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SIMULTANEOUS DETERMINATION OF TWO UNKNOWN THERMAL COEFFICIENTS OF A SEMI-INFINITE POROUS MATERIAL THROUGH A DESUBLIMATION MOVING BOUNDARY PROBLEM WITH COUPLED HEAT AND MOISTURE FLOWS

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Abstract

An analytical model of freezing (desublimation) of moisture in a porous medium with an overspecified condition at the fixed face is considered in order to determine simultaneously two unknown thermal coefficients of a semi-infinite phase-change material. When the evaporation front is experimentally determined, a moving boundary problem with coupled heat and moisture flows (Luikov type equations) with eight heat parameters can be considered. We obtain the explicit expression of the temperature of the two phases and the mass-transfer potential in the humid region, and we also give formulae for the two unknown thermal

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coefficients and the necessary and sufficient condition for the parameters in order to obtain the existence of a solution for 28 different cases.

1. Nomenclature

$a_i = \dfrac{k_i}{\rho c_i}$	thermal diffusivity of the phase-i
a_m	moisture diffusivity
c_i	specific heat capacity of the phase-i
k_i	thermal conductivity of the phase-i
$\mathcal{K}o = \dfrac{r u_0}{c_2(t_0 - t_v)}$	Kossovitch number
$\mathcal{L}u = \dfrac{a_m}{a_2}$	Luikov number
$\mathcal{P}n = \dfrac{\delta(t_0 - t_v)}{u_0}$	Posnov number
q_0	coefficient that characterizes the heat flux at $x = 0$
r	latent heat
$s(t)$	position of the evaporation front
t	time
T_i	temperature of the phase-i
t_0	initial temperature
t_s	temperature at the fixed face $x = 0$
t_v	phase-change temperature $(t_s < t_v < t_0)$
u	mass-transfer potential
u_0	initial mass-transfer potential

Greek symbols

ρ	mass density
δ	thermal gradient coefficient
σ	constant which characterizes the moving boundary (to be determined experimentally)
γ_i ($i = 1, \ldots, 36$)	parameters used in the text

Subscripts

$i = 1$	freezing region
$i = 2$	region in which there are coupled heat and moisture flows

2. Introduction

Heat and mass transfer with phase-change problems, taking place in a porous medium, such as evaporation, condensation, freezing, melting, sublimation and desublimation, have wide applications in separation processes, food technology, heat and mixture migration in solids and grounds, etc. Due to the non-linearity of the problem, solutions usually involve mathematical difficulties. Only a few exact solutions have been found for idealized cases (see [2], [3], [4], [5], [7], [18] for example). A large bibliography on free and moving boundary problems for heat-diffusion equation was given in [20]. The computation of temperature and moisture content fields in capillary porous media, from the knowledge of initial and boundary conditions, as well as of the thermophysical properties appearing in the formulation, constitutes a direct problem of heat and mass transfer. Mathematical formulation of the heat and mass transfer in capillary porous bodies has been established by Luikov ([9], [10]), and it was recently considered in [6], [8], [13], [14], [21], and [22]. Two different models were presented by Mikhailov [11] for solving the problem of evaporation of liquid moisture from a porous medium. For the problem of freezing (desublimation) of humid porous half-space, Mikhailov [12] also presented an exact solution. In [15] an exact solution was presented for temperature and moisture distributions in a humid porous half-space with a heat flux condition on the fixed face $x = 0$ of the type $\frac{q_0}{\sqrt{t}}$. In [16] the model developed in [12] and [15] was considered as a free boundary problem with an overspecified condition on the fixed face in order to obtain the temperatures of the two phases and the mass transfer potential in the humid region, the phase-change interface and one unknown thermal coefficient under certain specified restrictions upon data.

Now in this paper we consider the same model [16] with the phase-change interface as a moving boundary problem. That is to say, we consider that we experimentally know the position of the phase-change interface $x = s(t)$ given by the expression $s(t) = 2\sigma\sqrt{t}$ with $\sigma > 0$ a given constant. We consider a desublimation problem with an overspecified condition on the fixed face of the type given in [18]. So, we can calculate two unknown thermal coefficients under certain restrictions upon data, following the idea of [19] for one phase and [17] for two phases.

Let us consider the flow of heat and moisture through a porous half-space during freezing. The position of phase-change front at time t is given by $x = s(t)$. It divides the porous body into two regions. In the freezing region, $0 < x < s(t)$, there is no moisture movement and the temperature distribution is described by the heat equation:

$$\frac{\partial T_1}{\partial t}(x, t) = \frac{k_1}{\rho c_1} \frac{\partial^2 T_1}{\partial x^2}(x, t), \quad 0 < x < s(t), \ t > 0. \tag{1}$$

The region $s(t) < x < +\infty$ is humid capillary porous body in which there are coupled heat and moisture flows. The process is described by the well-known Luikov's system [10] for the case $\varepsilon = 0$ (ε is the phase conversion factor of liquid into vapor) given by

$$\frac{\partial T_2}{\partial t}(x, t) = \frac{k_2}{\rho c_2} \frac{\partial^2 T_2}{\partial x^2}(x, t), \quad x > s(t), \ t > 0, \tag{2}$$

$$\frac{\partial u}{\partial t}(x, t) = a_m \frac{\partial^2 u}{\partial x^2}(x, t), \quad x > s(t), \ t > 0. \tag{3}$$

The initial distributions of temperature and moisture are uniform

$$\begin{cases} T_2(x, 0) = T_2(+\infty, t) = t_0, \\ u(x, 0) = u(+\infty, t) = u_0. \end{cases} \tag{4}$$

It is assumed that on the surface of the half-space, the temperature is constant

$$T_1(0, t) = t_s, \tag{5}$$

where $t_s < t_v$.

On the freezing front, there exists the continuity of the temperatures:

$$T_1(s(t), t) = T_2(s(t), t) = t_v, \quad t > 0, \tag{6}$$

where $t_v < t_0$. Heat and moisture balance at the freezing front yields

$$k_1 \frac{\partial T_1}{\partial x}(s(t), t) - k_2 \frac{\partial T_2}{\partial x}(s(t), t) = \rho r u(s(t), t) \frac{ds}{dt}(t), \quad t > 0, \tag{7}$$

$$\frac{\partial u}{\partial x}(s(t), t) + \delta \frac{\partial T_2}{\partial x}(s(t), t) = 0, \quad t > 0. \tag{8}$$

We also consider an overspecified condition on the fixed face $x = 0$ [1]; considering that the heat flux depends on the time, like in [18]

$$k_1 \frac{\partial T_1}{\partial x}(0, t) = \frac{q_0}{\sqrt{t}}, \tag{9}$$

where $q_0 > 0$ is a coefficient which characterizes the heat flux at the fixed face $x = 0$.

The moving boundary $x = s(t)$ defined for $t > 0$ with $s(0) = 0$, is given by

$$s(t) = 2\sigma\sqrt{t}, \tag{10}$$

where $\sigma > 0$ is a given positive constant which must be experimentally determined. For example, for a desublimation experiment as described before we can use the regression method in order to determine the constant $\sigma = \dfrac{s(t)}{2\sqrt{t}}$ for n data obtained for times $t_1, t_2, \ldots, t_n$.

The goal of this paper is to find formulae for the simultaneous determination of two unknown thermal coefficients chosen among ρ (mass density), a_m (moisture diffusivity), c_1 (specific heat of the frozen region), c_2 (specific heat of the humid region), k_1 (thermal conductivity of the frozen region), k_2 (thermal conductivity of the humid region), δ (thermal gradient coefficient), r (latent heat) together with the temperatures T_1 (temperature of the freezing region), T_2 (temperature of the humid region) and the moisture u. We obtain the necessary and/or sufficient conditions for data in order to find explicit expressions for the two thermal unknown coefficients. This can be obtained in all of the 28 different cases.

3. Unknown Thermal Coefficients through a Moving Boundary Problem

The set of equations and conditions (1)-(10) is called *problem P*. Following [15], for the general case $\mathcal{L}u = \dfrac{a_m}{a_2} = \dfrac{\rho a_m c_2}{k_2} \neq 1$ we can obtain

$$T_1(x,t) = t_v - \frac{\sqrt{\pi} q_0}{\sqrt{\rho c_1 k_1}}\left[-\operatorname{erf}\left(\gamma_0 \frac{x}{2\sqrt{t}}\right) + \operatorname{erf}(\gamma_0 \sigma)\right], \quad 0 < x < s(t),\ t > 0, \tag{11}$$

$$T_2(x,t) = t_v + \frac{t_0 - t_v}{1 - \operatorname{erf}\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)}\left[\operatorname{erf}\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\frac{x}{2\sqrt{t}}\right) - \operatorname{erf}\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)\right], \quad x > s(t),\ t > 0, \tag{12}$$

$$u(x,t) = u_0 - \gamma_1\left\{\operatorname{erf}\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\frac{x}{2\sqrt{t}}\right) - \frac{\exp\left(\left(\frac{1}{\mathcal{L}u} - 1\right)\frac{\mathcal{L}u}{a_m}\sigma^2\right)\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{a_m t}}\right)\right)}{\sqrt{\mathcal{L}_u}}\right\}, \quad x > s(t),\ t > 0 \tag{13}$$

with

$$\gamma_0 = \sqrt{\frac{\rho c_1}{k_1}}, \qquad \gamma_1 = \frac{\delta \rho a_m c_2 (t_0 - t_v)}{k_2 - \rho a_m c_2} = \frac{\mathcal{P}n}{\frac{1}{\mathcal{L}u} - 1}, \tag{14}$$

where the two unknown thermal coefficients must satisfy the following system of transcendental equations:

$$\gamma_2 \exp(-(\gamma_0 \sigma)^2) - F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\,\sigma\right) = \gamma_3 \sigma \left\{ 1 - \gamma_1 \left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)} \right) \right\}, \tag{15}$$

$$\operatorname{erf}(\gamma_0 \sigma) = \frac{1}{\gamma_4} \tag{16}$$

with

$$\gamma_2 = \frac{\sqrt{\pi} q_0}{\sqrt{c_2 k_2 \rho}(t_0 - t_v)}, \qquad \gamma_3 = \sqrt{\frac{\pi \rho}{c_2 k_2}} \frac{r u_0}{(t_0 - t_v)} = \sqrt{\pi \frac{\mathcal{L}u}{a_m}}\,\mathcal{K}o,$$

$$\gamma_4 = \frac{\sqrt{\pi} q_0}{\sqrt{c_1 k_1 \rho}(t_v - t_s)}, \tag{17}$$

where the real functions F_1 and Q are defined by

$$F_1(x) = \frac{\exp(-x^2)}{(1 - \operatorname{erf}(x))}, \qquad Q(x) = \sqrt{\pi} x \exp(x^2)(1 - \operatorname{erf}(x)) \tag{18}$$

with the following properties:

$$F_1(0) = 1, \quad F_1(+\infty) = +\infty, \quad F_1'(x) > 0 \quad \forall x > 0, \tag{19}$$

$$Q(0) = 0, \quad Q(+\infty) = 1, \quad Q'(x) > 0 \quad \forall x > 0. \tag{20}$$

For the resolution of problem P, i.e., for the system (15) and (16), we can consider 28 different cases. The summary of these results are given in Appendix A, which shows also both the necessary and sufficient conditions to be verified by the data for the existence (and the uniqueness in several cases) of the solution of the problem and the expressions of the two unknown coefficients. Constants and functions used in this paper appear in Appendix B and Appendix C, respectively.

