AN INITIAL-BOUNDARY VALUE PROBLEM FOR THE ONE-DIMENSIONAL NON-CLASSICAL HEAT EQUATION IN A SLAB

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Abstract: A nonlinear problem for the one-dimensional heat equation in a bounded and homogeneous medium with temperature data on the boundary $x=0$ and $x=1$ is studied. It is considered a non- classical heat conduction problem because a uniform spatial heat source depending on the heat flux (or the temperature) on the boundary $x=0$ is taken into account. Existence and uniqueness for the solution are proved under suitable assumptions on the data. Comparisons results and asymptotic behavior for the solution regarding some particular cases for the heat source, initial, and boundary data are also obtained.

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1. INTRODUCTION

In this paper, we will consider initial and boundary value problems (IBVP), for the one-dimensional non-classical heat equation motivated by some phenomena regarding the design of thermal regulation devices that provides a heater or cooler effect [1, 5, 7, 6, 8, 9]. We first study a IBVP with Dirichlet boundary conditions and a heat source that depends on the heat flux at the fixed face $x=0$, and afterwards, we study a similar problem but with Neumann boundary conditions and a heat source that depends on the temperature on the fixed face $x=0$. We obtain in both cases existence of a solution through a system of second kind Volterra integral equations.

A heat conduction problem of the first type but for a semi-infinite material was analyzed in [8,9], where results on existence, uniqueness and asymptotic behavior for the solution were obtained. In other respects, a class of heat conduction problems characterized by a uniform heat source given as a multivalued function from $\mathbb{R}$ into itself was studied in [7] with results regarding existence, uniqueness and asymptotic behavior for the solution. Other references on the subject are [5,6]. Recently, free boundary problems (Stefan problems) for the non-classical heat equation have been given in [2-4], where some explicit solutions are also given.

2. PROBLEM (P1) - EXISTENCE AND UNIQUENESS

We study the following IBVP for the heat equation in the slab [0,1] (Problem (P1)):

$$
(P1)\begin{cases}
u_t - u_{xx} = - F(u_x(0,t),t), \ \ (x,t) \in \Omega \equiv \{(x,t): 0<x<1, \ \ 0<t<T\} & (2.1)\\
u(0,t)= f(t), \ 0< t <T & (2.2)\\
u(1,t) = g(t), \ 0< t <T & (2.3)\\
u(x,0) = h(x), \ 0 \le x \le 1 & (2.4)
\end{cases}
$$

where the unknown function $u = u(x,t)$ denotes the temperature profile for an homogeneous medium occupying the spatial region $0<x<1$, the boundary data f and g are real functions defined on $\mathbb{R}^+$, the initial temperature $h(x)$ is a real function defined on [0,1], and F is a given function of two real variables, which is related to the evolution of the heat flux $u_x(0,t)$ in this case.

For data $h=h(x)$, $g = g(t)$, $f = f(t)$ and F in problem (1.1)-(1.4) we shall consider the following assumptions:

(HA) g and f are continuously differentiable functions on $\mathbb{R}^+$;

(HB) h is a function $C^1[0,1]$, which verifies compatibility conditions: $h(0) = f(0), \quad h(1) = g(1)$;

(HC) The function $F = F(V,t)$ verifies:

(HC1) It is defined and continuous on the region $D = \mathbb{R}\times[0,T]$;

(HC2) For each $M>0$ and for $|V|\leq M$, it is uniformly Hölder continuous in variable t for each compact subset of $(0,T]$;

(HC3) For each bounded set B of D, there exists a bounded positive function $L_o = L_o(t)$, defined for $0< t \leq T$, such that

$$|F(V_2,t)-F(V_1,t)|\leq L_O(t)|V_2-V_1|, \forall (V_2,t),(V_1,t)\in B \quad ;$$

(HC4) It is bounded for bounded V for all $t \geq 0$;

(HD) $F(0,t)=0$, $0< t \leq T$.

(HE) $V F(V,t) > 0 \quad , \quad \forall\, V \neq 0, \quad \forall\, t>0$;

(HF) $f(t)\equiv 0 \quad \forall t>0 \quad , \quad g(t)\equiv u_{1_o} > 0 \quad \forall t>0 \quad , \quad h'(x)>0 \;\; \forall\, x\in[0,1] \quad , \quad h(1)\leq u_{1_o}$.

