\frac{\partial t_2}{\partial \tau}(x,\tau) = a_2 \frac{\partial^2 t_2}{\partial x^2}(x,\tau), \quad x > 0,\ \tau > 0, \tag{40}$$

$$\frac{\partial u}{\partial \tau}(x,\tau) = a_{\mathrm{m}} \frac{\partial^2 u}{\partial x^2}(x,\tau) + a_{\mathrm{m}}\delta \frac{\partial^2 t_2}{\partial x^2}(x,\tau), \quad x > 0,\ \tau > 0. \tag{41}$$

The initial distributions of temperature and moisture are uniform:

$$t_2(x,0) = t_2(+\infty,\tau) = t_0, \quad u(x,0) = u(+\infty,\tau) = u_0. \tag{42}$$

It is assumed that on the surface of the half-space the heat flux depends on the time in the following way, like in [12]:

$$k_2 \frac{\partial t_2}{\partial x}(0,\tau) = q_0/\sqrt{\tau}, \tag{43}$$

where $q_0 > 0$ is a coefficient which characterizes the heat flux at the fixed face $x = 0$. The set of equations and conditions (40)–(43) is called problem P_{hc}. Its solution is given by

$$t_2(x,\tau) = t_0 - (q_0\sqrt{\pi a_2}/k_2)\left(1 - \operatorname{erf}\left(x/2\sqrt{a_2 t}\right)\right). \tag{44}$$

We remark that

$$t_2(0,\tau) = t_0 - (q_0\sqrt{\pi a_2}/k_2) \tag{45}$$

which is a constant in time. Then we can say that there exists a phase change on this problem if and only if $t_2(0,t) < t_v$, that is to say, if $q_0 > (t_0 - t_v)k_2/\sqrt{\pi a_2}$. This fact certifies the result obtained in Theorem 1, but obviously it does not give us the corresponding explicit solution which must be obtained by the previous method.

The resolution for the problem for the moisture distribution is slightly different. It is assumed that on the surface of the half-space the moisture distribution is related to the temperature distribution in the following way:

$$\frac{\partial u}{\partial x}(0,\tau) + \delta \frac{\partial t_2}{\partial x}(0,\tau) = 0. \tag{46}$$

Then considering (44) the problem for the moisture distribution is

$$\begin{aligned} &\frac{\partial u}{\partial \tau}(x,\tau) = a_{\mathrm{m}} \frac{\partial^2 u}{\partial x^2}(x,\tau) - \frac{a_{\mathrm{m}}\delta q_0 x}{2a_2 k_2 t\sqrt{t}} \exp\left[-\left(x^2/4a_2 t\right)\right], \quad x>0,\ \tau>0, \\ &u(x,0) = u(+\infty,t) = u_0, \\ &\frac{\partial u}{\partial x}(0,\tau) = -\frac{\delta q_0}{k_2\sqrt{t}}. \end{aligned} \tag{47}$$

We consider a change of variable of the similarity type, that is to say, let $\eta = x/2\sqrt{t}$ and $u(x,t) = H(\eta)$ be. Then the problem appears in the following way:

$$\begin{aligned} &\frac{\partial^2 H}{\partial \eta^2}(\eta) - \frac{2}{a_{\mathrm{m}}}\eta \frac{\partial H}{\partial \eta}(\eta) = \frac{4\delta q_0}{a_2 k_2} \exp\left[-\left(\eta^2/a_2\right)\right], \quad \eta > 0, \\ &H(+\infty) = u_0, \\ &\frac{\partial H}{\partial \eta}(0) = -\frac{2\delta q_0}{k_2}. \end{aligned} \tag{48}$$

