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SIMULTANEOUS DETERMINATION OF UNKNOWN COEFFICIENTS THROUGH A PHASE-CHANGE PROCESS WITH TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY

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Abstract

Formulas are obtained for the simultaneous determination of two or three unknown thermal coefficients of a semi-infinite material with temperature-dependent thermal conductivity through a phase-change process with an overspecified condition on the fixed face through a moving boundary problem (inverse Stefan problem) or a free boundary problem (Stefan problem), respectively. We complete and improve the analysis done in Tarzia [24] and we also study the sensitivity of the solution depending on different thermal parameters, applied to aluminum.

Keywords and phrases: Stefan problem, unknown thermal coefficients, PCM, sensitivity analysis, inverse problem, free boundary problem, phase-change process, moving boundary problem.

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Nomenclature

c	Specific heat, $J/(kg^{\circ}C)$
h	Latent heat of fusion by unit of mass, J/kg
k	Thermal conductivity, $W/(m^{\circ}C)$
q_o	Coefficient that characterizes the heat flux at $x = 0$, $kg/s^{5/2}$
s	Position of the free or moving front, m
Ste	The Stefan number defined by (9), dimensionless
t	Time, s
T	Temperature, °C
x	Spatial coordinate, m

Greek symbols

α	Diffusivity coefficient, m^2/s
β	Coefficient that characterizes the thermal conductivity in equation (2), dimensionless
δ	Coefficient that characterizes the differential equation (4i), dimensionless
η	Similarity variable defined by (3), dimensionless
λ	Coefficient that characterizes the free boundary in equation (3ii), dimensionless
ρ	Density, kg/m^3
σ	Coefficient that characterizes the moving boundary in equation (3iibis), $m/s^{1/2}$

Subscripts

f	Fusion
o	Initial in time or in space

1. Introduction

Heat transfer problems with a phase-change such as melting and freezing have been studied in the last century due to their wide scientific and technological applications. A review of a long bibliography on moving and free boundary problems for phase-change materials (PCM) for the heat equation is shown in [25].

We consider the following solidification problem for a semi-infinite material with an overspecified condition on the fixed face $x = 0$ [1, 6, 8, 11]:

$$\begin{cases} \text{(i)} & \rho c T_t(x,t) = (k(T)T_x(x,t))_x,\ 0 < x < s(t),\ t > 0, \\ \text{(ii)} & T(0,t) = T_o < T_f,\ t > 0, \\ \text{(iii)} & k(T_o)T_x(0,t) = \dfrac{q_o}{\sqrt{t}},\ t > 0,\ q_o > 0, \\ \text{(iv)} & T(s(t),t) = T_f,\ t > 0, \\ \text{(v)} & k(T_f)T_x(s(t),t) = \rho h \dot{s}(t),\ t > 0, \end{cases} \tag{1}$$

where $T(x, t)$ is the temperature of the solid phase, $\rho > 0$ is the density of mass, $h > 0$ is the latent heat of fusion by unity of mass, $c > 0$ is the specific heat, $x = s(t)$ is the phase-change interface, T_f is the phase-change temperature, T_o is the temperature at the fixed face $x = 0$ and q_o is the coefficient that characterizes the heat flux at $x = 0$ given by (1(iii)) which must be obtained experimentally through a phase-change process. We suppose that the thermal conductivity has the following expression [9]:

$$k = k(T) = k_o[1 + \beta(T - T_o)/(T_f - T_o)],\ \ \beta \in \mathbb{R}. \tag{2}$$

We remark that the phase-change problem (1) with conditions ((i), (ii), (iv) and (v)) is a classical Stefan problem [6, 8]. We consider that the condition (1(iii)) is an overspecified condition at the fixed face $x = 0$, of the type given in [21], from which we can determine some unknown thermal coefficients [2-5, 14, 17]. We observe that if $\beta = 0$, then the problem (1) becomes the classical one-phase Lamé-Clapeyron-Stefan problem with an overspecified condition at the fixed face $x = 0$ and for this problem the corresponding simultaneous determination of thermal coefficients was studied in [22, 23]. The phase-change process with temperature-dependent thermal coefficient of the type (2) was firstly studied in [9]. Other papers related to determination of thermal coefficients are [12, 16, 18, 20, 26-29].

The solution to problem (1) is given by [9, 24]:

$$\begin{cases} \text{(i)} \quad T(x, t) = T_o + \dfrac{(T_f - T_o)}{\Phi(\lambda, \delta)}\Phi(\eta, \delta), \quad \eta = \dfrac{x}{2\sqrt{\alpha_o t}}, \ 0 < \eta < \lambda, \\ \text{(ii)} \quad s(t) = 2\lambda\sqrt{\alpha_o t}, \end{cases} \tag{3}$$

where $\alpha_o = k_o/\rho c$ is the coefficient of the diffusivity at the temperature T_o, $\Phi = \Phi(x, \delta)$ is the modified error function which is for all $\delta > -1$, the unique solution to the following boundary value problem in variable x, i.e.,

$$\begin{cases} \text{(i)} \quad \dfrac{\partial}{\partial x}[(1 + \delta\Phi_x(x, \delta))\Phi_x(x, \delta)] + 2x\Phi_x(x, \delta) = 0, \ x > 0, \ (\delta > -1), \\ \text{(ii)} \quad \Phi(0^+, \delta) = 0, \ \Phi(+\infty, \delta) = 1 \end{cases} \tag{4}$$

and the unknown thermal coefficients must satisfy the following system of equations [24]:

$$\beta - \delta\Phi(\lambda, \delta) = 0, \tag{5}$$

$$[1 + \delta\Phi(\lambda, \delta)]\frac{\Phi_x(\lambda, \delta)}{\lambda\Phi(\lambda, \delta)} - \frac{2h}{c(T_f - T_o)} = 0, \tag{6}$$

$$\frac{\Phi_x(0, \delta)}{\Phi(\lambda, \delta)} - \frac{2q_o}{(T_f - T_o)\sqrt{k_o\rho c}} = 0. \tag{7}$$