(HG) $f(t)\equiv 0 \quad \forall t>0 \quad , \quad g(t)\equiv 0 \quad \forall t>0 \quad , \quad h(x)>0 \;\; \forall\, x\in[0,1]$

Theorem 1. *Under the assumptions (HA) to (HD), the solution u to the problem (P1) has the expression*

$$\begin{aligned} u(x,t) = &\int_0^1 [\theta(x-\xi,t)-\theta(x+\xi,t)]h(\xi)d\xi - 2\int_0^t \theta_x(x,t-\tau)f(\tau)d\tau + 2\int_0^t \theta_x(x-1,t-\tau)g(\tau)d\tau \\ &-\int_0^t\left\{\int_0^1 [\theta(x-\xi,t-\tau)-\theta(x+\xi,t-\tau)]d\xi\right\}F(V(\tau),\tau)d\tau \end{aligned} \tag{2.5}$$

where V=V(t), defined by $V(t) = u_x(0,t)$ for all $t>0$, must satisfy the following second kind Volterra integral equation:

$$V(t) = 2\int_0^1 \theta(\xi,t)h'(\xi)d\xi - 2\int_0^t \theta(0,t-\tau)\dot{f}(\tau)d\tau + 2\int_0^t \theta(-1,t-\tau)\dot{g}(\tau)d\tau - \int_0^t \overline{K}(t-\tau)F(V(\tau),\tau)d\tau \tag{2.6}$$

and the functions $\theta = \theta(x,t)$, $K = K(x,t)$ and $\overline{K} = \overline{K}(t)$ are defined in the following way:

$$\theta(x,t) = K(x,t) + \sum_{j=1}^{\infty}[K(x+2j,t)+K(x-2j,t)],\; K(x,t) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}},\; \overline{K}(t) = 2(\theta(0,t)-\theta(1,t)), \quad t>0. \tag{2.7}$$

Theorem 2 *Under assumptions (HA), (HB), (HC1) and (HC4), there exists at least one solution $V(t) \in C^o(\mathbb{R}^+)$ to the integral equation (2.6), therefore we have at least one solution for the IBVP (2.1)-(2.4).*

Theorem 3 *Under the assumptions (HA) to (HD), there exists a unique solution to the problem (P1). Moreover, there exists a maximal time $\beta > 0$, such that the unique solution to (2.1) – (2.4) can be extended to the interval $0 \leq t \leq \beta$.*

3. Properties of the Solution to Problem (P1)

Theorem 4 *Under assumptions (HA) to (HD), the solution u to problem (P1) is bounded in terms of the initial and boundary data h, f and g.*

Lemma 5 *Let* $u_0(x,t)$ *be the solution to (2.1)-(2.4) with null heat source (i.e.* $F \equiv 0$*). Under the assumptions (HD), (HE) and (HF), we have that:*

a) $0 \le u(x,t) \le u_o(x,t), \quad \forall\, x \in [0,1], \forall t > 0.$ *b)* $\lim_{t\to+\infty} u(x,t) = 0, \forall x \in [0,1]$.

Now we will consider the continuous dependence of the functions *V=V(t)* and *u=u(x,t)* given by (2.5) and (2.6) respectively upon the data *f, g , h* and *F*. Let us denote by $V_i=V_i(t)$ (*i=1,2*) the solution to (2.6) and $u_i=u_i(x,t)$ given by (2.5) respectively for the data f_i, g_i, h_i and *F* (*i=1,2*) in problem (P1).

Theorem 6 *Considering problem (P1) under the assumptions (HA) to (HD), we obtain that* $V_2 - V_1$ *is bounded in terms of the initial and boundary data h, f and g. Therefore, the difference of the solutions* $u_2 - u_1$ *is also bounded in terms of the initial and boundary data h, f and g.*

Theorem 7 *Let* $u_i=u_i(x,t)$, $V_i=V_i(t)$ *(i=1,2) be the functions given by (2.5) and (2.6) for the data f, g, h and* F_i *(i=1,2) in problem (P1). Under the assumptions (HA) to (HD), we obtain the following estimation:*

$$\left|u_2(x,t)-u_1(x,t)\right| \le M_o \left\|F_2-F_1\right\|_{t,M}\left[t+\frac{2\left\|L_2\right\|_\infty}{\sqrt{\pi}}\sqrt{t}\; e^{\left\|L_2\right\|_\infty \frac{2\sqrt{t}}{\sqrt{\pi}}}\right] \qquad (3.1)$$

where $\left\|F_1-F_2\right\|_{t,M} = \sup_{\substack{\|z\|_t \le M \\ 0<\tau\le t}} \left|F_1(z(\tau),\tau)-F_2(z(\tau),\tau)\right|$, *and* M_o *is a positive constant which verifies the inequality*

$$\int_o^1 \left|\theta(x-\xi,t-\tau)-\theta(x+\xi,t-\tau)\right| d\xi \le M_o,\ 0<\tau<t\le T, \quad 0\le x\le 1. \qquad (3.2)$$

4. Problem (P2) - Existence and Uniqueness

Now, we will consider a new non-classical initial-boundary value problem (P2) for the heat equation in the slab [0,1], which is related to the previous problem (P1), given by:

$$(P2)\begin{cases} u_t - u_{xx} = -F(u(0,t),t), & (x,t) \in \Omega \equiv \{(x,t):\ 0<x<1,\ 0<t\le T\} \quad (4.1)\\ u_x(0,t)=f(t), & 0<t\le T \quad (4.2)\\ u_x(1,t)=g(t), & 0<t\le T \quad (4.3)\\ u(x,0)=h(x), & 0\le x\le 1. \quad (4.4)\end{cases}$$