The solution of this problem is obtained using the knowledge of the non-homogeneous ordinary differential equations, and it is given by

$$H(\eta) = u_0 + \frac{\sqrt{\pi a_2}\delta q_0 \mathscr{L}_u}{k_2(1-\mathscr{L}_u)} \operatorname{erf}\left(\frac{\eta}{\sqrt{a_2}}\right) + \frac{\sqrt{\pi a_{\mathrm{m}}}\delta q_0}{k_2\left(1+\sqrt{\mathscr{L}_u}\right)}\left(1 - \frac{1}{1-\sqrt{\mathscr{L}_u}}\operatorname{erf}\left(\frac{\eta}{\sqrt{a_{\mathrm{m}}}}\right)\right). \tag{49}$$

or returning to the primal variables by

$$u(x,t) = u_0 + \frac{\sqrt{\pi a_2}\delta q_0 \mathscr{L}_u}{k_2(1-\mathscr{L}_u)} \operatorname{erf}\left(\frac{x}{2\sqrt{a_2 t}}\right) + \frac{\sqrt{\pi a_{\mathrm{m}}}\delta q_0}{k_2\left(1+\sqrt{\mathscr{L}_u}\right)}\left(1 - \frac{1}{1-\sqrt{\mathscr{L}_u}}\operatorname{erf}\left(\frac{x}{2\sqrt{a_{\mathrm{m}} t}}\right)\right). \tag{50}$$

Then we have the following result:

Lemma 2. *If* $q_0 \leqslant k_2(t_0 - t_v)/\sqrt{\pi a_2}$ *the solution of the heat problem for the initial phase is given by* (44) *and* (50).

Remark 1. *The limit case* $q_0 = k_2(t_0 - t_v)/\sqrt{\pi a_2}$ *can be interpretated as the limit of the solution for the case* $q_0 > k_2(t_0 - t_v)/\sqrt{\pi a_2}$ *when* $r \to +\infty$ (*see Proposition* 3).

Remark 2. *In Section* 5 *we give some illustrative results corresponding to the previous results in order to see graphically* λ *versus* q_0, *and* $T_1\&T_2$ *versus* x *for different values of* q_0 *and a given time* $F_0 = 1$.

3. Statement of the problem $\widetilde{P}$

Now, if we replace the heat flux condition (5) by a constant boundary temperature condition at the fixed face $x = 0$ as

$$t_1(0, \tau) = t_s, \tag{5a}$$

where $t_s < t_v$, let problem $\widetilde{P}$ given by conditions (1)–(4), (5a), (6)–(16); it was studied by Luikov [9]. Its solution is given by:

$$\widetilde{T}_1(X, F_0) = \widetilde{A_1} + \widetilde{B_1}\operatorname{erf}\left(X/2\sqrt{a_{12}F_0}\right) = \widetilde{T}_s\left(1 - \frac{\operatorname{erf}\left(X/2\sqrt{a_{12}F_0}\right)}{\operatorname{erf}\left(\widetilde{\lambda}/\sqrt{a_{12}}\right)}\right), \quad 0 < X < \widetilde{S}(F_0), \tag{51}$$

$$\widetilde{T}_2(X, F_0) = \widetilde{A_2} + \widetilde{B_2}\operatorname{erf}\left(X/2\sqrt{F_0}\right) = \left(1 - \operatorname{erf}\widetilde{\lambda}\right)^{-1}\left(\operatorname{erf}\left(X/2\sqrt{F_0}\right) - \operatorname{erf}\widetilde{\lambda}\right), \quad X > \widetilde{S}(F_0), \tag{52}$$

$$\begin{aligned}\widetilde{\Theta}(X, F_0) = {} & [\mathscr{L}_u\mathscr{P}_n/(1 - \mathscr{L}_u)] \\ & \times\left(1 - \frac{\exp\left(\widetilde{\lambda}/\mathscr{L}_u\right)\operatorname{erf}\left(X/2\sqrt{\mathscr{L}_uF_0}\right)}{\sqrt{\mathscr{L}_u}\exp\left(\widetilde{\lambda}^2\right)\left[1 - \operatorname{erf}\widetilde{\lambda}\right]} + \frac{\operatorname{erf}\widetilde{\lambda}}{1 - \operatorname{erf}\widetilde{\lambda}} - \frac{\operatorname{erf}\left(X/2\sqrt{F_0}\right)}{1 - \operatorname{erf}\widetilde{\lambda}}\right),\end{aligned} \tag{53}$$