For the particular case $\delta = 0$, we have that $\Phi(x, \delta) = erf(x)$ is the error function, which is defined by:

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du. \tag{8}$$

We remark that if problem (1) is a free boundary problem (this case can be considered as a Stefan problem) with an overspecified condition on the fixed face $x = 0$, then the coefficient $\lambda > 0$ is an unknown coefficient. On the other hand, if problem (1) is a moving boundary problem (this case can be considered as an inverse Stefan problem) with an overspecified condition on the fixed face $x = 0$, then the phase-change interface will be given by

$$s(t) = 2\sigma\sqrt{t}, \tag{3iibis}$$

where σ must be obtained experimentally $(\sigma = \lambda\sqrt{\alpha_o})$ through a phase-change process [23].

When the coefficient $\delta = 0$, the corresponding determination of formulas for one or two unknown thermal coefficients were obtained in [22, 23]. When the coefficient $\delta \neq 0$ is given, the corresponding problem was analyzed in [24]; in this case, the necessary and sufficient conditions on the data were obtained in order to ensure the existence of the solution.

In this paper, we will consider that δ is an unknown coefficient and that the unknown thermal coefficients for the simultaneous determination will be chosen among: density (ρ), latent heat (h), specific heat (c), through 5 or 10 different cases for a free or moving boundary problem, respectively. In general, these types of problems are ill-posed [13], and for ill-posed problems a small perturbation in the input data may produce a big change in the output data. We will study the behavior of the solution parameters when a slightly modification is made in the data parameters.

We define the Stefan number by

$$Ste = \frac{c(T_f - T_o)}{h}. \tag{9}$$

The goal of the present paper is to consider that the coefficient δ is one of the unknown thermal coefficients; therefore the problem (1) consists of finding the coefficient δ simultaneously with two thermal coefficients chosen among: λ, β, k_o, ρ, c, h. In Section 3, for a one-phase Stefan problem, we determine the temperature $T(x, t)$, the free boundary interface $s(t)$ (i.e., the coefficient λ defined in (3(ii)) and the following parameters in 5 different cases for a free boundary problem (here $\lambda > 0$ is an unknown coefficient):

FB: (i) δ, λ, β (ii) δ, λ, k_o (iii) δ, λ, ρ (iv) δ, λ, h (v) δ, λ, c.

In Section 4, for a one-phase inverse Stefan problem (i.e. the interface $s(t)$ is given by (3iibis)), we determine the temperature $T(x, t)$ and the following parameters in 10 cases for a moving boundary problem (here $\sigma > 0$ is a known coefficient):

MB: (i) δ, β, k_o (ii) δ, β, ρ (iii) δ, β, h (iv) δ, β, c (v) δ, k_o, ρ

(vi) δ, k_o, h (vii) δ, k_o, c (viii) δ, ρ, h (ix) δ, ρ, c (x) δ, c, h.

We obtain the formulas for the unknown thermal coefficients and we give the proof of some of the cases. Table 1 and Table 2 summarize the formulas for the unknown thermal coefficients corresponding to the five or ten cases for the free or moving boundary problem (1), respectively. Finally, in Section 5, a sensitivity analysis, corresponding to the aluminum, is performed.

2. Auxiliary Functions

In order to give, case by case, the formulas for the unknown thermal coefficients and the restriction for data (when the case allows us), let us consider the following real functions, defined for $x > 0$:

$$F_1(x, \delta) = 1 + \delta\Phi(x, \delta); \qquad F_2(x, \delta) = \frac{x\Phi(x, \delta)}{\Phi_x(x, \delta)};$$

$$F_3(x, \delta) = [1 + \delta\Phi(x, \delta)]\Phi_x(x, \delta); \qquad F_4(x, \delta) = x\Phi(x, \delta);$$

$$F_5(x, \delta) = \frac{\Phi(x, \delta)}{x}; \qquad F_6(x, \delta) = \frac{x}{\Phi(x, \delta)}.$$

The properties of the function $\Phi = \Phi(x, \delta)$ and the properties of the functions F_1 to F_6 which, complement and improve the graphics given in [9], will be used to prove all our results.

3. Solution to the Five Cases through a Free Boundary Problem

3.1. Case 1 - Simultaneous determination of $\{\delta, \lambda, \beta\}$

Proposition 1. *The temperature and the free boundary for problem* (1) *with unknown thermal coefficients* $\{\delta, \lambda, \beta\}$ *are given by* (3), β *is given by the expression*:

$$\beta = \delta\Phi(\lambda, \delta) \tag{10}$$

and δ *and* λ *must satisfy the following system of equations* ($\lambda > 0$, $\delta > -1$):

$$(S1)\begin{cases} F_1(\lambda, \delta) = \dfrac{2}{Ste} F_2(\lambda, \delta), \\ \Phi(\lambda, \delta) = \dfrac{\sqrt{\rho c k_o}(T_f - T_o)}{2q_o} \Phi_x(0, \delta). \end{cases}$$

Proof. From equation (5), we obtain the equation (10), and from equations (6) and (7), we obtain that δ and λ must satisfy the system (S1).

3.2. Case 2 - Simultaneous determination of $\{\delta, \lambda, k_o\}$

Proposition 2. *The temperature and the free boundary for problem* (1) *with unknown thermal coefficients* $\{\delta, \lambda, k_o\}$ *are given by* (3), k_o *is given by the expression*:

$$k_o = \frac{1}{c\rho}\left(\frac{2q_o\Phi(\lambda, \delta)}{\Phi_x(0, \delta)(T_f - T_o)}\right)^2 \tag{11}$$

and δ *and* λ *must satisfy the following system of equations* $(\lambda > 0, \delta > -1)$:

$$(S2)\begin{cases} F_1(\lambda, \delta) = \dfrac{2}{Ste} F_2(\lambda, \delta), \\ \beta = \delta\Phi(\lambda, \delta). \end{cases}$$

Proof. From equation (7), we obtain the expression (11), and from equations (5) and (6), we obtain that δ and λ must satisfy the system (S2).

