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EXPLICIT SOLUTIONS FOR THE DESUBLIMATION PROBLEM IN A HUMID POROUS HALF-SPACE WITH A HEAT FLUX CONDITION

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SUMMARY

Exact solutions for the temperature and moisture distribution in a porous half-space with a heat flux condition of the type $q_0 / \sqrt{t}$ at the fixed face $x = 0$ are obtained. An inequality for the positive coefficient q_0 is necessary and sufficient in order to obtain that explicit solution. An equivalence between a phase change problem with a temperature condition and a phase change problem with a heat flux condition of that type on the surface is also obtained.

INTRODUCTION

Heat and mass transfer with phase change problems, taking place in a porous medium, such as evaporation, condensation, freezing, melting, sublimation and desublimation, have wide application in separation processes, food technology, heat and mixture migration in soils and grounds, etc. Due to the non-linearity of the problem, solutions usually involve mathematical difficulties. Only a few exact solutions have been found for idealized cases, for example Lamé and Clapeyron in 1831, Stefan in 1890, Carslaw an Jaeger in 1959, Luikov in 1968, Mikhailov in 1975 and 1976, and Tarzia in 1982.
Mathematical formulation of the heat and mass transfer in capillary porous bodies has been established by Luikov in 1966, 1968 and 1978. For solving the problem of evaporation of liquid moisture from a porous medium with two differents models was presented by Mikhailov in 1975. For the problem of freezing (desublimation) of humid porous half-space, Mikhailov also presented an exact solution in 1976. Lin presents in 1981 an exact solution of the desublimation problem in a porous medium for a temperature condition on a fixed face. Other problems in this direction are given by Fasano et al. in 1993 (and to appear), Gonzalez and Tarzia in 1996, and Santillan Marcus and Tarzia (to appear).
In the following, a desublimation process in a porous medium with heat flux condition at x = 0 of the type $q_0 / \sqrt{t}$ will be studied. An analytical model of the process is defined and exact solutions for temperature and moisture distributions, as well as the location of the moving desublimation, are obtained. An inequality for the coefficient q_0 is necessary and sufficient in order to obtain that explicit solution. Finally, an equivalence between a phase change problem with a temperature condition and a phase change problem with a heat flux condition of the type $q_0 / \sqrt{t}$ on the surface is obtained. Moreover, an inequality for the coefficient which characterizes the moving desublimation is also obtained when a temperature condition is given at the fixed face.

STATEMENT OF THE PROBLEM P

Let us consider a solid, rigid, porous half-space containing a uniform mixture of air and moisture in a vapor form. Initially the porous body is at a uniform moisture concentration C_i and an uniform temperature T_i'. The vapor is desublimated by maintaining a heat flux condition of the type $q_0 / \sqrt{t}$ in the surface at x = 0. For formulation of the problem, we made the same assumptions as Lin made in 1982:
(1) In the frozen region, $0 < x < s(t)$, there is no moisture movement, where $s = s(t)$ locates the moving desublimation front. In the vapor region, $s(t) < x < \infty$, there are heat and moisture mass flows.
(2) The convective terms in the vapor region are small and may be neglected.
(3) The thermophysical properties of the frozen and vapor regions remain respectively constant.
(4) The Soret effect, or the thermal diffusion, gives rise to a mass flux which is normally very small relative to the normal Fickian flux, and may be neglected.
The following differential equations describe the process of desublimation:

$$\frac{\partial T_1}{\partial t} = a_1 \frac{\partial^2 T_1(x,t)}{\partial x^2}, 0 < x < s(t) \qquad (1)$$

$$\frac{\partial T_2}{\partial t} = a_2 \frac{\partial^2 T_2(x,t)}{\partial x^2}, s(t) < x < \infty \qquad (2)$$

$$\frac{\partial C}{\partial t} = a_m \frac{\partial^2 C(x,t)}{\partial x^2}, s(t) < x < \infty \qquad (3)$$

where a_1 and a_2 are the volume averaged thermal diffusivities in the frozen and vapor regions, respectively, and a_m the volume averaged mass diffusivity of the vapor in the porous body.
The initial and boundary conditions can be described as follows:

$$T_2(x,0) = T_i,\ 0 < x < \infty \qquad (4)$$

$$C(x,0) = C_i,\ 0 < x < \infty \qquad (5)$$

$$s(0) = 0 \qquad (6)$$

$$k_1 \frac{\partial T_1}{\partial x}(0,t) = \frac{q_0}{\sqrt{t}} \qquad (7)$$

$$T_1(s(t),t) = T_2(s(t),t) = T_s \qquad (8)$$

$$C(s(t),t) = C_s < C_i \qquad (9)$$

$$T_2(\infty,t) = T_i \qquad (10)$$

$$C(\infty,t) = C_i \qquad (11)$$

The heat and moisture mass balance at the desublimation front $x = s(t)$ can be expressed by:

$$k_1 \frac{\partial T_1(s,t)}{\partial x} - k_2 \frac{\partial T_2}{\partial x} = C_0 L\, \dot{s}(t), \qquad (12)$$

$$a_m \frac{\partial C(s,t)}{\partial x} = (C_0 - C_s)\dot{s}(t) \qquad (13)$$

where k_1 and k_2 are the volume averaged thermal conductivities in the frozen and vapor regions, respectively, L is the latent heat of desublimation, C_s is the moisture concentration of the vapor phase at the desublimation front and $C_0 (C_0 > C_s)$ is the moisture concentration of the frozen phase, which is unknown and has to be determined as a part of the solution. The heat flux condition (7) was firstly considered by Tarzia in 1982 for a phase-change problem.

SOLUTION OF THE PROBLEM P

The system of equations (1), (2), (4), (6)-(8) and (10), describing the temperature distributions in the frozen and vapor regions in the desublimation process is the same as that describing the temperature distributions in the solid and liquid regions in a solidification process for a pure substance, with a heat flux condition of the type $q_0 / \sqrt{t}$. Therefore, Neumann's solution can be used to obtain the temperature distributions, as it has shown by Tarzia in 1982 and 1985. The solution of the equations (1) - (2) which satisfies the initial and boundary conditions (4), (6)-(8) can be obtained as:

$$T_1(x,t) = T_s + \frac{q_0\sqrt{\pi a_1}}{k_1}\left(erf\left(\frac{x}{2\sqrt{a_1 t}}\right) - erf\left(\sqrt{\frac{a_2}{a_1}}\lambda\right)\right) \qquad (14)$$

$$T_2(x,t) = T_i - \frac{T_i - T_s}{erfc\lambda}\ erfc\left(\frac{x}{2\sqrt{a_2 t}}\right) \qquad (15)$$

where $erf(z)$ is the error function defined by

$$erf(z) = \frac{2}{\pi}\int_0^z \exp(-x^2)dx$$

and $erfc(z)$ is the complimentary error function, defined by

$$erfc(z) = 1 - erf(z),$$

and λ is a dimensionless constant which characterizes the desublimation front given by:

$$s(t) = 2\lambda\sqrt{a_2 t} \qquad (16)$$

Similarly, the solution of equation (3) which satisfies the initial and boundary conditions (5), (9) and (11) can be expressed as:

$$C(x,t) = C_i - \frac{C_i - C_s}{erfc\left(\sqrt{\frac{a_2}{a_m}}\lambda\right)}\ erfc\left(\frac{x}{2\sqrt{a_m t}}\right) \qquad (17)$$

Functions (14), (15), (16) and (17) satisfy all the initial and boundary conditions except conditions (12) and (13) which are used for the determination of the unknown C_0 and λ. From (15) and (17) into equation (13) we obtain:

$$C_0 = C_s + \sqrt{\frac{a_m}{a_2}}\left(\frac{C_i - C_s}{\sqrt{\pi}\lambda}\right)F_1\left(\sqrt{\frac{a_m}{a_2}}\lambda\right) \qquad (18)$$

where F_1 is the real function defined by $F_1(z) = \frac{\exp(-z^2)}{erfc z}$.

Then, from (14)-(16) and (18) into condition (12), we obtain an equation for the determination of λ which is given by:

$$q_0 \exp\left(-\frac{a_2}{a_1}\lambda^2\right) - \frac{k_2(T_i - T_s)}{\sqrt{\pi a_2}}F_1(\lambda) - \sqrt{\frac{a_m}{\pi}}(C_i - C_s)LF_1\left(\sqrt{\frac{a_2}{a_m}}\lambda\right) = C_s L\sqrt{a_2}\,\lambda \qquad (19)$$

Now, we define the real functions $\Psi(x)$ and $\Phi(x)$ in the following way ($x > 0$):

$$\Psi(x) = q_0 \exp\left(-\frac{a_2}{a_1}x^2\right) - \frac{k_2(T_i - T_s)}{\sqrt{\pi\, a_2}} F_1(x) - \sqrt{\frac{a_m}{\pi}}(C_i - C_s) L F_1\left(\sqrt{\frac{a_2}{a_m}}x\right) \quad (20)$$

$$\Phi(x) = C_s L \sqrt{a_2}\, x. \quad (21)$$

We know that

$$F_1(0) = 1 \; ; \; F_1(+\infty) = +\infty \; ; \; F_1'(x) > 0 \;\; \forall x > 0,$$

then it results that $\Psi(x)$ is a strictly decreasing function, with the limit conditions

$$\Psi(0) = q_0 - \frac{k_2(T_i - T_s)}{\sqrt{\pi\, a_2}} - (C_i - C_s)L\sqrt{\frac{a_m}{\pi}};$$

$$\Psi(+\infty) = -\infty$$

Moreover, $\Phi(x)$ is a strictly increasing function, with

$$\Phi(0) = 0 \; ; \; \Phi(+\infty) = +\infty.$$

Then, we can enunciate the following:

<u>Theorem</u>. If the coefficient q_0 verifies the condition

$$q_0 > \frac{k_2(T_i - T_s)}{\sqrt{\pi\, a_2}} + (C_i - C_s)L\sqrt{\frac{a_m}{\pi}}, \quad (22)$$

then there exists one and only one solution $\lambda > 0$ of the equation (19). If $q_0 \le \frac{k_2(T_i - T_s)}{\sqrt{\pi\, a_2}} + (C_i - C_s)L\sqrt{\frac{a_m}{\pi}}$, then there is no solution of the problem (1)-(13) as a phase change problem; it is only a heat conduction problem for the initial phase.

<u>Proof.</u> It is sufficient to have $\Psi(0) > \Phi(0)$, which is inequality (22).

STATEMENT OF THE PROBLEM P^*

Now, if we replace the heat flux condition (7) by a constant boundary temperature condition at the fixed face x = 0 as:

$$T_1(0,t) = T_0 \quad (23)$$

where $T_o < T_s$, let the problem P^* be given by the conditions (1)-(6), (23), (8)-(13), which was previously studied (Lin, 1982). Its solution is given by:

$$T_1^*(x,t) = T_0 + \frac{T_i - T_0}{erf\left(\sqrt{\frac{a_2}{a_1}}\lambda^*\right)} erf\left(\frac{x}{2\sqrt{a_1 t}}\right), 0 < x < s^*(t) \quad (24)$$

$$T_2^*(x,t) = T_i - \frac{T_i - T_s}{erfc\left(\lambda^*\right)} erfc\left(\frac{x}{2\sqrt{a_2 t}}\right), x > s^*(t) \quad (25)$$

$$C^*(x,t) = C_i - \frac{C_i - C_s}{erfc\left(\sqrt{\frac{a_2}{a_m}}\lambda^*\right)} erfc\left(\frac{x}{2\sqrt{a_2 t}}\right), x > s^*(t) \quad (26)$$

$$s^*(t) = 2\lambda^* \sqrt{a_2 t} \quad (27)$$

where λ^* satisfies the following equation:

$$\frac{k_1(T_s - T_0)}{\sqrt{\pi\, a_1}} F_2\left(\sqrt{\frac{a_2}{a_1}}\lambda^*\right) - \frac{k_2(T_i - T_s)}{\sqrt{\pi\, a_2}} F_1\left(\lambda^*\right) - \frac{\sqrt{a_m} L (C_i - C_s)}{\sqrt{\pi}} F_1\left(\sqrt{\frac{a_2}{a_m}}\lambda^*\right) = \Phi\left(\lambda^*\right) \quad (28)$$

where $\Phi(x)$ is given by (21) and F_2 is the real function defined by $F_2(z) = \frac{\exp\left(-z^2\right)}{erf\, z}$ which satisfies the following properties:

$$F_2(0^+) = +\infty \; ; \; F_2(+\infty) = 0 \; ; \; F_2'(x) < 0 \;\; \forall x > 0$$

If we define, as before, the following function:

$$\Theta(x) = \frac{k_1(T_s - T_0)}{\sqrt{\pi\, a_1}} F_2\left(\sqrt{\frac{a_2}{a_1}}x\right) - \frac{k_2(T_i - T_s)}{\sqrt{\pi\, a_2}} F_1(x) - \sqrt{\frac{a_m}{\pi}} L (C_i - C_s) F_1\left(\sqrt{\frac{a_2}{a_m}}x\right) \quad (29)$$

then we notice that $\Theta(x)$ is a continuous decreasing function, with the following properties:

$$\Theta(0^+) = +\infty \; ; \; \Theta(+\infty) = -\infty \; ;$$
$$\Theta'(x) < 0, x > 0$$

Therefore, there is one and only one solution λ^* of the equation (28).