Now, we shall give necessary and sufficient conditions in order to obtain solution to above system (15) and (16) in all the cases.

First, we have the following preliminary lemma:

Lemma 1. *We have*

$$E(x) = \frac{m^2 - 1}{1 - \dfrac{Q(mx)}{Q(x)}} < 0 \qquad \forall x > 0, \ \forall m > 0, \ m \neq 1.$$

Proof. See [16].

Theorem 1 (Case 1: Determination of the unknown coefficients c_1 and k_1). *If*

$$\gamma_5 = \frac{1}{\gamma_2}\left\{F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right) + \gamma_3\sigma\left(1 - \gamma_1\left(1 - \frac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{\mathcal{L}u}{a_m}}\sigma\right)}\right)\right)\right\} < 1 \tag{21}$$

with γ_1, γ_2 *and* γ_3 *defined in* (14) *and* (17), *and* F_1 *and* Q *defined in* (18), *then there exists a unique solution to problem* P *which is given by* (11)-(13), *and the thermal coefficients* k_1 *and* c_1 *are given by the following expressions*:

$$k_1 = \frac{\sqrt{\pi}q_0}{(t_v - t_s)} F_2\left(\sqrt{\frac{1}{\log\left(\dfrac{1}{\gamma_5}\right)}}\right), \tag{22}$$

$$c_1 = \frac{\sqrt{\pi}q_0}{\sigma\rho(t_v - t_s)} F_3\left(\sqrt{\frac{1}{\log\left(\dfrac{1}{\gamma_5}\right)}}\right), \tag{23}$$

where the real functions F_2 *and* F_3 *are defined by*

$$F_2(x) = \frac{\operatorname{erf}(x)}{x}, \qquad F_3(x) = x \operatorname{erf}(x). \tag{24}$$

Proof. Considering $x = \sqrt{\dfrac{\rho c_1}{k_1}}\sigma$ and $y = \dfrac{\sqrt{\rho c_1 k_1}(t_v - t_s)}{\sqrt{\pi}q_0}$, the system

(15)-(16) can be written in the following way:

$$\exp(-x^2) = \gamma_5, \tag{25}$$

$$\operatorname{erf}(x) = y. \tag{26}$$

By considering that γ_5 is always a positive number by Lemma 1 from (25) we obtain that $x = \sqrt{-\log(\gamma_5)}$ by imposing the condition $\gamma_5 < 1$, that is to say (21). So, from (26) it follows that $y = \operatorname{erf}(\sqrt{-\log(\gamma_5)})$. Finally, after some calculations we have the expressions (22) and (23) for the thermal coefficients c_1 and k_1, respectively.

Theorem 2 (Case 2: Determination of the unknown coefficients ρ and a_m). *If*

$$\gamma_6 = \frac{k_1(t_v - t_s)}{\sqrt{\pi}\sigma q_0} < \frac{2}{\sqrt{\pi}} \tag{27}$$

and

$$\gamma_8 = \frac{1}{\mathcal{P}n}\left(1 - \frac{\sigma c_1 q_0}{k_1 r u_0}\,\frac{\exp[-(F_2^{-1}(\gamma_6))^2]}{(F_2^{-1}(\gamma_6))^2} + \frac{F_1(\gamma_7)}{\sqrt{\pi}K_0\gamma_7}\right) \in (-\infty, -1) \cup [0, +\infty)$$

with $\gamma_7 = \sqrt{\frac{c_1 k_2}{c_2 k_1}}\,\frac{1}{F_2^{-1}(\gamma_6)}$, *then the solution to problem P is given by* (11)-(13), *and the thermal coefficients* ρ *and* a_m *are given by*

$$\rho = \frac{k_1}{\sigma^2 c_1}[F_2^{-1}(\gamma_6)]^2, \tag{28}$$

$$a_m = \frac{\sigma^2}{\xi^2}, \tag{29}$$

where ξ *is the unique solution of the equation*

$$F_4(x) = F_5(x), \quad x > 0, \tag{30}$$

where

$$F_4(x) = \gamma_8\left(\frac{c_1 k_2}{c_2 k_1}\,\frac{x^2}{[F_2^{-1}(\gamma_6)]^2} - 1\right), \quad F_5(x) = 1 - \frac{Q(x)}{Q\left(\sqrt{\frac{c_1 k_2}{c_2 k_1}}\,\frac{1}{[F_2^{-1}(\gamma_6)]}\right)}.$$

Proof. From (16) and imposing (27), we have that (28) holds. Then, from (15) we have

$$\frac{1}{\mathcal{P}n}\left(1-\frac{\sigma c_1 q_0}{k_1 r u_0}\frac{\exp[-(F_2^{-1}(\gamma_6))^2]}{[F_2^{-1}(\gamma_6)]^2}+\frac{F_1(\gamma_7)}{\sqrt{\pi}K_0\gamma_7}\right)\left(\frac{c_1 k_2}{c_2 k_1}\frac{x^2}{[F_2^{-1}(\gamma_6)]^2}-1\right)$$

$$=1-\frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q(\gamma_7)},$$

where

$$\gamma_7=\sqrt{\frac{c_1 k_2}{c_2 k_1}}\frac{1}{[F_2^{-1}(\gamma_6)]},$$

that is to say (30). It is easy to see that if $\gamma_8 \geq 0$, then there always exists a unique solution ξ to (30), but if $\gamma_8 < 0$, then we can only have a unique solution ξ when $\gamma_8 < (-1)$. In both cases a_m is given by (29).

Theorem 3 (Case 3: Determination of the unknown coefficients ρ and c_2). *If* (27) *holds and*

$$\gamma_{10}=\frac{\sigma c_1 q_0}{k_1 r u_0}\frac{\exp[-(F_2^{-1}(\gamma_6))^2]}{(F_2^{-1}(\gamma_6))^2}>1, \tag{31}$$

then there exists at least one solution to problem P which is given by (11)-(13), *where the coefficient* ρ *is given by* (28) *and the coefficient* c_2 *is given by the expression*

$$c_2=\frac{c_1 k_2}{k_1}\frac{\xi^2}{[F_2^{-1}(\gamma_6)]}, \tag{32}$$

where ξ *is a solution of the equation*

$$F_6(x)=F_7(x), \quad x>0 \tag{33}$$

with

$$F_6(x)=\gamma_{10}-\gamma_{11}H(x), \quad H(x)=xF_1(x),$$

$$\gamma_{11}=\frac{c_1 k_2}{\sqrt{\pi}r u_0 k_1}\frac{(t_0-t_v)}{[F_2^{-1}(\gamma_6)]^2}, \quad F_7(x)=1-\frac{a_m\mathcal{P}n x^2}{\sigma^2-a_m x^2}\frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q(x)}.$$

Proof. Working in the same way as in Theorem 2, from (16) and imposing (27), it follows that (28) holds. Then, from (15) and taking into account that $x = \sqrt{\frac{c_1 k_2}{c_2 k_1}} \frac{1}{[F_2^{-1}(\gamma_6)]}$ we have that

$$1 - \frac{\sigma c_1 q_0}{k_1 r u_0} \frac{\exp[-(F_2^{-1}(\gamma_6))^2]}{(F_2^{-1}(\gamma_6))^2} + \frac{c_1 k_2}{\sqrt{\pi} r u_0 k_1} \frac{(t_0 - t_v)}{[F_2^{-1}(\gamma_6)]^2} x F_1(x)$$

$$= \frac{a_m \mathcal{P}n x^2}{\sigma^2 - a_m x^2} \left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q(x)} \right),$$

that is to say (33). Function F_7 has the following properties:

$$F_7(0^+) = 1, \qquad F_7(+\infty) = 1 + \mathcal{P}n\left(1 - Q\left(\frac{\sigma}{\sqrt{a_m}}\right)\right) > 1. \tag{34}$$

So we can say that we have at least a solution ξ of (33) if we impose that $\gamma_{10} > 1$. It is easy to see that (32) follows from the definition of x.

Theorem 4 (Case 4: Determination of the unknown coefficients r and a_m). *If*

$$\gamma_{12} = \frac{t_0 - t_v}{t_v - t_s} \sqrt{\frac{c_2 k_2}{c_1 k_1}} \frac{F_1\left(\sqrt{\frac{\rho c_2}{k_2}} \sigma\right)}{F_2(\gamma_0)} < 1, \tag{35}$$

then problem P has infinite solutions given by (11)-(13), *where the coefficient r is given by*

$$r = \frac{\gamma_{13}\left(\gamma_2 \exp(-(\gamma_0 \sigma)^2) - F_1\left(\sqrt{\frac{\rho c_2}{k_2}} \sigma\right)\right)}{1 - \frac{\mathcal{P}n}{\frac{k_2}{\rho a_m c_2} - 1}\left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\rho c_2}{k_2}} \sigma\right)}\right)} \tag{36}$$

for each $a_m \in \mathbb{R}^+$, where $\gamma_{13} = \frac{(t_0 - t_v)}{\sigma u_0} \sqrt{\frac{c_2 k_2}{\pi \rho}}$.

Proof. If problem data verify (16), first we fix $a_m \in \mathbb{R}^+$, and after some calculations in (15) we have (36). Taking into account the previous Lemma, it is easy to check that $r > 0$ if we impose

$$F_1\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right) < \gamma_2 \exp(-(\gamma_0\sigma)^2). \tag{37}$$

Combining (16) and (37) we obtain (35). This analysis can be done for each given $a_m > 0$.