3.3. Case 3 - Simultaneous determination of $\{\delta, \lambda, \rho\}$

Proposition 3. *The temperature and the free boundary for problem* (1) *with unknown thermal coefficients* $\{\delta, \lambda, \rho\}$ *are given by* (3), ρ *is given by the expression*:

$$\rho = \frac{1}{ck_o}\left(\frac{2q_o\Phi(\lambda, \delta)}{\Phi_x(0, \delta)(T_f - T_o)}\right)^2 \tag{12}$$

and δ *and* λ *must satisfy the system of equations* (S2).

Proof. It is similar to the proof of Proposition 2.

3.4. Case 4 - Simultaneous determination of $\{\delta, \lambda, h\}$

Proposition 4. *The temperature and the free boundary for problem* (1) *with unknown thermal coefficients* $\{\delta, \lambda, h\}$ *are given by* (3), h *is given by the expression*:

$$h = \frac{c(T_f - T_o)(1+\beta)}{2F_2(\lambda, \delta)} \tag{13}$$

and δ *and* λ *must satisfy the following system of equations* $(\lambda > 0, \delta > -1)$:

$$\text{(S3)} \begin{cases} \beta = \delta\Phi(\lambda, \delta), \\ \Phi(\lambda, \delta) = \dfrac{\sqrt{\rho c k_o}(T_f - T_o)}{2q_o}\Phi_x(0, \delta). \end{cases}$$

Proof. From equation (6), we obtain the expression (13), and from equations (5) and (7), we obtain the system (S3).

3.5. Case 5 - Simultaneous determination of $\{\delta, \lambda, c\}$

Proposition 5. *The temperature and the free boundary for problem* (1) *with unknown thermal coefficients* $\{\delta, \lambda, c\}$ *are given by* (3), *c is given by the expression*:

$$c = \frac{1}{\rho k_o}\left(\frac{2q_o\Phi(\lambda, \delta)}{\Phi_x(0, \delta)(T_f - T_o)}\right)^2 \tag{14}$$

and δ *and* λ *must satisfy the following system of equations* $(\lambda > 0, \delta > -1)$:

$$\text{(S4)} \begin{cases} \beta = \delta\Phi(\lambda, \delta), \\ F_3(\lambda, \delta) = \dfrac{\rho h k_o(T_f - T_o)}{2q_o^2}(\Phi_x(0, \delta))^2 F_6(\lambda, \delta). \end{cases}$$

A necessary condition for the existence of solution $\lambda > 0$ *and* $\delta > -1$ *for system* (S4) *is that the data* $\beta, k_o, \rho, h, q_o, T_f$ *and* T_o *verify the following restriction*:

$$\frac{(T_f - T_o)\rho h k_o}{2q_o^2} < 1. \tag{R1}$$

Proof. From equation (7), we obtain the expression (14). The first equation of (S4) is obtained from (5), and from equations (6) and (7) we obtain the second equation in system (S4). The necessary condition (R1) is given by Property 3 (case 4) in [24].

In Table 1, we summarize the formulas for the unknown thermal coefficients corresponding to the five cases for a free boundary problem:

Table 1. Formulas for the unknown thermal coefficients for the five cases with free boundary formulation

Case N°	Unknown coeff.	Solution
FB-(i)	δ, λ, β	$\beta = \delta\Phi(\lambda, \delta)$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} F_1(\lambda, \delta) = \frac{2}{Ste} F_2(\lambda, \delta), \\ \Phi(\lambda, \delta) = \frac{\sqrt{\rho c k_o}(T_f - T_o)}{2q_o} \Phi_x(0, \delta) \end{cases}$
FB-(ii)	δ, λ, k_o	$k_o = \frac{1}{c\rho}\left(\frac{2q_o\Phi(\lambda, \delta)}{(T_f - T_o)\Phi_x(0, \delta)}\right)^2$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} F_1(\lambda, \delta) = \frac{2}{Ste} F_2(\lambda, \delta), \\ \beta = \delta\Phi(\lambda, \delta) \end{cases}$
FB-(iii)	δ, λ, ρ	$\rho = \frac{1}{ck_o}\left(\frac{2q_o\Phi(\lambda, \delta)}{(T_f - T_o)\Phi_x(0, \delta)}\right)^2$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} F_1(\lambda, \delta) = \frac{2}{Ste} F_2(\lambda, \delta), \\ \beta = \delta\Phi(\lambda, \delta) \end{cases}$
FB-(iv)	δ, λ, h	$h = \frac{c(T_f - T_o)(1 + \beta)}{2F_2(\lambda, \delta)}$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} \beta = \delta\Phi(\lambda, \delta), \\ \Phi(\lambda, \delta) = \frac{\sqrt{\rho c k_o}(T_f - T_o)}{2q_o} \Phi_x(0, \delta) \end{cases}$
FB-(v)	δ, λ, c	$c = \frac{1}{\rho k_o}\left(\frac{2q_o\Phi(\lambda, \delta)}{(T_f - T_o)\Phi_x(0, \delta)}\right)^2$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} \beta = \delta\Phi(\lambda, \delta), \\ F_3(\lambda, \delta) = \frac{\rho h k_o (T_f - T_o)}{2q_o^2} (\Phi_x(0, \delta))^2 F_6(\lambda, \delta) \end{cases}$

4. Solution to the Ten Cases through a Moving Boundary Problem

In this case, the moving boundary $s(t)$ is given by the expression (3iibis) with a given $\sigma > 0$ (which can be determined experimentally) and the unknown thermal coefficients must solve the following system of equations:

$$\beta - \delta\Phi(\sigma\sqrt{\rho c/k_o}, \delta) = 0, \tag{15}$$

$$[1 + \delta\Phi(\sigma\sqrt{\rho c/k_o}, \delta)]\frac{\sqrt{k_o}\Phi_x(\sigma\sqrt{\rho c/k_o}, \delta)}{\sigma\sqrt{\rho c}\Phi(\sigma\sqrt{\rho c/k_o}, \delta)} - \frac{2h}{c(T_f - T_o)} = 0, \tag{16}$$

$$\frac{\Phi_x(0, \delta)}{\Phi(\sigma\sqrt{\rho c/k_o}, \delta)} - \frac{2q_o}{(T_f - T_o)\sqrt{k_o \rho c}} = 0. \tag{17}$$

We present the solution for the different ten cases and the proof of some of them.