RELATIONSHIP BETWEEN PROBLEMS WITH TEMPERATURE AND HEAT FLUX AT THE FIXED FACE

Now we are back to the initial problem P with heat flux condition, considering the case $q_0 > \frac{k_2(T_i - T_s)}{\sqrt{\pi\, a_2}} + (C_i - C_s)L\sqrt{\frac{a_m}{\pi}}$. Evaluating (14) in $x = 0$, we obtain:

$$T_1(0,t) = T_s - \frac{q_0\sqrt{\pi\, a_1}}{k_1} erf\left(\sqrt{\frac{a_2}{a_1}}\lambda\right) \quad (30)$$

Then we can consider the problem P^* putting

$$T_0 = T_s - \frac{q_0\sqrt{\pi a_1}}{k_1} erf\left(\sqrt{\frac{a_2}{a_1}}\lambda\right) < T_s \quad (31)$$

The solution of the problem is given by (24)-(27) where λ^* is solution of the equation (28). We know that for this problem exists one and only one $\lambda^* > 0$ such as $\Theta(\lambda^*) = \Phi(\lambda^*)$.

Now, we want to demonstrate that $\lambda^* = \lambda$; for that we shall prove that λ^* is also solution of the equation (19).

$$\Phi(\lambda^*) = \Theta(\lambda^*) = \frac{k_1(T_s - (T_s - \frac{q_0\sqrt{\pi a_1}}{k_1} erf(\sqrt{\frac{a_2}{a_1}}\lambda))}{\sqrt{\pi a_2}} F_2\left(\sqrt{\frac{a_2}{a_1}}\lambda^*\right)$$

$$-\frac{k_2(T_i - T_s)}{\sqrt{\pi a_2}} F_1(\lambda^*) - \sqrt{\frac{a_m}{\pi}} L(C_i - C_s) F_1(\sqrt{\frac{a_2}{a_1}}\lambda^*) = \Psi(\lambda^*)$$

that is λ^* is a solution of the equation (19) which has a unique solution λ, then $\lambda^* = \lambda$.

Therefore we obtained that the solutions of the problem P^* are the same of the initial problem, that is to say:

$$T_1 = T_1^* \; ; \; T_2 = T_2^* \; ;$$
$$C = C^* \; ; \; s = s^*$$

Next, we can enunciate the following property:

Property. A phase change problem for temperature and moisture distributions in a porous half-space with a heat flux condition on the surface x = 0 verifying the condition (22), is equivalent to a phase change problem with a temperature condition considering

$$T_0 = T_1(0,t) = T_s - \frac{q_0\sqrt{\pi a_1}}{k_1} erf\left(\sqrt{\frac{a_2}{a_1}}\lambda\right) < T_s \quad (32)$$

Moreover, the relationship among q_0, T_s and T_0 is given by (31), where λ is the coefficient which characterizes the free boundary.

As a consequence of this property, we can translate the inequality (30) for q_0 for problem P to an inequality for λ for problem P^*, that is to say

$$\left(\sqrt{\frac{a_2}{a_1}}\lambda^*\right) = \frac{(T_s - T_0)k_1}{q_0\sqrt{\pi a_1}} < \frac{(T_s - T_0)\frac{k_1}{\sqrt{a_1}}}{\frac{(T_i - T_s)k_2}{\sqrt{a_2}} + (C_i - C_s)L\sqrt{a_m}}.$$

This last inequality is valid for the problem P^*, and it has a physical sense when the right side of the inequation is minor than one, that is to say:

Corollary. When data for the problem P^* verifies the inequality

$$\frac{\frac{k_1}{\sqrt{a_1}}(T_s - T_0)}{\frac{(T_i - T_s)k_2}{\sqrt{a_2}} + (C_i - C_s)L\sqrt{a_m}} < 1 \quad (33)$$

then the coefficient λ^* of the free boundary $s^*(t) = 2\lambda^*\sqrt{t}$ satisfies the inequality

$$\lambda^* < \sqrt{\frac{a_1}{a_2}}\, erf^{-1}\left(\frac{(T_s - T_0)\frac{k_1}{\sqrt{a_1}}}{\frac{(T_i - T_s)k_2}{\sqrt{a_2}} + (C_i - C_s)L\sqrt{a_m}}\right). \quad (34)$$

CONCLUSION

Exact solutions are obtained for temperature and moisture distribution in a porous half-space with a heat flux condition at $x = 0$ of the type $q_0/\sqrt{t}$. An inequality for the coefficient q_0 is necessary and sufficient in order to obtain that explicit solution. Next, we have introduced the problem P^*, which is the problem P changing the condition (7) by a temperature condition (23) in x = 0. Finally we have established an equivalence between a phase change problem with a temperature condition and a phase change problem with a heat flux condition of the type $q_0/\sqrt{t}$ on the surface, finding an inequality that the coefficient which characterizes the free boundary must verify. This paper generalizes the results obtained for the temperature of the solid-liquid phase-change problem by Tarzia in 1982, for the desublimation problem in a humid porous half-space with a heat flux at the fixed face (with coupled temperature and concentration distributions).

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