Theorem 5 (Case 5: Determination of the unknown coefficients c_2 and k_2). *If*

$$\gamma_{14} = \frac{t_v - t_s}{\sigma r u_0}\sqrt{\frac{c_1 k_1}{\rho}} F_8(\gamma_0\sigma) > 1, \tag{38}$$

where the function F_8 is defined by

$$F_8(x) = \frac{\exp(-x^2)}{\operatorname{erf}(x)}, \tag{39}$$

then there exist infinite solutions to problem P which are given by (11)-(13), *the coefficient c_2 is given by*

$$c_2 = \frac{k_2}{\sigma^2\rho}\xi^2, \tag{40}$$

where ξ is a solution of equation

$$F_7(x) = F_9(x), \quad x > 0 \tag{41}$$

for each $k_2 \in \mathbb{R}^+$ with

$$F_9(x) = \frac{q_0}{\sigma\rho r u_0}\exp[-(\gamma_0\sigma)^2] + \frac{k_2(t_0 - t_v)}{\sqrt{\pi}\sigma^2\rho r u_0} H(x). \tag{42}$$

Proof. First, as in Theorem 4, problem data must verify (16). Then from (15) and considering $x = \sqrt{\frac{\rho c_2}{k_2}}\sigma$ we have

$$\frac{q_0}{\sigma\rho r u_0}\exp[-(\gamma_0\sigma)^2] + \frac{k_2(t_0 - t_v)}{\sqrt{\pi}\sigma^2\rho r u_0} x F_1(x) = 1 - \frac{a_m \mathcal{P} n x^2}{\sigma^2 - a_m x^2}\frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q(x)},$$

that is to say (41). Function F_9 has the following properties:

$$F_9(0^+) = \frac{q_0}{\sigma\rho r u_0}\exp[-(\gamma_0\sigma)^2], \quad F_9(+\infty) = -\infty, \quad F_9'(x) < 0 \ \ \forall x > 0. \tag{43}$$

Then, taking into account (34), we have that both functions would meet in at least one $x > 0$ if $\frac{q_0 \exp[-(\gamma_0\sigma)^2]}{\sigma\rho r u_0} > 1$. Then, considering (16), we find a solution of (41) if (38) holds. As before, it is easy to see that (40) follows from the definition of x. This analysis can be done for each given $k_2 > 0$.

Theorem 6 (Case 6: Determination of the unknown coefficients r and δ). *If*

$$\gamma_{15} = \frac{(t_0 - t_v)}{(t_v - t_s)}\sqrt{\frac{c_2 k_2}{c_1 k_1}}\,\frac{F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)}{F_5(\gamma_0\sigma)} < 1, \tag{44}$$

then there exist infinite solutions to problem P which are given by (11)-(13), *the coefficient r is given by*

$$r = \frac{\gamma_{13}\left(\gamma_2 \exp(-(\gamma_0\sigma)^2) - F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)\right)}{1 - \frac{\delta(t_0 - t_v)}{u_0\left(\frac{1}{\mathcal{L}u} - 1\right)}\left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)}\right)} \tag{45}$$

for each $\delta \in \mathbb{R}^+$.

Proof. As before, if problem data verify equation (16), we have that if we take a given $\delta \in \mathbb{R}^+$, (45) follows from (15) after some calculations. Then we have to prove that $r > 0$. Considering here again the results of the previous Lemma, it is easy to see that if we impose that

$$F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right) < \gamma_2 \exp(-(\gamma_0\sigma)^2) \tag{46}$$

we have $r > 0$. Combining (16) and (46) we obtain (44). This analysis can be done for each given $\delta > 0$.

Theorem 7 (Case 7: Determination of the unknown coefficients ρ and k_1). *There always exists a unique solution to problem P which is given by* (11)-(13), *the coefficients* k_1 *and* ρ *are given by*

$$k_1 = \frac{\rho\sigma^2 c_1}{\log\left(\frac{1}{F_{10}(\rho)}\right)}, \tag{47}$$

where ρ *is a unique solution to the equation*

$$F_{19}(x) = \frac{\sigma c_1 (t_v - t_s)}{\sqrt{\pi} q_0} x, \quad x > 0 \tag{48}$$

with

$$F_{10}(x) = \sqrt{\frac{c_2 k_2}{\pi}} \frac{(t_0 - t_v)}{q_0} \sqrt{x} \left\{ F_1\left(\sqrt{\frac{c_2}{k_2}} \sigma \sqrt{x}\right) \right.$$

$$\left. + \frac{\sqrt{\pi c_2} \mathcal{K} o \sigma}{\sqrt{k_2}} \left[1 - \frac{a_m c_2 \mathcal{P} n x}{k_2 - a_m c_2 x} \left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{c_2}{k_2}} \sigma \sqrt{x}\right)} \right) \right] \right\}$$

and

$$F_{19}(x) = F_3\left(\sqrt{\log\left(\frac{1}{F_{10}(x)}\right)}\right).$$

Proof. From (15) we obtain

$$\sqrt{\frac{\pi}{\rho c_2 k_2}} \frac{q_0 \exp\left(-\sqrt{\frac{\rho c_1}{k_1}} \sigma\right)}{(t_0 - t_v)}$$

$$= F_1\left(\frac{\sqrt{\rho c_2}}{k_2} \sigma\right) + \frac{\sqrt{\pi} \rho r u_0 \sigma}{\sqrt{c_2 k_2} (t_0 - t_v)} \left[1 - \frac{a_m c_2 \mathcal{P} n \rho}{k_2 - a_m c_2 \rho} \left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\rho c_2}{k_2}} \sigma\right)} \right) \right]$$

and thus (47). Hence from (16) we have (48). Function F_{10} has the following properties:

$$F_{10}(0^+) = 0, \quad F_{10}(+\infty) = +\infty, \quad F_{10}'(x) > 0 \quad \forall x > 0.$$

Then, there would exist a $\hat{\rho}$ such that $F_{10}(\hat{\rho}) = 1$, and then $0 < F_{10}(x) < 1$ for all $x \in (0, \hat{\rho})$. Therefore, function F_{19} has the following properties:

$$F_{19}(0^+) = +\infty, \quad F_{19}(\hat{\rho}) = 0, \quad F_{10}^{A'}(x) < 0 \quad \forall x \in (0, \hat{\rho}).$$

It follows easily that (48) would always have a unique solution that belongs to $(0, \hat{\rho})$ and therefore $k_1 > 0$.

Theorem 8 (Case 8: Determination of the unknown coefficients ρ and k_2). *If* (27) *holds, and*

$$\gamma_{17} = \gamma_{10} - \frac{(t_0 - t_v)}{u_0}\left[\frac{c_2}{r} + \delta\left(1 - Q\left(\frac{\sigma}{\sqrt{a_m}}\right)\right)\right] > 1, \tag{49}$$

then there exists at least one solution to problem P which is given by (11)-(13), *where the coefficient ρ is given by* (28) *and the coefficient k_2 is given by*

$$k_2 = \frac{1}{\xi^2}, \tag{50}$$

where ξ is a solution to the equation

$$F_{11}(x) = 1, \quad x > 0 \tag{51}$$

with

$$F_{11}(x) = \frac{1}{\gamma_{16}}\left\{\frac{\rho a_m c_2 \mathcal{P}n x^2}{1 - \rho a_m c_2 x^2}\left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q(\sqrt{\rho c_2}\sigma x)}\right) - \frac{1}{\mathcal{K}o Q(\sqrt{\rho c_2}\sigma x)}\right\} \tag{52}$$

and

$$\gamma_{16} = 1 - \frac{q_0}{\rho\sigma r u_0}\exp(-(\gamma_0\sigma)^2). \tag{53}$$

Proof. From (16) and imposing (27), we have (28). Now let $x = \sqrt{\dfrac{1}{k_2}}$ and from (15) it follows that

$$1 - \frac{q_0}{\rho\sigma r u_0} \exp(-(\gamma_0\sigma)^2) = \frac{\rho a_m c_2 \mathcal{P}n x^2}{1 - \rho a_m c_2 x^2}\left(1 - \frac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q(\sqrt{\rho c_2}\sigma x)}\right) - \frac{1}{\mathcal{K}o Q(\sqrt{\rho c_2}\sigma x)}.$$

According to (52) and (53) we can write the last equation as (51). Function F_{11} has the following properties:

$$F_{11}(0^+) = -\infty, \quad F_{11}(+\infty) = \frac{1}{\gamma_{16}}\left\{\mathcal{P}n\left(1 - Q\left(\frac{\sigma}{\sqrt{a_m}}\right)\right) - \frac{1}{\mathcal{K}o}\right\}.$$

Therefore, (52) would admit at least one solution if

$$\frac{1}{\gamma_{16}}\left\{\mathcal{P}n\left(1 - Q\left(\frac{\sigma}{\sqrt{a_m}}\right)\right) - \frac{1}{\mathcal{K}o}\right\} > 1. \tag{54}$$

Combining (28) and (53) with (54) it follows that (49) must hold.

Theorem 9 (Case 9: Determination of the unknown coefficients ρ and r). *If* (27) *holds, and*

$$\gamma_{21} = \frac{1}{\gamma_{19}} F_1\left(\sqrt{\frac{c_2 k_1}{c_1 k_2}} F_2^{-1}(\gamma_6)\right) \exp[(F_2^{-1}(\gamma_6))^2] < 1, \tag{55}$$

then there exists a unique solution to problem P which is given by (11)-(13), *where the coefficient* ρ *is given by* (28) *and the coefficient r is given by*

$$r = \frac{\gamma_{18}\left(\gamma_{19} \exp(-(F_2^{-1}(\gamma_6))^2) - F_1\left(\sqrt{\dfrac{c_2 k_1}{c_1 k_2}} F_2^{-1}(\gamma_6)\right)\right)}{1 - \dfrac{\mathcal{P}n}{\gamma_{20} - 1}\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{c_2 k_1}{c_1 k_2}} F_2^{-1}(\gamma_6)\right)}\right)} \tag{56}$$

with

$$\gamma_{18} = \sqrt{\frac{c_1c_2k_2}{\pi k_1}}\,\frac{(t_0 - t_v)}{\sigma^2 u_0 F_2^{-1}(\gamma_6)}, \quad \gamma_{19} = \sqrt{\frac{\pi c_1}{c_2k_1k_2}}\,\frac{\sigma q_0}{F_2^{-1}(\gamma_6)},$$

$$\gamma_{20} = \frac{\sigma^2 c_1 k_2}{a_m c_2 k_1 [F_2^{-1}(\gamma_6)]^2}.$$

Proof. As before, from (16) and imposing (27), we have that (28) holds. In addition, (56) can be drawn from (15). Taking into account the previous Lemma, we impose

$$\gamma_{19}\exp(-(F_2^{-1}(\gamma_6))^2) - F_1\left(\sqrt{\frac{c_2k_1}{c_1k_2}}F_2^{-1}(\gamma_6)\right) > 1$$

or equivalently (55) in order to get $r > 0$.