4.1. Case 1 - Simultaneous determination of $\{\delta, \beta, k_o\}$

Proposition 6. *If the moving boundary is given by* (3iibis), *then the temperature of problem* (1), *with unknown thermal coefficients* $\{\delta, \beta, k_o\}$, *is given by* (3*i*), β *is given by* (10) *and* k_o *is given by the expression*:

$$k_o = \frac{\sigma^2 \rho c}{\lambda^2}, \tag{18}$$

where δ *and* λ *must satisfy the following system of equations*:

$$(S5)\begin{cases} F_1(\lambda, \delta) = \dfrac{2}{Ste} F_2(\lambda, \delta), \\ F_4(\lambda, \delta) = \dfrac{\rho c \sigma (T_f - T_o)}{2q_o}\Phi_x(0, \delta). \end{cases}$$

Proof. In order to solve the system (15)-(17), we define the auxiliary unknown variable:

$$\lambda = \sigma\sqrt{\rho c/k_o}. \tag{19}$$

From equations (15) and (19), we obtain the expressions for β and k_o, depending on δ and λ. Therefore, from equations (16) and (17), we obtain the system (S5).

4.2. Case 2 - Simultaneous determination of $\{\delta, \beta, \rho\}$

Proposition 7. *If the moving boundary is given by* (3iibis), *then the temperature of problem* (1), *with unknown thermal coefficients* $\{\delta, \beta, \rho\}$, *is given by* (3i), β *is given by* (10) *and* ρ *is given by the expression*:

$$\rho = \frac{\lambda^2 k_o}{\sigma^2 c}, \tag{20}$$

where δ *and* λ *must satisfy the following system of equations*:

$$\text{(S6)} \begin{cases} F_1(\lambda, \delta) = \dfrac{2}{Ste} F_2(\lambda, \delta), \\ F_5(\lambda, \delta) = \dfrac{(T_f - T_o) k_o}{2 q_o \sigma} \Phi_x(0, \delta). \end{cases}$$

Proof. It is similar to the proof of Proposition 6.

4.3. Case 3 - Simultaneous determination of $\{\delta, \beta, h\}$

Proposition 8. *If the moving boundary is given by* (3iibis), *then the temperature of problem* (1), *with unknown thermal coefficients* $\{\delta, \beta, h\}$, *is given by* (3i), β *is given by* (10) *and h is given by the expression*:

$$h = \frac{c(T_f - T_o) F_1(\lambda, \delta)}{2 F_2(\lambda, \delta)} \tag{21}$$

and δ *is the solution to the following equation* $(\delta > -1)$:

$$\Phi(\lambda, \delta) = \frac{(T_f - T_o)\sqrt{k_o \rho c}}{2 q_o} \Phi_x(0, \delta), \tag{22}$$

where λ *is the known coefficient given by* (19).

Proof. In order to solve the system (15)-(17), we define the auxiliary unknown variable (19). From equations (15) and (16), we obtain the expressions for β and h, depending on δ and λ. Therefore, from equations (17) and (19), we obtain equation (22).

4.4. Case 4 - Simultaneous determination of $\{\delta, \beta, c\}$

Proposition 9. *If the moving boundary is given by* (3iibis), *then the temperature of problem* (1), *with unknown thermal coefficients* $\{\delta, \beta, c\}$, *is given by* (3i), β *is*

given by (10) *and c is given by the expression*:

$$c = \frac{\lambda^2 k_o}{\sigma^2 \rho}, \tag{23}$$

where δ *and* λ *must satisfy the following system of equations*:

$$\text{(S7)} \begin{cases} F_3(\lambda, \delta) = \dfrac{h\sigma\rho}{q_o} \Phi_x(0, \delta), \\ F_5(\lambda, \delta) = \dfrac{(T_f - T_o)k_o}{2q_o\sigma} \Phi_x(0, \delta). \end{cases}$$

A necessary condition for the existence of a solution $\lambda > 0$ *and* $\delta > -1$ *for the system* (S7) *is that data* σ, k_o, ρ, h, q_o, T_f *and* T_o *verify the following restriction*:

$$\frac{\rho h \sigma}{q_o} < 1. \tag{R2}$$

Proof. We make the change of variable (19). From equations (15) and (19) we obtain the expressions for β and c, given by (10) and (23), respectively. From equations (16) and (17), we obtain the following equations for δ and λ:

$$\begin{cases} F_3(\lambda, \delta) = \dfrac{2h\sigma^2\rho}{(T_f - T_o)k_o} F_5(\lambda, \delta), \\ F_5(\lambda, \delta) = \dfrac{(T_f - T_o)k_o}{2q_o\sigma} \Phi_x(0, \delta) \end{cases}$$

which is equivalent to the system of equations (S7). The necessary condition is given by Property 4 (case 6) in [24].