Theorem 8 *Under the assumptions (HA) to (HD), the solution u to the problem (P2) has the expression*

$$\begin{aligned} u(x,t) = {} & \int_0^1\left[\theta(x-\xi,t)+\theta(x+\xi,t)\right]h(\xi)d\xi - 2\int_0^t \theta(x,t-\tau)f(\tau)d\tau + 2\int_0^t \theta(x-1,t-\tau)g(\tau)d\tau \\ & -\int_0^t\left\{\int_0^1\left[\theta(x-\xi,t-\tau)+\theta(x+\xi,t-\tau)\right]d\xi\right\}F(V(\tau),\tau)d\tau \end{aligned} \qquad (4.5)$$

where V=V(t), defined by V(t)=u(0,t), must satisfy the following second kind Volterra integral equation:

$$V(t)=2\int_0^1\theta(\xi,t)h(\xi)d\xi-2\int_0^t\theta(0,t-\tau)f(\tau)d\tau+2\int_0^t\theta(-1,t-\tau)g(\tau)d\tau-2\int_0^t\int_0^1\theta(\xi,t-\tau)d\xi F(V(\tau),\tau)d\tau. \qquad (4.6)$$

Theorem 9 *Under the assumptions (HA) to (HD), there exists a unique solution to the problem (P2). Moreover, there exists a maximal time β > T > 0, such that the unique solution to (4.1) – (4.4) can be extended to the interval 0 ≤ t ≤ β.*

5. PROPERTIES OF THE SOLUTION TO PROBLEM (P2)

Theorem 10 *Under the assumptions (HA) to (HD), the solution u to problem (P2) is bounded in terms of the initial and boundary data h, f and g.*

Theorem 11 *Let us define V_i and u_i (i=1,2) as in Theorem 6, with respect to the problem (P2). Under the assumptions (HA) to (HD), we obtain that V_2-V_1 is bounded in terms of the initial and boundary data h, f and g. Therefore, the difference of the solutions u_2-u_1 is also bounded in terms of the initial and boundary data h, f and g.*

Theorem 12 *Let us define V_i and u_i (i=1,2) as in Theorem 7, with respect to the problem (P2).Under the assumptions (HA) to (HD), then we obtain the following estimation:*

$$|u_2(x,t)-u_1(x,t)|\le M_1\|F_2-F_1\|_{t,M}\,t\left[1+\|L_2\|_t C_3\exp(C_3\|L_2\|_t)\right]. \qquad (4.7)$$

where M_1 is a positive constant which verifies the inequality

$$\int_o^1 |\theta(x-\xi,t-\tau)+\theta(x+\xi,t-\tau)|d\xi\le M_1,\ 0<\tau<t\le T\ ,\ 0\le x\le 1. \qquad (4.8)$$

Theorem 13 *Under the hypotheses (HG) and (HE), we have that* $0<u(x,t)<\|h\|_\infty,\ \forall x\in[0,1],\ \forall t\ge 0.$

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REFERENCES

[1] L.R. BERRONE , D.A. TARZIA, L.T.VILLA, *Asymptotic behavior of a Non –classical Heat Conduction Problem for a Semi-infinite Material*, Mathematical Methods in the Applied Sciences, 23 (2000), pp. 1161-1177.

[2] A.C. BRIOZZO, D.A. TARZIA, *Existence and uniqueness of a one-phase Stefan problem for a non-classical heat equation with temperature boundary condition at the fixed face*, Electronic Journal of Differential Equations, 2006 (2006) No. 21, pp1-16.

[3] A.C. BRIOZZO, D.A. TARZIA, *A one-phase Stefan problem for a non-classical heat equation with a heat flux condition on the fixed face*, Applied Mathematics and Computation, 182 (2006), pp. 809-819.

[4] A.C. BRIOZZO, D.A. TARZIA, *Exact solutions for non-classical Stefan problems*, International Journal of Differential Equations, 2010 (2010), Article ID 868059, pp. 1-19.

[5] K. GLASHOFF, J. SPREKELS, *The regulation of temperature by thermostats and set-valued integral equations*, J. Integral Eq., 4 (1982), pp. 95-112.

[6] N. KENMOCHI, *Heat conduction with a class of automatic heat source controls*, Pitman Research Notes in Mathematics Series #186 (1990), pp. 471-474.

[7] N. KENMOCHI, M. PRIMICERIO, *One-dimensional heat conduction with a class of automatic heat source controls*, IMA J. Appl. Math., 40 (1998), pp. 205-216.

[8] D.A. TARZIA, L.T VILLA, *Some nonlinear heat conduction problems for a semi-infinite strip with a non-uniform heat source*, Rev. Un. Mat. Argentina, 41 (1998), pp. 99-114.

[9] L.T. VILLA, *Problemas de control para una ecuación unidimensional del calor*, Rev. Un. Mat. Argentina, 32 (1986), pp. 163-169.