Theorem 10 (Case 10: Determination of the unknown coefficients ρ and c_1). *If* (27) *holds, and*

$$\gamma_{22} = \frac{\mathcal{P}n\left(1 - Q\left(\frac{\sigma}{\sqrt{a_m}}\right)\right)}{\left(\frac{\sigma^2}{a_m} - 1\right)\left(1 + \frac{c_2(t_0 - t_v)}{ru_0}\right)} < 1, \tag{57}$$

then there exists at least one solution to problem P which is given by (11)-(13), *where the coefficient* c_1 *is given by*

$$c_1 = \frac{k_1}{\rho\sigma^2}[F_2^{-1}(\gamma_6)]^2 \tag{58}$$

and the coefficient ρ *is given by*

$$\rho = \frac{k_2}{\sigma^2 c_2}\xi^2, \tag{59}$$

where ξ *is a solution to the equation*

$$F_7(x) = F_{12}(x), \quad x > 0 \tag{60}$$

with

$$F_{12}(x) = \frac{\sigma c_2 q_0}{k_2 r u_0}\frac{1}{x^2}\left\{\exp[-(F_2^{-1}(\gamma_6))^2] - \frac{k_2(t_0 - t_v)}{\sqrt{\pi}\sigma q_0}H(x)\right\}.$$

Proof. From (16) and imposing (27), we obtain (58). Let $x = \sqrt{\frac{\rho c_2}{k_2}}\sigma$. Now (15) becomes (60). Function F_{12} has the following properties:

$$F_{12}(0^+) = +\infty, \quad F_{12}(+\infty) = -\frac{c_2(t_0 - t_v)}{r u_0}, \quad F_{12}'(x) < 0 \quad \forall x > 0.$$

Then, taking into account (34) it follows that (60) always has a unique solution ξ if $\frac{\sigma^2}{a_m} - 1 < 0$, but if $\frac{\sigma^2}{a_m} - 1 > 0$, then it would be a unique solution ξ when (57) holds. Of course, (57) holds when $\frac{\sigma^2}{a_m} - 1 < 0$.

Theorem 11 (Case 11: Determination of the unknown coefficients ρ and δ). *If* (27) *holds, and*

$$\gamma_{23} = \frac{1 + \frac{\sigma^2}{r}\gamma_{18}F_1\left(\sqrt{\frac{c_2 k_1}{c_1 k_2}}F_2^{-1}(\gamma_6)\right)}{\gamma_{10}} < 1 \tag{61}$$

then there exists a unique solution to problem P which is given by (11)-(13), *where the coefficients* ρ *and* δ *are given by* (28) *and*

$$\delta = \frac{1 - \gamma_{10} + \frac{\sigma^2}{r}\gamma_{18}F_1\left(\sqrt{\frac{c_2 k_1}{c_1 k_2}}F_2^{-1}(\gamma_6)\right)}{\frac{(t_0 - t_v)}{u_0(\gamma_{20} - 1)}\left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{c_2 k_1}{c_1 k_2}}F_2^{-1}(\gamma_6)\right)}\right)}. \tag{62}$$

Proof. From (16) and imposing (27) we obtain that (28) holds. From (15) we have

$$\frac{\sigma c_1 q_0}{k_1 r u_0}\frac{\exp[-(F_2^{-1}(\gamma_6))^2]}{(F_2^{-1}(\gamma_6))^2} - \sqrt{\frac{c_1 c_2 k_2}{\pi k_1}}\frac{(t_0 - t_v)}{r u_0 F_2^{-1}(\gamma_6)}F_1\left(\sqrt{\frac{c_2 k_1}{c_1 k_2}}F_2^{-1}(\gamma_6)\right)$$

$$= 1 - \frac{\delta(t_0 - t_v)}{u_0(\gamma_{20} - 1)}\left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{c_2 k_1}{c_1 k_2}}F_2^{-1}(\gamma_6)\right)}\right)$$

and then (62). Regard the previous Lemma 1, we have $\delta > 0$ when $1 - \gamma_{10} + \frac{\sigma^2}{r}\gamma_{18}F_1\left(\sqrt{\frac{c_2 k_1}{c_1 k_2}}F_2^{-1}(\gamma_6)\right) > 0$, that is to say (61).

Theorem 12 (Case 12: Determination of the unknown coefficients a_m and k_1). *If*

$$\gamma_{25} = \frac{1}{\mathcal{P}n}\left\{1 + \frac{F_1\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right) - \gamma_2 \exp[-(F_3^{-1}(\gamma_{24}))^2]}{\gamma_3 \sigma}\right\} \in (-\infty, -1) \cup [0, +\infty) \tag{63}$$

with

$$\gamma_{24} = \frac{\sigma\rho c_1(t_v - t_s)}{\sqrt{\pi} q_0}, \tag{64}$$

then there exists a unique solution to problem P which is given by (11)-(13), *the coefficient* k_1 *is given by*

$$k_1 = \frac{\rho\sigma^2 c_1}{[F_3^{-1}(\gamma_{24})]^2} \tag{65}$$

and the coefficient a_m *is given by*

$$a_m = \frac{1}{\xi^2}, \tag{66}$$

where ξ *is the unique solution to the equation*

$$F_{13}(x) = F_{14}(x), \quad x > 0 \tag{67}$$

with

$$F_{13}(x) = 1 - \frac{Q(\sigma x)}{Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}, \tag{68}$$

$$F_{14}(x) = \gamma_{25}\left(\frac{k_2}{\rho c_2}x^2 - 1\right). \tag{69}$$

Proof. From (16) we have

$$F_3\left(\sqrt{\frac{\rho c_1}{k_1}}\sigma\right) = \gamma_{24}$$

and then (65). We can rewrite (15) as

$$1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)} = \gamma_{25}\left(\frac{k_2}{\rho a_m c_2} - 1\right).$$

We have (67) letting $x = \frac{1}{\sqrt{a_m}}$. Functions F_{13} and F_{14} have the following properties:

$$F_{13}(0^+) = 1, \quad F_{13}(+\infty) = 1 - \frac{1}{Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)} < 1, \quad F_{13}'(x) > 0 \quad \forall x > 0,$$

$$F_{14}(0^+) = -\gamma_{25}, \quad F_{14}(+\infty) = \operatorname{sgn}(\gamma_{25}) \cdot \infty, \quad F_{14}'(x) = \frac{2k_2\gamma_{25}}{\rho c_2}x.$$

So, the behaviour of F_{14} depends on the sign of γ_{25}. There always exists a unique solution ξ to (67) if $\gamma_{25} > 0$, but if $\gamma_{25} < 0$ we just can assure the existence of a unique solution ξ if $\gamma_{25} < (-1)$. Then, combining these inequalities we conclude that if (63) holds, then there exists a unique solution ξ to (67).

Theorem 13 (Case 13: Determination of the unknown coefficients δ and a_m). *If*

$$\gamma_4 < 1 \tag{70}$$

and

$$\gamma_{26} = \frac{\gamma_2 \exp\left[-\left(\operatorname{erf}^{-1}\left(\frac{1}{\gamma_4}\right)\right)^2\right] - F_1\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}{\gamma_3\sigma} > 1, \tag{71}$$

then there exist infinite solutions to problem P given by (11)-(13), *where the coefficient* δ *is given by*

$$\delta = \frac{1-\gamma_{26}}{\dfrac{(t_0 - t_v)}{u_0\left(\dfrac{k_2}{\rho a_m c_2} - 1\right)}\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{\rho c_2}{k_2}}\sigma\right)}\right)} \tag{72}$$

for each $a_m \in \mathbb{R}^+$.

Proof. If problem data verifies (16), we can write $\gamma_0\sigma = \operatorname{erf}^{-1}\left(\dfrac{1}{\gamma_4}\right)$, and therefore $\gamma_4 > 1$. Then $a_m \in \mathbb{R}^+$ and from (15) we obtain (72). We are now in a position to show that $\delta > 0$. This follows easily from (71), considering the previous Lemma 1, and this analysis can be done for each given $a_m > 0$.

Theorem 14 (Case 14: Determination of the unknown coefficients r and k_1). *If*

$$\gamma_{27} = \frac{\gamma_2 \exp[-(F_3^{-1}(\gamma_{24}))^2]}{F_1\left(\sqrt{\dfrac{\mathcal{L}u}{a_m}}\sigma\right)} > 1, \tag{73}$$

then there exists a unique solution to problem P which is given by (11)-(13), *and the thermal coefficients* k_1 *and r are given by* (65) *and*

$$r = \frac{\gamma_{13}\left[\gamma_2 \exp(-(F_3^{-1}(\gamma_{24}))^2) - F_1\left(\sqrt{\dfrac{\mathcal{L}u}{a_m}}\sigma\right)\right]}{1 - \gamma_1\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{\mathcal{L}u}{a_m}}\sigma\right)}\right)}, \tag{74}$$

respectively.

Proof. As in the proof of Theorem 12 we obtain (65). Then, from (15) we get (74). So, we are now in a position to show that $r > 0$. This follows easily from (73) considering the previous Lemma.

Theorem 15 (Case 15: Determination of the unknown coefficients δ and k_1). *If*

$$\gamma_{28} = \frac{\gamma_2 \exp[-(F_3^{-1}(\gamma_{24}))^2] - F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)}{\gamma_3\sigma} > 1, \tag{75}$$

then there exists a unique solution to problem P which is given by (11)-(13), *and the thermal coefficients k_1 and r are given by* (65) *and*

$$\delta = \frac{1-\gamma_{28}}{\dfrac{(t_0 - t_v)}{u_0\left(\dfrac{1}{\mathcal{L}u} - 1\right)}\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{\mathcal{L}u}{a_m}}\sigma\right)}\right)}, \tag{76}$$

respectively.

Proof. We follow Theorem 14.

Theorem 16 (Case 16: Determination of the unknown coefficients δ and c_1). *If* (27) *holds, and*

$$\gamma_{29} = \frac{\gamma_2 \exp[-(F_2^{-1}(\gamma_6))^2] - F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)}{\gamma_3\sigma} > 1, \tag{77}$$

then there exists a unique solution to problem P which is given by (11)-(13), *where the coefficients c_1 and δ are given by* (58) *and*

$$\delta = \frac{1-\gamma_{29}}{\dfrac{(t_0 - t_v)}{u_0\left(\dfrac{1}{\mathcal{L}u} - 1\right)}\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{\mathcal{L}u}{a_m}}\sigma\right)}\right)}, \tag{78}$$

respectively.

Proof. As in the proof of Theorem 10 from (16) and imposing (27) we obtain (58). Then, from (15) we get (76). So, we are now in a position to show that $\delta > 0$. This follows easily from (77) considering the previous Lemma.

Theorem 17 (Case 17: Determination of the unknown coefficients r and c_1). *If* (27) *holds, and*

$$\gamma_{30} = \frac{\gamma_2 \exp[-(F_2^{-1}(\gamma_6))^2]}{F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)} > 1, \tag{79}$$

then there exists a unique solution to problem P which is given by (11)-(13), *where the coefficients c_1 and r are given by* (58) *and*

$$r = \frac{\gamma_{13}\left(\gamma_2 \exp(-(F_2^{-1}(\gamma_6))^2) - F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)\right)}{1 - \gamma_1\left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)}\right)}, \tag{80}$$

respectively.

Proof. We follow Theorem 16.