4.5. Case 5 - Simultaneous determination of $\{\delta, k_o, \rho\}$

Proposition 10. *If the moving boundary is given by* (3iibis), *then the temperature of problem* (1), *with unknown thermal coefficients* $\{\delta, k_o, \rho\}$, *is given by* (3i), k_o *and* ρ *are given by the expressions*:

$$k_o = \frac{2\sigma q_o}{T_f - T_o} \frac{F_5(\lambda, \delta)}{\Phi_x(0, \delta)}, \tag{24}$$

$$\rho = \frac{2q_o\lambda}{c\sigma(T_f - T_o)} \frac{F_5(\lambda, \delta)}{\Phi_x(0, \delta)}, \tag{25}$$

where δ *and* λ *must satisfy the following system of equations*:

$$(S8)\begin{cases}\delta\Phi(\lambda, \delta) = \beta, \\ \Phi_x(\lambda, \delta) = \dfrac{2h}{c(T_f - T_o)(1+\beta)} F_4(\lambda, \delta).\end{cases}$$

Proof. We make the change of variable (19). From equations (17) and (19), we obtain the expressions for k_o and ρ, given by (24) and (25), respectively. From equations (15) and (16), we obtain the system of equations (S8) for δ and λ.

4.6. Case 6 - Simultaneous determination of $\{\delta, k_o, h\}$

Proposition 11. *If the moving boundary is given by* (3iibis), *then the temperature of problem* (1), *with unknown thermal coefficients* $\{\delta, k_o, h\}$, *is given by* (3i), k_o *is given by* (18) *and h is given by the following expression*:

$$h = \frac{c(T_f - T_o)}{2} \frac{1+\beta}{F_2(\lambda, \delta)}, \tag{26}$$

where δ *and* λ *must satisfy the following system of equations*:

$$(S9)\begin{cases}\beta = \delta\Phi(\lambda, \delta), \\ F_4(\lambda, \delta) = \dfrac{\rho c\sigma(T_f - T_o)}{2q_o} \Phi_x(0, \delta).\end{cases}$$

Moreover, the solution to the system (S9) *is given by*

$$\lambda = \frac{\rho c\sigma(T_f - T_o)}{2\beta q_o} \delta\Phi_x(0, \delta) \tag{27}$$

and δ *must be the solution to the following equation*:

$$\delta\Phi\left(\frac{\rho c\sigma(T_f - T_o)}{2\beta q_o} \delta\Phi_x(0, \delta), \delta\right) = \beta, \quad \delta > -1. \tag{28}$$

Proof. We make the change of variable (19) and we obtain the expression (18). From equations (16) and (19), we obtain the expression (26). From equations (15) and (17), we obtain the system of equations (S9) for δ and λ whose solution is given by (27) for λ and δ must be the solution to the equation (28).

4.7. Case 7 - Simultaneous determination of $\{\delta, k_o, c\}$

Proposition 12. *If the moving boundary is given by* (3iibis), *then the temperature of problem* (1), *with unknown thermal coefficients* $\{\delta, k_o, c\}$, *is given by* (3i), k_o *is given by* (25) *and c is given by the expression*:

$$c = \frac{2q_o}{\rho\sigma(T_f - T_o)} \frac{F_4(\lambda, \delta)}{\Phi_x(0, \delta)}, \tag{29}$$

where δ *and* λ *must satisfy the following system of equations*:

$$(S10) \begin{cases} \beta = \delta\Phi(\lambda, \delta), \\ \Phi_x(\lambda, \delta) = \dfrac{h\sigma\rho}{q_o(1+\beta)} \Phi_x(0, \delta). \end{cases}$$

A necessary condition for the existence of solution $\lambda > 0$ *and* $\delta > -1$ *for the system* (S10), *particularly for the second equation, is that the data* σ, k_o, ρ, h, q_o, T_f *and* T_o *verify the restriction* (R2).

Proof. It is similar to the proof of Proposition 9.

4.8. Case 8 - Simultaneous determination of $\{\delta, \rho, h\}$

Proposition 13. *If the moving boundary is given by* (3iibis), *then the temperature of problem* (1), *with unknown thermal coefficients* $\{\delta, \rho, h\}$, ρ *and h are given by* (20) *and* (26), *respectively, where* δ *and* λ *must satisfy the following system of equations*:

$$(S11) \begin{cases} \beta = \delta\Phi(\lambda, \delta), \\ F_5(\lambda, \delta) = \dfrac{(T_f - T_o)k_o}{2q_o\sigma} \Phi_x(0, \delta). \end{cases}$$

Moreover, the solution to the system (S11) *is given by*

$$\lambda = \frac{2\sigma\beta q_o}{k_o(T_f - T_o)} \frac{1}{\delta\Phi_x(0, \delta)} \tag{30}$$

and δ *must be the solution to the following equation*:

$$\delta\Phi\left(\frac{2\sigma\beta q_o}{k_o(T_f - T_o)} \frac{1}{\delta\Phi_x(0, \delta)}, \delta\right) = \beta, \quad \delta > -1. \tag{31}$$

Proof. It is similar to the proof of Proposition 11.

4.9. Case 9 - Simultaneous determination of $\{\delta, \rho, c\}$

Proposition 14. *If the moving boundary is given by* (3iibis), *then the temperature of problem* (1), *with unknown thermal coefficients* $\{\delta, \rho, c\}$, *is given by* (3i), ρ *and* c *are given by the expressions*:

$$\rho = \frac{k_o(1+\beta)(T_f - T_o)}{2\sigma^2 h\beta}\lambda\delta\Phi_x(\lambda, \delta), \tag{32}$$

$$c = \frac{2h}{(1+\beta)(T_f - T_o)}F_2(\lambda, \delta), \tag{33}$$

where δ *and* λ *must satisfy the system of equations* (S11).

Proof. It is similar to the proof of Proposition 11.

4.10. Case 10 - Simultaneous determination of $\{\delta, c, h\}$

Proposition 15. *If the moving boundary is given by* (3iibis), *then the temperature of problem* (1), *with unknown thermal coefficients* $\{\delta, c, h\}$, *is given by* (3i), c *is given by* (33) *and* h *is given by the following expression*:

$$h = \frac{k_o(T_f - T_o)(1+\beta)}{2\rho\sigma^2}\frac{\lambda^2}{F_2(\lambda, \delta)}, \tag{34}$$

where δ *and* λ *must satisfy the system of equations* (S11).