$$\widetilde{S}(F_0) = 2\widetilde{\lambda}\sqrt{F_0}, \tag{54}$$

where $\widetilde{T}_s = (t_s - t_v)/(t_0 - t_v) < 0$, and $\widetilde{\lambda}$ satisfies the following equation:

$$\alpha(x) = \beta(x), \quad x > 0, \tag{55}$$

where

$$\alpha(x) = -k_{12}\widetilde{T}_s(a_{12})^{-1/2}\big(\exp\big(-x^2/a_{12}\big)/\mathrm{erf}(x/\sqrt{a_{12}})\big), \quad x > 0$$

and $\beta(x)$ is given by (34).

The function $\alpha(x)$ is a continuous decreasing function, with the following properties:

$$\alpha(0) = +\infty; \quad \alpha(+\infty) = -\infty; \quad \alpha'(x) < 0, \; x > 0$$

(Note that $[\mathscr{L}_u\mathscr{P}_n/(1-\mathscr{L}_u)]H'(x) > 0 \; \forall \mathscr{L}_u \neq 1$.) Therefore, there is one and only one solution of Eq. (55).

If now we consider the latent heat r as a parameter in problem $\widetilde{P}$, then we have a dependence of the solutions about this new parameter, say

$$\widetilde{T}_1 = \widetilde{T}_{1r}; \quad \widetilde{T}_2 = \widetilde{T}_{2r}; \quad \widetilde{\Theta} = \widetilde{\Theta}_r; \quad \widetilde{S} = \widetilde{S}_r; \quad \widetilde{\lambda} = \widetilde{\lambda}(r)$$

and we can easily see the following statement:

Proposition 3. *If $r \to +\infty$, the parameter $\widetilde{\lambda} = \widetilde{\lambda}(r) \to 0$, and the solutions of the $\widetilde{P}$ problem tends to*:

$$\lim_{r\to+\infty} \widetilde{T}_{2r}(X, F_0) = \mathrm{erf}\Big(X/2\sqrt{F_0}\Big), \quad X > 0, \tag{56}$$

$$\lim_{r\to+\infty} \widetilde{\Theta}_r(X, F_0) = [\mathscr{L}_u\mathscr{P}_n/(1-\mathscr{L}_u)]\Big(1 - \mathrm{erf}\Big(X/2\sqrt{F_0}\Big) - (\mathscr{L}_u)^{-1/2}\mathrm{erf}\Big(X/2\sqrt{\mathscr{L}_u F_0}\Big)\Big), \quad X > 0, \tag{57}$$

$$\lim_{r\to+\infty} \widetilde{S}_r(t) = 0. \tag{58}$$

Remark 3. (i) *We remark that* (56)–(58) *is the solution for the heat transfer problem without a phase change process for the initial phase with constant temperature condition on the surface $x = 0$.* (ii) *We can interpretate* (51)–(54) *for the case $r \to +\infty$, saying that the initial phase cannot reach the another phase.* (iii) *The solution of problem P for the limit case $q_0 = k_2(t_0 - t_v)/\sqrt{\pi a_2}$ is given by* (56)–(58).