Theorem 18 (Case 18: Determination of the unknown coefficients c_1 and a_m). *If* (27) *holds, and*

$$\gamma_{31} = \frac{1}{\mathcal{P}n}\left\{1 + \frac{F_1\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right) - \gamma_2 \exp[-(F_2^{-1}(\gamma_6))^2]}{\sqrt{\frac{\pi\rho c_2}{k_2}}\mathcal{K}o\sigma}\right\} \in (-\infty, -1) \cup [0, +\infty), \tag{81}$$

then there exists a unique solution to problem P which is given by (11)-(13), *where the coefficient c_1 is given by* (58) *and the coefficient a_m is given by*

$$a_m = \frac{1}{\xi^2}, \tag{82}$$

where ξ is a unique solution to

$$F_{13}(x) = F_{15}(x), \quad x > 0 \tag{83}$$

with

$$F_{15}(x) = \gamma_{31}\left(\frac{k_2}{\rho c_2}x^2 - 1\right). \tag{84}$$

Proof. As in the proof of Theorem 10 from (16) and imposing (27) we obtain (58). Then, from (15) we get

$$1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)} = \gamma_{31}\left(\frac{k_2}{\rho a_m c_2} - 1\right).$$

We obtain (83) letting $x = \frac{1}{\sqrt{a_m}}$. Function F_{15} has the following properties:

$$F_{15}(0^+) = -\gamma_{31}, \quad F_{15}(+\infty) = \operatorname{sgn}(\gamma_{31}) \cdot \infty, \quad F_{15}'(x) = \frac{2k_2\gamma_{31}}{\rho c_2}x.$$

So, the behaviour of F_{15} depends on the sign of γ_{31}. There always exists a unique solution ξ to (83) if $\gamma_{31} > 0$, but if $\gamma_{31} < 0$ we just can assure the existence of a unique solution ξ if $\gamma_{31} < (-1)$. Then, combining these inequalities we conclude that if (81) holds, then there exists a unique solution ξ to (83).

Theorem 19 (Case 19: Determination of the unknown coefficients c_1 and c_2). *If* (27) *holds, and*

$$\gamma_{32} = \frac{q_0}{\sigma \rho r u_0}\exp(-(F_2^{-1}(\gamma_6))^2) > 1, \tag{85}$$

then there exists at least one solution to problem P which is given by (11)-(13), *where the coefficient c_1 is given by* (58) *and the coefficient c_2 is given by* (40), *where ξ is a solution to the equation*

$$F_7(x) = F_{16}(x), \quad x > 0 \tag{86}$$

with

$$F_{16}(x) = \gamma_{32} - \frac{k_2(t_0 - t_v)}{\sigma^2\rho\sqrt{\pi} r u_0}H(x). \tag{87}$$

Proof. As in the proof of Theorem 10 from (16) and imposing (27) we obtain (58). Then, from (15) and letting $x = \sqrt{\frac{\rho c_2}{k_2}}\sigma$ we have

$$\frac{q_0}{\sigma\rho r u_0}\exp[-(F_2^{-1}(\gamma_6))^2] + \frac{k_2(t_0 - t_v)}{\sqrt{\pi}\sigma^2\rho r u_0}xF_1(x) = 1 - \frac{a_m \mathcal{P} n x^2}{\sigma^2 - a_m x^2}\frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q(x)},$$

that is to say (86). Function F_{16} has the following properties:

$$F_{16}(0^+) = \frac{q_0}{\sigma\rho r u_0}\exp[-(F_2^{-1}(\gamma_6))^2]; \quad F_{16}(+\infty) = -\infty; \quad F_{16}'(x) < 0 \quad \forall x > 0.$$

Then, taking into account (34), if (85) holds, then both functions would meet in at least one $\xi > 0$. It is easy to check that (40) follows from the definition of x.

Theorem 20 (Case 20: Determination of the unknown coefficients δ and c_2). *If*

$$\gamma_{14} > \sqrt{\pi}, \tag{88}$$

then there exist infinite solutions to problem P which are given by (11)-(13), *the coefficient δ is given by*

$$\delta = \frac{\gamma_{16} + \sqrt{\frac{c_2 k_2}{\pi\rho}}\frac{(t_0 - t_v)}{\sigma r u_0}F_1\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}{\frac{\rho a_m c_2(t_0 - t_v)}{u_0(k_2 - \rho a_m c_2)}\left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}\right)} \tag{89}$$

for each $c_2 \in \left(0, \frac{k_2}{\rho\sigma^2}[H^{-1}(\gamma_{33})]^2\right)$ *with*

$$\gamma_{33} = \frac{\sigma^2\rho\sqrt{\pi} r u_0}{k_2(t_0 - t_v)}\gamma_{16}. \tag{90}$$

Proof. From (15) we get (89). We have to assure that

$$\gamma_{16} > \sqrt{\frac{c_2 k_2}{\pi\rho}}\frac{(t_0 - t_v)}{\sigma r u_0}F_1\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)$$

to have $\delta > 0$, or equivalently

$$H\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right) < \gamma_{33}. \tag{91}$$

Let $x = \sqrt{\frac{\rho c_2}{k_2}}\sigma$. If $\gamma_{33} > 0$, then there exists such x verifying (91), and consequently

$$\gamma_{16} > 1. \tag{92}$$

Problem data verify (16), then we have $q_0 = \sqrt{\frac{\rho c_1 k_1}{\pi}} \frac{t_v - t_s}{\operatorname{erf}(\gamma_0 \sigma)}$, and this together with (92) leads to $\frac{1}{\sqrt{\pi}}\gamma_{14} > 1$, that is to say (88) holds. Therefore $x \in (0, H^{-1}(\gamma_{33}))$, and this clearly yields $c_2 \in \left(0, \frac{k_2}{\rho\sigma^2}[H^{-1}(\gamma_{33})]^2\right)$.

Theorem 21 (Case 21: Determination of the unknown coefficients r and c_2). *There always exists a solution to problem P which is given by* (11)-(13), *where the coefficient r is given by*

$$r = \frac{1 - \gamma_{16} - \dfrac{c_2(t_0 - t_v)}{u_0 Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}}{1 - \dfrac{\rho a_m c_2 \mathcal{P}n}{k_2 - \rho a_m c_2}\left(1 - \dfrac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}\right)} \tag{93}$$

for each $c_2 \in \left(0, \frac{k_2}{\rho\sigma^2}[H^{-1}(\gamma_{34}F_8(\gamma_0\sigma))]^2\right)$ *with*

$$\gamma_{34} = \frac{\sigma(t_v - t_s)}{k_2(t_0 - t_v)}\sqrt{\rho c_1 k_1}. \tag{94}$$

Proof. We can now proceed analogously to the proof of Theorem 20. From (15) we get (93). We have to assure that $1 - \gamma_{16} > \dfrac{c_2(t_0 - t_v)}{u_0 Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}$ to have $r > 0$, or equivalently

$$H\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right) < \frac{\sqrt{\pi}\sigma q_0}{k_2(t_0 - t_v)} \exp(-(\gamma_0\sigma)^2). \tag{95}$$

Let $x = \sqrt{\frac{\rho c_2}{k_2}}\sigma$. The right side of (95) is always positive, so there always exists such x that verifies (95). Problem data verify (16), so we can write $q_0 = \sqrt{\frac{\rho c_1 k_1}{\pi}} \frac{t_v - t_s}{\operatorname{erf}(\gamma_0\sigma)}$. This together with (95) leads to

$$H\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right) < \gamma_{34} F_8(\gamma_0\sigma),$$

and then we have $c_2 \in \left(0, \frac{k_2}{\rho\sigma^2}[H^{-1}(\gamma_{34}F_8(\gamma_0\sigma))]^2\right)$.

Theorem 22 (Case 22: Determination of the unknown coefficients a_m and k_2). *If*

$$\gamma_{14} > 1, \tag{96}$$

then there exists at least one solution to problem P which is given by (11)-(13), *the coefficient* k_2 *is given by*

$$k_2 = \frac{\sigma^2 \rho c_2}{\xi^2}, \tag{97}$$

where ξ *is a solution to* (41) *for each* $a_m \in \mathbb{R}^+$.

Proof. The proof follows by the same method as in Theorem 5.

Theorem 23 (Case 23: Determination of the unknown coefficients c_2 and k_1). *If* (27) *and* (85) *hold, then there exists a unique solution to problem P which is given by* (11)-(13), *where the coefficient* k_1 *is given by*

$$k_1 = \frac{\rho\sigma^2 c_1}{[F_2^{-1}(\gamma_6)]^2} \tag{98}$$

and the coefficient c_2 *is given by* (40), *where* ξ *is a solution to* (86).

Proof. As in the proof of Theorem 10 from (16) and imposing (27) we obtain (98). Then, from (15) and following by the same method as in Theorem 19, we have (40).

Theorem 24 (Case 24: Determination of the unknown coefficients k_1 and k_2). *If*

$$\gamma_{35} = \frac{q_0}{\sigma\rho r u_0}\exp(-(F_3^{-1}(\gamma_{24}))^2) > \gamma_{36} = 1 + \frac{1}{\mathcal{K}o} - \mathcal{P}n\left(1 - Q\left(\sqrt{\frac{\sigma}{a_m}}\right)\right), \tag{99}$$

then there exists at least a solution to problem P which is given by (11)-(13), *where the coefficient* k_1 *is given by* (65) *and the coefficient* k_2 *is given by* (97), *where* ξ *is a solution to*

$$F_7(x) = F_{17}(x), \quad x > 0 \tag{100}$$

with

$$F_{17}(x) = \gamma_{35} - \frac{1}{\mathcal{K}oQ(x)}. \tag{101}$$

Proof. As in the proof of Theorem 12 we obtain (65). Then let $x = \sqrt{\frac{\rho c_2}{k_2}}\sigma$, and (15) leads to

$$\frac{q_0}{\sigma\rho r u_0}\exp(-(F_3^{-1}(\gamma_{24}))^2) - \frac{1}{\mathcal{K}oQ(x)} = 1 - \frac{a_m \mathcal{P}n x^2}{\sigma^2 - a_m x^2}\frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q(x)},$$

that is to say (100). Function F_{17} has the following properties:

$$F_{17}(0^+) = -\infty, \quad F_{17}(+\infty) = \gamma_{35} - \frac{1}{\mathcal{K}o}, \quad F'_{17}(x) > 0 \quad \forall x > 0.$$

Then, taking into account (34), we have that both functions would meet in at least one $\xi > 0$ if (99) holds. As before, (97) follows from the definition of x.