Proof. It is similar to the proof of Proposition 11.

Now, in Table 2, we summarize the formulas for the unknown thermal coefficients corresponding to the ten cases for a moving boundary problem (i.e., an inverse Stefan problem).

Table 2. Restrictions and formulas for the unknown thermal coefficients for the cases of moving boundary

Case N°	Unknown coeff.	Solution
MB-(i)	δ, β, k_o	$\beta = \delta\Phi(\lambda, \delta)$, $k_o = \dfrac{\sigma^2 \rho c}{\lambda^2}$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} F_1(\lambda, \delta) = \dfrac{2}{Ste} F_2(\lambda, \delta), \\ F_4(\lambda, \delta) = \dfrac{\rho c \sigma (T_f - T_o)}{2q_o} \Phi_x(0, \delta) \end{cases}$
MB-(ii)	δ, β, ρ	$\beta = \delta\Phi(\lambda, \delta)$, $\rho = \dfrac{\lambda^2 k_o}{\sigma^2 c}$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} F_1(\lambda, \delta) = \dfrac{2}{Ste} F_2(\lambda, \delta), \\ F_5(\lambda, \delta) = \dfrac{(T_f - T_o) k_o}{2q_o \sigma} \Phi_x(0, \delta) \end{cases}$
MB-(iii)	δ, β, h	$\beta = \delta\Phi(\lambda, \delta)$, $h = \dfrac{c(T_f - T_o) F_1(\lambda, \delta)}{2F_2(\lambda, \delta)}$, $\lambda = \sigma\sqrt{\rho c / k_o}$, where δ is the solution to the equation: $\Phi(\lambda, \delta) = \dfrac{(T_f - T_o)\sqrt{k_o \rho c}}{2q_o} \Phi_x(0, \delta)$
MB-(iv)	δ, β, c	$\beta = \delta\Phi(\lambda, \delta)$, $c = \dfrac{\lambda^2 k_o}{\sigma^2 \rho}$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} F_3(\lambda, \delta) = \dfrac{h\sigma\rho}{q_o} \Phi_x(0, \delta), \\ F_5(\lambda, \delta) = \dfrac{(T_f - T_o) k_o}{2q_o \sigma} \Phi_x(0, \delta) \end{cases}$
MB-(v)	δ, k_o, ρ	$k_o = \dfrac{2\sigma q_o}{T_f - T_o} \dfrac{F_5(\lambda, \delta)}{\Phi_x(0, \delta)}$, $\rho = \dfrac{2q_o \lambda}{c\sigma(T_f - T_o)} \dfrac{F_5(\lambda, \delta)}{\Phi_x(0, \delta)}$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} \delta\Phi(\lambda, \delta) = \beta, \\ \Phi_x(\lambda, \delta) = \dfrac{2h}{c(T_f - T_o)(1 + \beta)} F_4(\lambda, \delta) \end{cases}$

MB-(vi)	δ, k_o, h	$k_o = \dfrac{\sigma^2 \rho c}{\lambda^2}$, $h = \dfrac{c(T_f - T_o)}{2} \dfrac{(1+\beta)}{F_2(\lambda, \delta)}$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} \beta = \delta \Phi(\lambda, \delta), \\ F_4(\lambda, \delta) = \dfrac{\rho c \sigma (T_f - T_o)}{2 q_o} \Phi_x(0, \delta) \end{cases}$
MB-(vii)	δ, k_o, c	$k_o = \dfrac{2\sigma q_o}{T_f - T_o} \dfrac{F_5(\lambda, \delta)}{\Phi_x(0, \delta)}$, $c = \dfrac{2 q_o}{\rho \sigma (T_f - T_o)} \dfrac{F_4(\lambda, \delta)}{\Phi_x(0, \delta)}$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} \beta = \delta \Phi(\lambda, \delta), \\ \Phi_x(\lambda, \delta) = \dfrac{h \sigma \rho}{q_o (1+\beta)} \Phi_x(0, \delta) \end{cases}$
MB-(viii)	δ, ρ, h	$\rho = \dfrac{\lambda^2 k_o}{\sigma^2 c}$, $h = \dfrac{c(T_f - T_o)}{2} \dfrac{(1+\beta)}{F_2(\lambda, \delta)}$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} \beta = \delta \Phi(\lambda, \delta), \\ F_5(\lambda, \delta) = \dfrac{(T_f - T_o) k_o}{2 q_o \sigma} \Phi_x(0, \delta) \end{cases}$
MB-(ix)	δ, ρ, c	$\rho = \dfrac{k_o (1+\beta)(T_f - T_o)}{2\sigma^2 h \beta} \lambda \delta \Phi_x(\lambda, \delta)$, $c = \dfrac{2h}{(1+\beta)(T_f - T_o)} F_2(\lambda, \delta)$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} \beta = \delta \Phi(\lambda, \delta), \\ F_5(\lambda, \delta) = \dfrac{(T_f - T_o) k_o}{2 q_o \sigma} \Phi_x(0, \delta) \end{cases}$
MB-(x)	δ, c, h	$c = \dfrac{\lambda^2 k_o}{\sigma^2 \rho}$, $h = \dfrac{k_o (T_f - T_o)(1+\beta)}{2\rho\sigma^2} \dfrac{\lambda^2}{F_2(\lambda, \delta)}$, where $\{\delta, \lambda\}$ is the solution to the system: $\begin{cases} \beta = \delta \Phi(\lambda, \delta), \\ F_5(\lambda, \delta) = \dfrac{(T_f - T_o) k_o}{2 q_o \sigma} \Phi_x(0, \delta) \end{cases}$

5. Sensitivity Analysis

We use the free software SCILAB for the numerical analysis. For each case, first we have to determine the solution to the corresponding system of equations. The command bvodeS was used to solve the differential problem (4), which allowed us to evaluate the modified error function at the necessary points. To find the solution to the system, we minimize the sum of the squares of the equations of the nonlinear system, using the Levenberg-Marquardt algorithm. Secondly, using the approximately solution to the system, we evaluated the unknown thermal coefficients.