4. Relationship between heat transfer problems with temperature and heat flux at the fixed face

Now, we are back to the initial problem P with heat flux condition, considering the case $q_0 > k_2(t_0 - t_v)/\sqrt{\pi a_2}$. Evaluating (28) in $x = 0$, we obtain

$$T_1(0, \tau) = -[\sqrt{\pi a_1}q_0/(k_1(t_0 - t_v))]\,\mathrm{erf}(\lambda/\sqrt{a_{12}}). \tag{59}$$

Returning to the dimensionless variables, we observe that

$$t_1(0,\tau) = -(t_0 - t_v)(\sqrt{a_{12}}/k_{12})\,\mathrm{erf}(\lambda/\sqrt{a_{12}}) + t_v < t_v. \tag{60}$$

Then we can consider the $\widetilde{P}$ problem putting

$$t_s = -[\sqrt{\pi a_1}q_0/k_1]\,\mathrm{erf}(\lambda/\sqrt{a_{12}}) + t_v < t_v. \tag{61}$$

The solution of this problem is given by (51)–(54) where $\widetilde{\lambda}$ is solution of Eq. (55). We know that for this problem exists one and only one $\widetilde{\lambda} > 0$ such as $\alpha(\widetilde{\lambda}) = \beta(\widetilde{\lambda})$. Now we want to demonstrate that $\widetilde{\lambda} = \lambda$; for that we shall prove that $\widetilde{\lambda}$ is also solution of Eq. (35). We have

$$\begin{aligned}\beta\left(\widetilde{\lambda}\right) = \alpha\left(\widetilde{\lambda}\right) &= (k_{12}\sqrt{\pi a_1}q_0/[k_1(t_0 - t_v)\sqrt{a_{12}}])\,\mathrm{erf}\left(\widetilde{\lambda}/\sqrt{a_{12}}\right)\frac{\exp\left(-\widetilde{\lambda}^2/a_{12}\right)}{\mathrm{erf}\left(\widetilde{\lambda}/\sqrt{a_{12}}\right)} - F_1\left(\widetilde{\lambda}\right)\\ &= [\sqrt{\pi a_2}q_0/(k_2(t_0 - t_v))]\exp\left(-\widetilde{\lambda}^2/a_{12}\right) - F_1\left(\widetilde{\lambda}\right) = \mu\left(\widetilde{\lambda}\right),\end{aligned} \tag{62}$$

that is $\widetilde{\lambda}$ is a solution of Eq. (35) which has a unique solution λ, then $\widetilde{\lambda} = \lambda$.

Besides, the constant $\widetilde{A}_1$ and $\widetilde{B}_1$ belonging to the solution $\widetilde{T}_1$ are

$$\widetilde{A}_1 = -[\sqrt{\pi a_1}q_0/(k_1(t_0 - t_v))]\,\mathrm{erf}(\lambda/\sqrt{a_{12}}) = A_1, \tag{63}$$

$$\widetilde{B}_1 = -\widetilde{T}_s^{-1}\,\mathrm{erf}(\lambda/\sqrt{a_{12}}) = \sqrt{\pi a_1}q_0/(k_1(t_0 - t_v)) = B_1 \tag{64}$$

and A_2, A_3, B_2, B_3 are the same as in (22), (25) and (26). It is obvious that if $\lambda = \widetilde{\lambda}$, then $S = \widetilde{S}$.

Therefore, we obtained that the solutions of problem $\widetilde{P}$ are the same of the initial problem, that is to say, $T_1 = \widetilde{T}_1$; $T_2 = \widetilde{T}_2$; $\theta = \widetilde{\theta}$; $S = \widetilde{S}$. This immediately implies that $t_1 = \widetilde{t}_1$; $t_2 = \widetilde{t}_2$; $u = \widetilde{u}$ and $s = \widetilde{s}$. Next, we can enunciate the following property:

Theorem 4. *A phase change problem for temperature and moisture distributions in a porous half-space with a heat flux condition on the surface $x = 0$ verifying condition* (36), *is equivalent to a phase change problem with a temperature condition considering*

$$t_1(0,\tau) = t_s < t_v. \tag{65}$$

Moreover, the relationship among q_0, t_v and t_s is given by

$$t_s = t_v - [\sqrt{\pi a_1}q_0/k_1]\,\mathrm{erf}(\lambda/\sqrt{a_1}), \tag{66}$$

where λ is the coefficient which characterizes the free boundary.