Theorem 25 (Case 25: Determination of the unknown coefficients c_1 and k_2). *If* (27) *holds, and*

$$\gamma_{32} > \gamma_{36}, \tag{102}$$

then there exists at least a solution to problem P which is given by (11)-(13), *where the coefficient* c_1 *is given by* (58) *and the coefficient* k_2 *is given by* (97), *where* ξ *is a solution to*

$$F_7(x) = F_{18}(x), \quad x > 0 \tag{103}$$

with

$$F_{18}(x) = \gamma_{32} - \frac{1}{\mathcal{K}o\,Q(x)}. \tag{104}$$

Proof. As in the proof of Theorem 10 from (16) and imposing (27) we obtain (58). Then, from (15) and letting $x = \sqrt{\frac{\rho c_2}{k_2}}\,\sigma$, we have

$$\frac{q_0}{\sigma\rho r u_0}\exp(-(F_2^{-1}(\gamma_6))^2) - \frac{1}{\mathcal{K}o\,Q(x)} = 1 - \frac{a_m \mathcal{P}n x^2}{\sigma^2 - a_m x^2}\,\frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q(x)},$$

that is to say (103). Function F_{18} has the following properties:

$$F_{18}(0^+) = -\infty, \quad F_{18}(+\infty) = \gamma_{32} - \frac{1}{\mathcal{K}o}, \quad F_{18}'(x) > 0 \quad \forall x > 0.$$

So, taking into account (34), if

$$\gamma_{32} - \frac{1}{\mathcal{K}o} > 1 + \mathcal{P}n\left(1 - Q\left(\frac{\sigma}{\sqrt{a_m}}\right)\right)$$

or equivalently (102), then both functions would meet in at least one $\xi > 0$. As before, (97) follows from the definition of x.

Theorem 26 (Case 26: Determination of the unknown coefficients r and k_2). *If*

$$\gamma_{14} > \frac{\sigma}{\sqrt{\pi}\,\mathcal{K}o}, \tag{105}$$

then there exist infinite solutions to problem P which are given by (11)-(13), *where the coefficient r is given by*

$$r = \frac{1 - \gamma_{16} - \dfrac{1}{\pi \mathcal{K}o Q\left(\sqrt{\dfrac{\rho c_2}{k_2}}\sigma\right)}}{1 - \dfrac{\rho a_m c_2 \mathcal{P}n}{k_2 - \rho a_m c_2}\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{\rho c_2}{k_2}}\sigma\right)}\right)} \tag{106}$$

for each $k_2 \in \left(0, \dfrac{\rho\sigma^2 c_2}{\left[Q^{-1}\left(\dfrac{\sigma}{\sqrt{\pi}\mathcal{K}o\gamma_{14}}\right)\right]^2}\right)$.

Proof. From (15) we get (106). Our next goal is to determine that $r > 0$. In order to get this, we have to assure that

$$1 - \gamma_{16} > \frac{1}{\pi \mathcal{K}o Q\left(\sqrt{\dfrac{\rho c_2}{k_2}}\sigma\right)}$$

or equivalently

$$Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right) > \frac{\sigma\rho c_2(t_0 - t_v)}{\pi q_0}\exp(\gamma_0\sigma)^2. \tag{107}$$

Problem data verify equation (16), so we can say $q_0 = \sqrt{\dfrac{\rho c_1 k_1}{\pi}}\dfrac{t_v - t_s}{\operatorname{erf}(\gamma_0\sigma)}$. Substituting this into (107) we have

$$Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right) > \frac{\sigma}{\sqrt{\pi}\gamma_{14}\mathcal{K}o}. \tag{108}$$

Therefore, we have to ask for $\dfrac{\sigma}{\sqrt{\pi}\gamma_{14}\mathcal{K}o} < 1$, or equivalently (105), to have the existence of such k_2 that verifies (108). Then, considering (20), we conclude that $k_2 \in \left(0, \dfrac{\rho\sigma^2 c_2}{\left[Q^{-1}\left(\dfrac{\sigma}{\sqrt{\pi}\mathcal{K}o\gamma_{14}}\right)\right]^2}\right)$.

Theorem 27 (Case 27: Determination of the unknown coefficients δ and k_2). *If*

$$\gamma_{14} > \sqrt{\pi}\left(1 + \frac{1}{\pi Ko}\right), \tag{109}$$

then there exist infinite solutions to problem P which are given by (11)-(13), *where the coefficient* δ *is given by*

$$\delta = \frac{\gamma_{16} + \dfrac{1}{\pi Ko Q\left(\sqrt{\dfrac{\rho c_2}{k_2}}\sigma\right)}}{\dfrac{\rho a_m c_2(t_0 - t_v)}{u_0(k_2 - \rho a_m c_2)}\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{\rho c_2}{k_2}}\sigma\right)}\right)} \tag{110}$$

for each $k_2 \in \left(0, \dfrac{\rho\sigma^2 c_2}{\left[Q^{-1}\left(1/\left[\pi Ko\left(\frac{\gamma_{14}}{\sqrt{\pi}} - 1\right)\right]\right)\right]^2}\right)$.

Proof. From (15) we get (110). Our next goal is to determine that $\delta > 0$. In order to get this, we have to assure that $\gamma_{16} + \left[\pi Ko Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)\right]^{-1} < 0$ or equivalently

$$Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right) > \left[\pi Ko\left(\frac{q_0}{\sigma\rho r u_0}\exp(-(\gamma_0\sigma)^2) - 1\right)\right]^{-1}. \tag{111}$$

Problem data verify (16), so we can say $q_0 = \sqrt{\frac{\rho c_1 k_1}{\pi}}\frac{t_v - t_s}{\operatorname{erf}(\gamma_0\sigma)}$. Substituting this into (111) we have

$$Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right) > \left[\pi Ko\left(\frac{\gamma_{14}}{\sqrt{\pi}} - 1\right)\right]^{-1}. \tag{112}$$

Therefore, we have to ask for $0 < \left[\pi Ko\left(\frac{\gamma_{14}}{\sqrt{\pi}} - 1\right)\right]^{-1} < 1$, or equivalently

(109), to have the existence of such k_2 that verifies (112). Then, considering (20), we conclude that $k_2 \in \left(0, \dfrac{\rho\sigma^2 c_2}{\left[Q^{-1}\left(1/\left[\pi Ko\left(\frac{\gamma_{14}}{\sqrt{\pi}} - 1\right)\right]\right)\right]^2}\right)$.

Theorem 28 (Case 28: Determination of the unknown coefficients c_2 and a_m). *If* (96) *holds, then there exist infinite solutions to problem P which are given by* (11)-(13), *where the coefficient* c_2 *is given by* (40), *where* ξ *is a solution to*

$$F_7(x) = F_9(x), \quad x > 0 \tag{113}$$

for each $a_m \in \mathbb{R}^+$.

Proof. Problem data must verify (16). Then fix $a_m \in \mathbb{R}^+$ and let $x = \sqrt{\frac{\rho c_2}{k_2}}\sigma$. From (15) we obtain (113). Taking into account (34) and (43), we assure that we have at least one ξ that verifies (113) if

$$1 < \frac{q_0}{\sigma \rho r u_0} \exp(-(\gamma_0 \sigma)^2). \tag{114}$$

We can say $q_0 = \sqrt{\frac{\rho c_1 k_1}{\pi}} \frac{t_v - t_s}{\operatorname{erf}(\gamma_0 \sigma)}$. Substituting this into (114) we deduce that it would be at least one ξ that verifies (113) if (96) holds. As before, (40) follows from the definition of x. This analysis can be done for each given $a_m > 0$.

4. Conclusions

We considered an analytical model of freezing (desublimation) of moisture in a porous medium with an overspecified condition at the fixed face in order to determine simultaneously two unknown thermal coefficients of a semi-infinite phase-change material. This model has Luikov type equations with eight heat parameters, and it can be considered as a moving boundary problem with coupled heat and moisture flows. If the phase-change interface is given by $s(t) = 2\sigma\sqrt{t}$, where σ is a positive constant experimentally determined. We obtain the explicit expression of the temperature of the two phases T_1 and T_2, the mass-transfer potential in the humid region u, and we also determine formulae for the two unknown thermal coefficients chosen among ρ (mass density), a_m (moisture diffusivity), c_1 (specific heat of the frozen region), c_2 (specific heat of the humid region), k_1 (thermal conductivity of the frozen region), k_2 (thermal conductivity of the humid region), δ (thermal gradient coefficient), r (latent heat), together with the necessary and sufficient condition for the existence of such a solution for 28 different cases.

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Appendix A: Summary of the formulae of the simultaneous two unknown thermal coefficients

Case	Restrictions	Solution
(1) c_1, k_1	$\gamma_5 < 1$	$c_1 = \frac{\sqrt{\pi} q_0}{\sigma\rho(t_v - t_s)} F_3\left(\sqrt{\frac{1}{\log\left(\frac{1}{\gamma_5}\right)}}\right)$ $k_1 = \frac{\sqrt{\pi} q_0}{(t_v - t_s)} F_2\left(\sqrt{\frac{1}{\log\left(\frac{1}{\gamma_5}\right)}}\right)$
(2) a_m, ρ	$\gamma_6 < \frac{2}{\sqrt{\pi}}$ $\gamma_8 < (-1)$ Or $\gamma_8 \geq 0$	$\rho = \frac{k_1}{\sigma^2 c_1}[F_2^{-1}(\gamma_6)]^2$ $a_m = \frac{\sigma^2}{\xi^2}$, where ξ is the unique solution to $F_4(x) = F_5(x),\ x > 0$
(3) c_2, ρ	$\gamma_6 < \frac{2}{\sqrt{\pi}}$ $\gamma_{10} > 1$	$\rho = \frac{k_1}{\sigma^2 c_1}[F_2^{-1}(\gamma_6)]^2$ $c_2 = \frac{c_1 k_2}{k_1} \frac{\xi^2}{[F_2^{-1}(\gamma_6)]}$, where ξ is a solution to $F_6(x) = F_7(x),\ x > 0$
(4) a_m, r	$\gamma_{12} < 1$	$r = \frac{\gamma_{13}\left(\gamma_2 \exp(-(\gamma_0\sigma)^2) - F_1\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)\right)}{1 - \frac{\mathcal{P}n}{\frac{k_2}{\rho a_m c_2} - 1}\left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}\right)}$ and any $a_m \in \mathbb{R}^+$
(5) c_2, k_2	$\gamma_{14} > 1$	$c_2 = \frac{k_2}{\sigma^2\rho}\xi^2$, where ξ is a solution to $F_7(x) = F_9(x),\ x > 0$ and any $k_2 \in \mathbb{R}^+$