In each case, we have used the corresponding data, from the following set of values that satisfy equations (5)-(7). The data corresponding to aluminum near its melting point is:

$$\beta = 0{,}0318778, \quad \delta = 0{,}1177546, \quad \lambda = 0{,}2433491, \quad k_o = 293{,}1882\ \mathrm{W/m^\circ C},$$

$$\rho = 2698{,}4\ \mathrm{kg/m^3}, \quad c = 783{,}6192\ \mathrm{J/kg^\circ C}, \quad h = 388000\ \mathrm{J/kg},$$

$$q_o = 3179226{,}8\ \mathrm{kg/s^{5/2}}, \quad T_f = 660^\circ\mathrm{C}, \quad T_o = 600^\circ\mathrm{C}.$$

In order to determine the influence of known parameters over unknown coefficients, we define the *normalized sensitivity* by the following expression [19]:

$$S(p, q_i) = \frac{q_i}{p}\left[\frac{\partial p}{\partial q_i}\right], \tag{35}$$

where p is a particular solution parameter (e.g., the dimensionless parameters δ or λ, or the initial thermal conductivity k_o in Case 2 of free boundary problems), q_i is one of the given parameters (e.g., β, ρ, c, h in Case 2). Thanks to its dimensionless nature, we can compare the sensitivity of parameters of different magnitudes. The normalized sensitivity indicates the percentage change on the value of the parameter p, when the variable q_i increases or decreases 1% of its value [10, 15]. We will approximate $S(p, q_i)$ by the following way:

$$S(p, q_i)^+ \approx \frac{q_i}{p(q)} \cdot \frac{p(\hat{q}^+) - p(q)}{\hat{q}_i^+ - q_i} \text{ right normalized sensitivity,}$$

$$S(p, q_i)^- \approx \frac{q_i}{p(q)} \cdot \frac{p(\hat{q}^-) - p(q)}{\hat{q}_i^- - q_i} \text{ left normalized sensitivity,} \tag{36}$$

where q is the vector $(q_1, ..., q_i, ...)$, $p(q) = p(q_1, ..., q_i, ...)$ and

$$\hat{q}_j^+ = q_j,\ j \neq i \quad \text{and} \quad \hat{q}_i^+ = q_i + \varepsilon|q_i|,$$

$$\hat{q}_j^- = q_j,\ j \neq i \quad \text{and} \quad \hat{q}_i^- = q_i - \varepsilon|q_i| \tag{37}$$

and $\varepsilon = 0.01$ or $\varepsilon = 0.001$, depending on the different cases. Here the right normalized sensitivity represents the change on the parameter p when q_i increases a 1% of its value, and the left normalized sensitivity represents the change on the parameter p when q_i decreases 1%.

5.1. Determination of coefficients through a free boundary problem

We have analyzed the relationship between the solution and the different parameters. Table 3 shows the right and the left normalized sensitivities, in each case, taking $\varepsilon = 0.01$. If the sensitivity is negative, then it means that the parameter p is decreasing with respect to q_i, and if it is positive, then it means that the parameter p is increasing with respect to q_i.

Table 3. Left and right normalized sensitivities in the five cases of the free boundary problems

Case N°	Unknown	β		k_o		ρ		c		h	
1	δ	-	-	–64.2	–63.9	–64.2	–63.9	–5.2	–5.2	–58.7	–58.9
	λ	-	-	–0.5	–0.5	–0.5	–0.5	0.4	0.4	–0.9	–0.9
	β	-	-	–65.5	–64.2	–65.5	–64.2	–4.8	–4.8	–60.6	–59.4
2	δ	0.9	0.9	-	-	0	0	–0.4	–0.4	0.4	–0.4
	λ	0.007	0.008	-	-	0	0	0.4	–0.4	–0.4	–0.4
	k_o	–0.01	–0.01	-	-	–1	–1	–0.07	–0.07	–0.9	–0.9
3	δ	0.9	0.9	0	0	-	-	–0.4	–0.4	0.4	–0.4
	λ	0.007	0.008	0	0	-	-	0.4	–0.4	–0.4	–0.4
	ρ	–0.01	–0.01	–1	–1	-	-	–0.07	–0.07	–0.9	–0.9
4	δ	0.9	0.9	–0.4	–0.4	–0.4	–0.4	–0.4	–0.4	-	-
	λ	0.01	0.01	0.5	0.5	0.5	0.5	0.5	0.5	-	-
	h	–0.01	–0.01	–1	–1	–1	–1	–0.08	–0.08	-	-
5	δ	1	1	5.6	6.7	5.6	6.7	-	-	5.6	6.7
	λ	–0.09	–0.09	–6.3	–6.6	–6.3	–6.6	-	-	–6.3	–6.6
	c	–0.2	–0.2	–13.6	–13.3	–13.6	–13.3	-	-	–12.5	–12.4

5.1.1. Case 1

In this case, we can observe that the solution behaves in the same way when we modify k_o or ρ. Figure 1 shows the relationship between the parameter solutions and the given parameters $\{k_o, \rho\ c \text{ and } h\}$. The parameters δ and β are more sensitive to changes in k_o or ρ, and the parameter λ is more sensitive to changes with respect to changes produced in h.