As a consequence of Theorem 4, we can translate inequality (36) for q_0 for problem P to an inequality for λ for problem $\widetilde{P}$, that is to say,

$$k_2(t_0-t_v)(\pi a_2)^{-1/2} < q_0 = k_1 \frac{\partial}{\partial x} t_1(0,\tau)\sqrt{\tau} = k_1\sqrt{F_0/a_2}(t_0-t_v)\frac{\partial}{\partial X} T_1(0,F_0)$$
$$= k_1(t_s-t_v)/[\sqrt{\pi a_1}\operatorname{erf}(\lambda/\sqrt{a_{12}})]. \quad (67)$$

Therefore, we obtain the inequality

$$\operatorname{erf}(\lambda/\sqrt{a_{12}}) < k_{12}(t_v-t_s)/\sqrt{a_{12}}(t_0-t_v) \quad (68)$$

which is valid for the $\widetilde{P}$ problem. This quotation has sense when the right side of the equation is minor than 1, that is to say.

Corollary 5. *When data for problem $\widetilde{P}$ verifies the inequality*

$$k_{12}(t_v-t_s)/\sqrt{a_{12}}(t_0-t_v) < 1 \quad (69)$$

then the coefficient λ of the free boundary $\widetilde{S}(F_0) = 2\widetilde{\lambda}\sqrt{F_0}$ satisfies the inequality

$$\lambda < \sqrt{a_{12}}\operatorname{erf}^{-1}[k_{12}(t_v-t_s)/\sqrt{a_{12}}(t_0-t_v)]. \quad (70)$$

5. Some illustrative results

In order to study the effects of the parameter q_0 (coefficient which characterizes the heat flux on the fixed surface $x = 0$) over our process we shall give firstly the graphics of the function λ vs q_0, where λ is the dimensionless coefficient which characterizes the free boundary (22). Let $q_0^* = q_0\sqrt{\pi a_2}/k_2(t_0-t_v)$ be the dimensionless corresponding coefficient. For a given positive time, Fig. 1 shows the behavior of λ as a function of q_0^* considering all the other parameters fixed. We

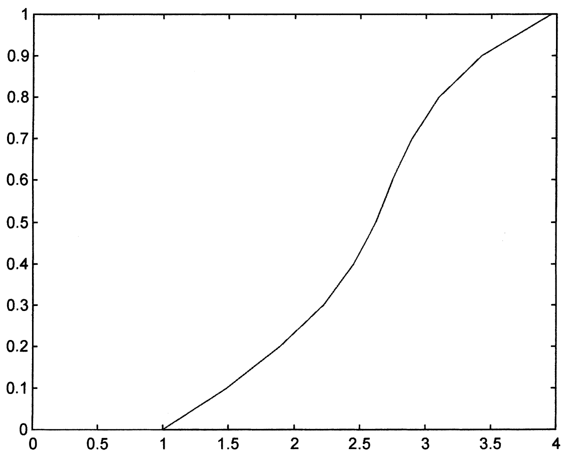

Fig. 1. Behavior of λ as a function of q_0^*.

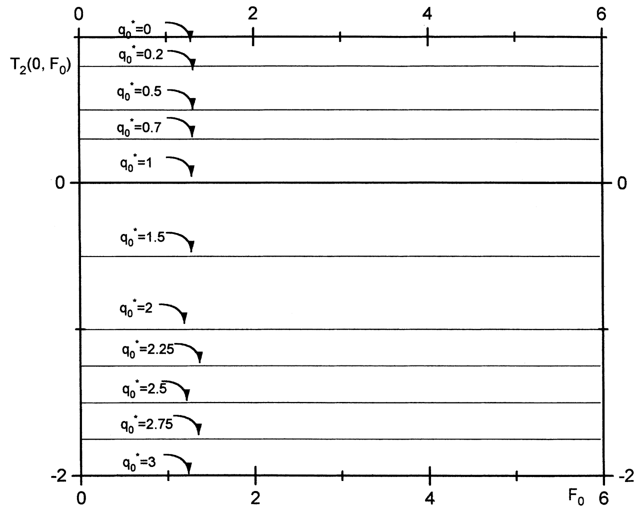

Fig. 2. Behavior of the constant temperature distribution $T_2(0, F_0)$ as a function of the dimensionless time F_0 considering $q_0^* > 0$.