(6)	$r,\ \delta$	$\gamma_{15} < 1$	$r = \dfrac{\gamma_{13}\left(\gamma_2 \exp(-(\gamma_0\sigma)^2) - F_1\left(\sqrt{\dfrac{\mathcal{L}u}{a_m}}\sigma\right)\right)}{1 - \dfrac{\delta(t_0 - t_v)}{u_0\left(\dfrac{1}{\mathcal{L}u} - 1\right)}\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{\mathcal{L}u}{a_m}}\sigma\right)}\right)}$ and any $\delta \in \mathbb{R}^+$
(7)	$k_1,\ \rho$	---	$k_1 = \dfrac{\rho\sigma^2 c_1}{\log\left(\dfrac{1}{F_{10}(\rho)}\right)}$, where ρ is a solution to $F_{19}(x) = \dfrac{\sigma c_1(t_v - t_s)}{\sqrt{\pi}q_0}x,\ x > 0$
(8)	$k_2,\ \rho$	$\gamma_6 < \dfrac{2}{\sqrt{\pi}}$ $\gamma_{17} > 1$	$\rho = \dfrac{k_1}{\sigma^2 c_1}[F_2^{-1}(\gamma_6)]^2$ $k_2 = \dfrac{1}{\xi^2}$, where ξ is a solution to $F_{11}(x) = 1,\ x > 0$
(9)	$r,\ \rho$	$\gamma_6 < \dfrac{2}{\sqrt{\pi}}$ $\gamma_{21} < 1$	$\rho = \dfrac{k_1}{\sigma^2 c_1}[F_2^{-1}(\gamma_6)]^2$ $r = \dfrac{\gamma_{18}\left(\gamma_{19}\exp(-(F_2^{-1}(\gamma_6))^2) - F_1\left(\sqrt{\dfrac{c_2k_1}{c_1k_2}}F_2^{-1}(\gamma_6)\right)\right)}{1 - \dfrac{\mathcal{P}n}{\gamma_{20} - 1}\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{c_2k_1}{c_1k_2}}F_2^{-1}(\gamma_6)\right)}\right)}$
(10)	$c_1,\ \rho$	$\gamma_6 < \dfrac{2}{\sqrt{\pi}}$ $\gamma_{22} < 1$	$c_1 = \dfrac{k_1}{\rho\sigma^2}[F_2^{-1}(\gamma_6)]^2$ $\rho = \dfrac{k_2}{c_2\sigma^2}\xi^2$, where ξ is a solution to $F_7(x) = F_{12}(x),\ x > 0$
(11)	$\delta,\ \rho$	$\gamma_6 < \dfrac{2}{\sqrt{\pi}}$ $\gamma_{23} < 1$	$\rho = \dfrac{k_1}{\sigma^2 c_1}[F_2^{-1}(\gamma_6)]^2$ $\delta = \dfrac{1 - \gamma_{10} + \dfrac{\sigma^2}{r}\gamma_{18}F_1\left(\sqrt{\dfrac{c_2k_1}{c_1k_2}}F_2^{-1}(\gamma_6)\right)}{\dfrac{(t_0 - t_v)}{u_0(\gamma_{20} - 1)}\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{c_2k_1}{c_1k_2}}F_2^{-1}(\gamma_6)\right)}\right)}$

(12)	k_1, a_m	$\gamma_{25} < (-1)$ or $\gamma_{25} \geq 0$	$k_1 = \dfrac{\rho\sigma^2 c_1}{[F_3^{-1}(\gamma_{24})]^2}$ $a_m = \dfrac{1}{\xi^2}$, where ξ is the unique solution to $F_{13}(x) = F_{14}(x),\ x > 0$
(13)	δ, a_m	$\gamma_4 < 1$ $\gamma_{26} > 1$	$\delta = \dfrac{1 - \gamma_{26}}{\dfrac{(t_0 - t_v)}{u_0\left(\dfrac{k_2}{\rho a_m c_2} - 1\right)}\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{\rho c_2}{k_2}}\sigma\right)}\right)}$ and any $a_m \in \mathbb{R}^+$
(14)	r, k_1	$\gamma_{27} > 1$	$k_1 = \dfrac{\rho\sigma^2 c_1}{[F_3^{-1}(\gamma_{24})]^2}$ $r = \dfrac{\gamma_{13}\left[\gamma_2 \exp(-(F_3^{-1}(\gamma_{24}))^2) - F_1\left(\sqrt{\dfrac{\mathcal{L}u}{a_m}}\sigma\right)\right]}{1 - \gamma_1\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{\mathcal{L}u}{a_m}}\sigma\right)}\right)}$
(15)	δ, k_1	$\gamma_{28} > 1$	$k_1 = \dfrac{\rho\sigma^2 c_1}{[F_3^{-1}(\gamma_{24})]^2}$ $\delta = \dfrac{1 - \gamma_{28}}{\dfrac{(t_0 - t_v)}{u_0\left(\dfrac{1}{\mathcal{L}u} - 1\right)}\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{\mathcal{L}u}{a_m}}\sigma\right)}\right)}$
(16)	δ, c_1	$\gamma_6 < \dfrac{2}{\sqrt{\pi}}$ $\gamma_{29} > 1$	$c_1 = \dfrac{k_1}{\sigma^2\rho}[F_2^{-1}(\gamma_6)]^2$ $\delta = \dfrac{1 - \gamma_{29}}{\dfrac{(t_0 - t_v)}{u_0\left(\dfrac{1}{\mathcal{L}u} - 1\right)}\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{\mathcal{L}u}{a_m}}\sigma\right)}\right)}$

(17)	r, c_1	$\gamma_6 < \frac{2}{\sqrt{\pi}}$ $\gamma_{30} > 1$	$c_1 = \frac{k_1}{\sigma^2\rho}[F_2^{-1}(\gamma_6)]^2$ $r = \dfrac{\gamma_{13}\left(\gamma_2\exp(-(F_2^{-1}(\gamma_6))^2) - F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)\right)}{1-\gamma_1\left(1-\dfrac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)}\right)}$
(18)	a_m, c_1	$\gamma_6 < \frac{2}{\sqrt{\pi}}$ $\gamma_{31} < (-1)$ or $\gamma_{31} \geq 0$	$c_1 = \frac{k_1}{\sigma^2\rho}[F_2^{-1}(\gamma_6)]^2$ $a_m = \frac{1}{\xi^2}$, where ξ is the unique solution to $F_{13}(x) = F_{15}(x),\ x > 0$
(19)	c_1, c_2	$\gamma_6 < \frac{2}{\sqrt{\pi}}$ $\gamma_{32} > 1$	$c_1 = \frac{k_1}{\sigma^2\rho}[F_2^{-1}(\gamma_6)]^2$ $c_2 = \frac{k_1}{\sigma^2\rho}\xi^2$, where ξ is a solution to $F_7(x) = F_{16}(x),\ x > 0$
(20)	c_2, δ	$\gamma_{14} > \sqrt{\pi}$	$\delta = \dfrac{\gamma_{16} + \sqrt{\frac{c_2 k_2}{\pi\rho}}\,\frac{(t_0 - t_v)}{\sigma r u_0}\,F_1\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}{\dfrac{\rho a_m c_2(t_0 - t_v)}{u_0(k_2 - \rho a_m c_2)}\left(1-\dfrac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}\right)}$ and any $c_2 \in \left(0, \frac{k_2}{\rho\sigma^2}[H^{-1}(\gamma_{33})]^2\right)$
(21)	c_2, r	---	$r = \dfrac{1-\gamma_{16} - \dfrac{c_2(t_0 - t_v)}{u_0 Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}}{1-\dfrac{\rho a_m c_2 \mathcal{P}n}{k_2 - \rho a_m c_2}\left(1-\dfrac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}\right)}$ and any $c_2 \in \left(0, \frac{k_2}{\rho\sigma^2}[H^{-1}(\gamma_{34}F_8(\gamma_0\sigma))]^2\right)$

(22) a_m, k_2	$\gamma_{14} > 1$	$k_2 = \dfrac{\sigma^2 \rho c_2}{\xi^2}$, where ξ is a solution to $F_7(x) = F_9(x),\ x > 0$ and any $a_m \in \mathbb{R}^+$
(23) c_2, k_1	$\gamma_6 < \dfrac{2}{\sqrt{\pi}}$ $\gamma_{32} > 1$	$k_1 = \dfrac{\rho\sigma^2 c_1}{[F_2^{-1}(\gamma_6)]^2}$ $c_2 = \dfrac{k_2}{\sigma^2\rho}\xi^2$, where ξ is the unique solution to $F_7(x) = F_{16}(x),\ x > 0$
(24) k_1, k_2	$\gamma_{35} > \gamma_{36}$	$k_1 = \dfrac{\sigma^2\rho c_1}{[F_3^{-1}(\gamma_{24})]^2}$ $k_2 = \dfrac{\sigma^2\rho c_2}{\xi^2}$, where ξ is a solution to $F_7(x) = F_{17}(x),\ x > 0$
(25) c_1, k_2	$\gamma_6 < \dfrac{2}{\sqrt{\pi}}$ $\gamma_{32} > \gamma_{36}$	$c_1 = \dfrac{k_1}{\sigma^2\rho}[F_2^{-1}(\gamma_6)]^2$ $k_2 = \dfrac{\sigma^2\rho c_2}{\xi^2}$, where ξ is a solution to $F_7(x) = F_{18}(x),\ x > 0$
(26) r, k_2	$\gamma_{14} > \dfrac{\sigma}{\sqrt{\pi}Ko}$	$r = \dfrac{1 - \gamma_{16} - \dfrac{1}{\pi Ko Q\left(\sqrt{\dfrac{\rho c_2}{k_2}}\sigma\right)}}{1 - \dfrac{\rho a_m c_2 \mathcal{P}n}{k_2 - \rho a_m c_2}\left(1 - \dfrac{Q\left(\dfrac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\dfrac{\rho c_2}{k_2}}\sigma\right)}\right)}$ and any $k_2 \in \left(0, \dfrac{\rho\sigma^2 c_2}{\left[Q^{-1}\left(\dfrac{\sigma}{\sqrt{\pi}Ko\gamma_{14}}\right)\right]^2}\right)$

(27)	k_2, δ	$\gamma_{14} > \sqrt{\pi}\left(1 + \frac{1}{\sqrt{\pi}\mathcal{K}o}\right)$	$\delta = \dfrac{\gamma_{16} + \dfrac{1}{\pi \mathcal{K}o Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}}{\dfrac{\rho a_m c_2(t_0 - t_v)}{u_0(k_2 - \rho a_m c_2)}\left(1 - \dfrac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}\right)}$ and any $k_2 \in \left(0, \dfrac{\rho\sigma^2 c_2}{\left[Q^{-1}\left(1/\left[\pi\mathcal{K}o\left(\frac{\gamma_{14}}{\sqrt{\pi}} - 1\right)\right]\right)\right]^2}\right)$
(28)	a_m, c_2	$\gamma_{14} > 1$	$c_2 = \dfrac{k_2}{\sigma^2\rho}\xi^2$, where ξ is a solution to $F_7(x) = F_9(x),\ x > 0$ and any $a_m \in \mathbb{R}^+$.