put $a_{12} = 1$, $K_0 = 1.2$, $\mathscr{L}_u = 0.4$, $\mathscr{P}_n = 1$. If $q_0^* > 1$, λ increases as q_0^* increases. If $q_0^* \leqslant 1$, $\lambda = 0$, and there is no phase change.

Fig. 2 shows the behavior of the constant temperature distribution on the fixed face $x = 0$ (see (45)) as a function of the dimensionless time F_0 given certain values of $0 \leqslant q_0^* \leqslant 1$ and $q_0^* > 1$. Notice that

$$T_2(0, \tau) = \frac{t_2(0, t) - t_\mathrm{v}}{t_0 - t_\mathrm{v}} = 1 - q_0^*,$$

and when $q_0^* = 0$, we have $T_2 = 1$, and when $q_0^* = 1$, we have $T_2 = 0$. Here we notice when q_0^* tends to $+\infty$, T_2 tends to $-\infty$.

Finally, Fig. 3 shows the behavior of the dimensionless temperature distributions T_1 and T_2 as a function of the position, given a fixed time (see (28) and (29)). Here we consider $F_0 = 1$, and the values of the other parameters are the same as above. Notice the change of derivative when temperature distributions reaches from above and below the value 0 (the dimensionless phase change temperature) when $q_0^* > 1$. Furthermore, when the temperature distribution begins from a positive value, there is no phase change in the process (that is to say, $0 < q_0^* \leqslant 1$).

From all these figures we have verified numerically the theoretical results obtained before analytically.

6. Conclusion

Exact solutions for temperature and moisture distribution in a porous half-space with a heat flux condition at $x = 0$ of the type $q_0/\sqrt{t}$ are obtained. An inequality for the coefficient q_0 is necessary and sufficient in order to obtain that explicit solution. Next, we introduce the problem

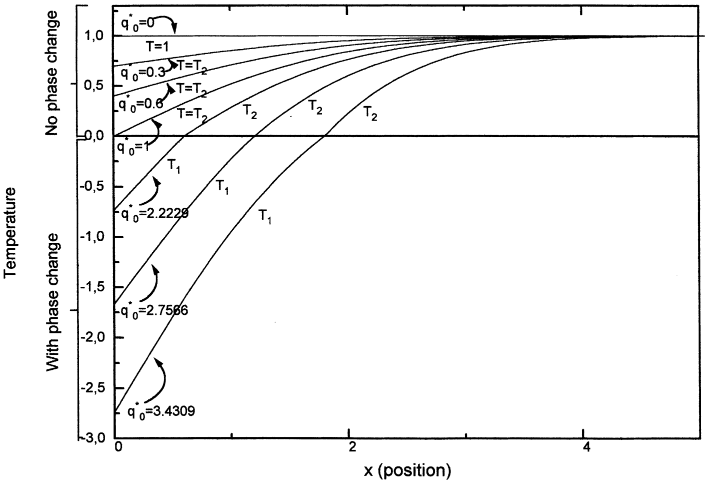

Fig. 3. Behavior of the temperature distributions as a fuction at the position x, taking $F_0 = 1$.

$\widetilde{P}$, which is the problem P changing the heat flux condition by a temperature condition on the fixed face $x = 0$, and we study the behavior of the solution of this problem considering when the parameter latent heat r tends to infinity. Finally we establish an equivalence between the above two phase change problems finding an inequality that the coefficient which characterizes the free boundary verifies.

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