Appendix B

The complementary constants used in the text are the following ones:

$$\gamma_0 = \sqrt{\frac{\rho c_1}{k_1}}\sigma; \quad \gamma_1 = \frac{\delta\rho a_m c_2(t_0 - t_v)}{k_2 - \rho a_m c_2} = \frac{\mathcal{P}n}{\dfrac{1}{\mathcal{L}u} - 1}; \quad \gamma_2 = \frac{\sqrt{\pi}q_0}{\sqrt{c_2 k_2 \rho}(t_0 - t_v)};$$

$$\gamma_3 = \sqrt{\frac{\pi\rho}{c_2 k_2}}\frac{r u_0}{(t_0 - t_v)} = \sqrt{\pi\frac{\mathcal{L}u}{a_m}}\mathcal{K}o; \quad \gamma_4 = \sqrt{\frac{\pi}{\rho c_1 k_1}}\frac{q_0}{(t_v - t_s)};$$

$$\gamma_5 = \frac{1}{\gamma_2}\left\{F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right) + \gamma_3\sigma\left[1 - \gamma_1\left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)}\right)\right]\right\};$$

$$\gamma_6 = \frac{k_1(t_v - t_s)}{\sqrt{\pi}\sigma q_0}; \quad \gamma_7 = \sqrt{\frac{c_1 k_2}{c_2 k_1}}\frac{1}{F_2^{-1}(\gamma_6)};$$

$$\gamma_8 = \frac{1}{\mathcal{P}n}\left\{1 - \frac{\sigma c_1 q_0}{k_1 r u_0}\frac{\exp(-(F_2^{-1}(\gamma_6))^2)}{F_2^{-1}(\gamma_6)} + \frac{1}{\sqrt{\pi}\mathcal{K}o}\frac{F_1(\gamma_7)}{\gamma_7}\right\};$$

$$\gamma_9 = \frac{1}{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}; \quad \gamma_{10} = \frac{\sigma c_1 q_0}{k_1 r u_0} \frac{\exp(-(F_2^{-1}(\gamma_6))^2)}{F_2^{-1}(\gamma_6)}; \quad \gamma_{11} \frac{c_1 k_2 (t_0 - t_v)}{\sqrt{\pi} k_1 r u_0 (F_2^{-1}(\gamma_6))^2};$$

$$\gamma_{12} = \frac{(t_0 - t_v)}{(t_v - t_s)} \sqrt{\frac{c_2 k_2}{c_1 k_1}} \frac{F_1\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}{F_5(\gamma_0 \sigma)}; \quad \gamma_{13} = \frac{(t_0 - t_v)}{\sigma u_0} \sqrt{\frac{c_2 k_2}{\pi \rho}};$$

$$\gamma_{14} = \frac{(t_v - t_s)}{\sigma r u_0} \sqrt{\frac{c_1 k_1}{\rho}} F_8(\gamma_0 \sigma); \quad \gamma_{15} = \frac{(t_0 - t_v)}{(t_v - t_s)} \sqrt{\frac{c_2 k_2}{c_1 k_1}} \frac{F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)}{F_5(\gamma_0 \sigma)};$$

$$\gamma_{16} = 1 - \frac{q_0}{\rho \sigma r u_0} \exp(-(\gamma_0 \sigma)^2); \quad \gamma_{17} = \gamma_{10} - \frac{(t_0 - t_v)}{u_0} \left[\frac{c_2}{r} + \delta\left(1 - Q\left(\frac{\sigma}{\sqrt{a_m}}\right)\right)\right];$$

$$\gamma_{18} = \sqrt{\frac{c_1 c_2 k_2}{\pi k_1}} \frac{(t_0 - t_v)}{\sigma^2 u_0 F_2^{-1}(\gamma_6)}; \quad \gamma_{19} = \sqrt{\frac{\pi c_1}{c_2 k_1 k_2}} \frac{\sigma q_0}{F_2^{-1}(\gamma_6)};$$

$$\gamma_{20} = \frac{\sigma^2 c_1 k_2}{a_m c_2 k_1 [F_2^{-1}(\gamma_6)]^2}; \quad \gamma_{21} = \frac{1}{\gamma_{19}} F_1\left(\sqrt{\frac{c_2 k_1}{c_1 k_2}} F_2^{-1}(\gamma_6)\right) \exp[(F_2^{-1}(\gamma_6))^2];$$

$$\gamma_{22} = \frac{\mathcal{P}n\left(1 - Q\left(\frac{\sigma}{\sqrt{a_m}}\right)\right)}{\left(\frac{\sigma^2}{a_m} - 1\right)\left(1 + \frac{c_2(t_0 - t_v)}{r u_0}\right)}; \quad \gamma_{23} = \frac{1 + \frac{\sigma^2}{r}\gamma_{18} F_1\left(\sqrt{\frac{c_2 k_1}{c_1 k_2}} F_2^{-1}(\gamma_6)\right)}{\gamma_{10}};$$

$$\gamma_{24} = \frac{\sigma \rho c_1 (t_v - t_s)}{\sqrt{\pi} q_0}; \quad \gamma_{25} = \frac{1}{\mathcal{P}n}\left\{1 + \frac{F_1\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right) - \gamma_2 \exp[-(F_3^{-1}(\gamma_{24}))^2]}{\gamma_3 \sigma}\right\};$$

$$\gamma_{26} = \frac{\gamma_2 \exp\left[-\left(\operatorname{erf}^{-1}\left(\frac{1}{\gamma_4}\right)\right)^2\right] - F_1\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}{\gamma_3 \sigma};$$

$$\gamma_{27} = \frac{\gamma_2 \exp[-(F_3^{-1}(\gamma_{24}))^2]}{F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)};$$

$$\gamma_{28} = \frac{\gamma_2 \exp[-(F_3^{-1}(\gamma_{24}))^2] - F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)}{\gamma_3\sigma};$$

$$\gamma_{29} = \frac{\gamma_2 \exp[-(F_2^{-1}(\gamma_6))^2] - F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)}{\gamma_3\sigma};$$

$$\gamma_{30} = \frac{\gamma_2 \exp[-(F_2^{-1}(\gamma_6))^2]}{F_1\left(\sqrt{\frac{\mathcal{L}u}{a_m}}\sigma\right)};$$

$$\gamma_{31} = \frac{1}{\mathcal{P}n}\left\{1 + \frac{F_1\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right) - \gamma_2 \exp[-(F_2^{-1}(\gamma_6))^2]}{\sqrt{\frac{\pi\rho c_2}{k_2}}\mathcal{K}o\sigma}\right\};$$

$$\gamma_{32} = \frac{q_0}{\sigma\rho r u_0} \exp(-(F_2^{-1}(\gamma_6))^2);$$

$$\gamma_{33} = \frac{\sigma^2\rho\sqrt{\pi} r u_0}{k_2(t_0 - t_v)}\gamma_{16}; \qquad \gamma_{34} = \frac{\sigma(t_v - t_s)}{k_2(t_0 - t_v)}\sqrt{\rho c_1 k_1};$$

$$\gamma_{35} = \frac{q_0}{\sigma\rho r u_0} \exp(-(F_3^{-1}(\gamma_{24}))^2);$$

$$\gamma_{36} = 1 + \frac{1}{\mathcal{K}o} - \mathcal{P}n\left(1 - Q\left(\sqrt{\frac{\sigma}{a_m}}\right)\right).$$

Appendix C

The real functions used in the text, defined for $x > 0$, are the following ones:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)\,du;$$

$$Q(x) = \sqrt{\pi}x \exp(x^2)(1 - \operatorname{erf}(x)); \quad F_1(x) = \frac{\exp(-x^2)}{1 - \operatorname{erf}(x)};$$

$$H(x) = xF_1(x); \quad F_2(x) = \frac{\operatorname{erf}(x)}{x}; \quad F_3(x) = x \operatorname{erf}(x);$$

$$F_4(x) = \gamma_8(\gamma_7^2 x^2 - 1); \quad F_5(x) = 1 - \gamma_9 Q(x);$$

$$F_6(x) = \gamma_{10} - \gamma_{11} H(x); \quad F_7(x) = 1 - \frac{a_m \mathcal{P} n x^2}{\sigma^2 - a_m x^2}\left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q(x)}\right);$$

$$F_8(x) = \frac{\exp(-x^2)}{\operatorname{erf}(x)}; \quad F_9(x) = \frac{q_0}{\rho\sigma r u_0}\exp(-(\gamma_0\sigma)^2) - \frac{k_2(t_0 - t_v)}{\sigma^2\rho\sqrt{\pi} r u_0} H(x);$$

$$F_{10}(x) = \sqrt{\frac{c_2 k_2}{\pi}}\,\frac{(t_0 - t_v)}{q_0}\sqrt{x}\left\{F_1\left(\sqrt{\frac{c_2}{k_2}}\sigma\sqrt{x}\right)\right.$$

$$\left. + \frac{\sqrt{\pi c_2}\mathcal{K}o\sigma}{\sqrt{k_2}}\left[1 - \frac{a_m c_2 \mathcal{P} n x}{k_2 - a_m c_2 x}\left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q\left(\sqrt{\frac{c_2}{k_2}}\sigma\sqrt{x}\right)}\right)\right]\right\};$$

$$F_{11}(x) = \frac{1}{\gamma_{16}}\left\{\frac{\rho a_m c_2 \mathcal{P} n x^2}{1 - \rho a_m c_2 x^2}\left(1 - \frac{Q\left(\frac{\sigma}{\sqrt{a_m}}\right)}{Q(\sqrt{\rho c_2}\sigma x)}\right) - \frac{1}{\mathcal{K}o Q(\sqrt{\rho c_2}\sigma x)}\right\};$$

$$F_{12}(x) = \frac{\sigma c_2 q_0}{k_2 r u_0} \frac{1}{x^2} \left\{ \exp[-(F_2^{-1}(\gamma_6))^2] - \frac{k_2(t_0 - t_v)}{\sqrt{\pi}\sigma q_0} H(x) \right\},$$

$$F_{13}(x) = 1 - \frac{Q(\sigma x)}{Q\left(\sqrt{\frac{\rho c_2}{k_2}}\sigma\right)}; \quad F_{14}(x) = \gamma_{25}\left(\frac{k_2}{\rho c_2} x^2 - 1\right),$$

$$F_{15}(x) = \gamma_{31}\left(\frac{k_2}{\rho c_2} x^2 - 1\right); \quad F_{16}(x) = \gamma_{32} - \frac{k_2(t_0 - t_v)}{\sigma^2 \rho \sqrt{\pi} r u_0} H(x);$$

$$F_{17}(x) = \gamma_{35} - \frac{1}{Ko Q(x)}; \quad F_{18}(x) = \gamma_{32} - \frac{1}{Ko Q(x)};$$

$$F_{19}(x) = F_3\left(\sqrt{\log\left(\frac{1}{F_{10}(x)}\right)}\right), \text{ defined for } x \text{ such that } 0 < F_{10}(x) < 1.$$

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