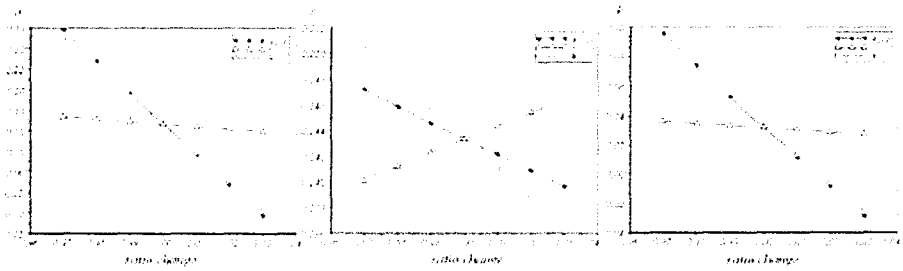

Figure 1. Parametric change of $\{\delta, \lambda, \beta\}$ versus the relative change of $\{k_o, \rho, c, h\}$ in Case 1 of the free boundary problem.

5.1.2. Cases 2 and 3

These two cases are analyzed together, because when we modify the parameters h, β or c, the corresponding δ and λ are the same for both cases. This can be explained observing that the system of equations to be solved is the same in both cases (S2 in our case). Another remark is that, in the previous system, the parameters k_o or ρ do not appear explicitly. Therefore, the parameters δ and λ remain constant when we modify the values of k_o or ρ. Figures 2 and 3 show the relationship between the parameter solutions and the given parameters $\{\beta, k_o, \rho, c \text{ and } h\}$ in Cases 2 and 3, respectively. The parameter δ is more sensitive to changes in β, the parameter λ is more sensitive to changes in h and the parameters k_o and ρ are more sensitive to changes in ρ and k_o, respectively.

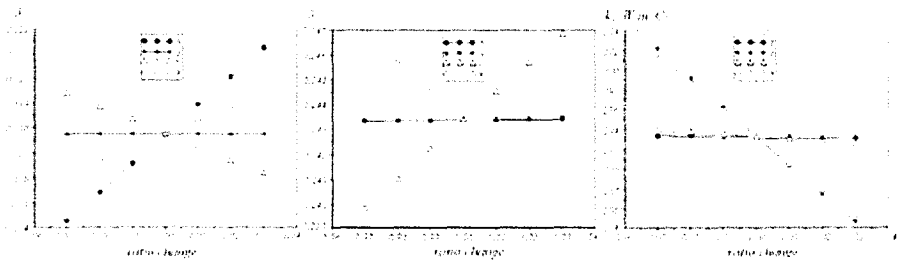

Figure 2. Parametric change of $\{\delta, \lambda, k_o\}$ versus the relative change of $\{\beta, \rho, c, h\}$ in Case 2 of the free boundary problem.

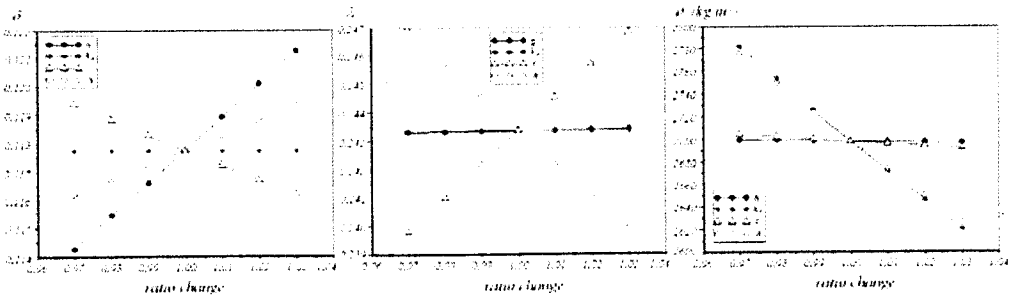

Figure 3. Parametric change of $\{\delta, \lambda, \rho\}$ versus the relative change of $\{\beta, k_o, c, h\}$ in Case 3 of the free boundary problem.

5.1.3. Case 4

As in Case 1, we can observe that the solution to system (S3) behaves in the same way when we modify k_o, ρ or c. Figure 4 shows the relationship between the parameter solutions and the given parameters $\{\beta, k_o, \rho \text{ and } c\}$. The parameter δ is more sensitive to changes in β, the parameter λ is more sensitive to changes in k_o, ρ and c, and h was more sensitive to changes in ρ and k_o.

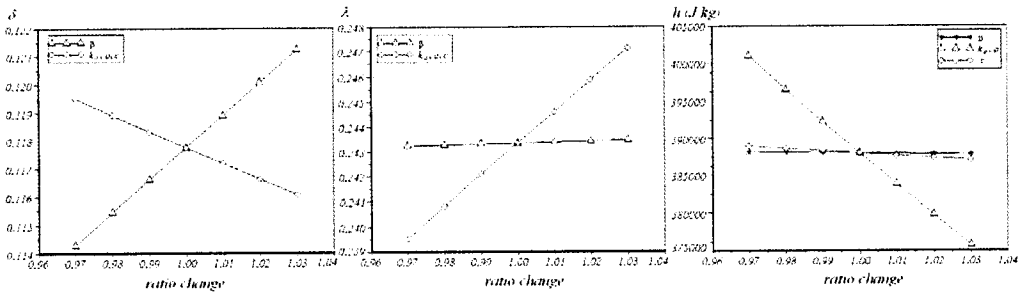

Figure 4. Parametric change of $\{\delta, \lambda, h\}$ versus the relative change of $\{\beta, k_o, \rho, c\}$ in Case 4 of the free boundary problem.

5.1.4. Case 5

We can observe that the solution to system (S4) behaves in the same way when we modify k_o, ρ or h. The restriction (R1) is verified until we increase a 1% the parameters. Figure 5 shows the relationship between the parameter solutions and the given parameters $\{\beta, k_o, \rho \text{ and } h\}$. The parameter δ is more sensitive to changes in k_o, ρ and h; the parameter λ is more sensitive to changes in k_o and ρ; and the parameter c is more sensitive to changes in ρ and k_o.

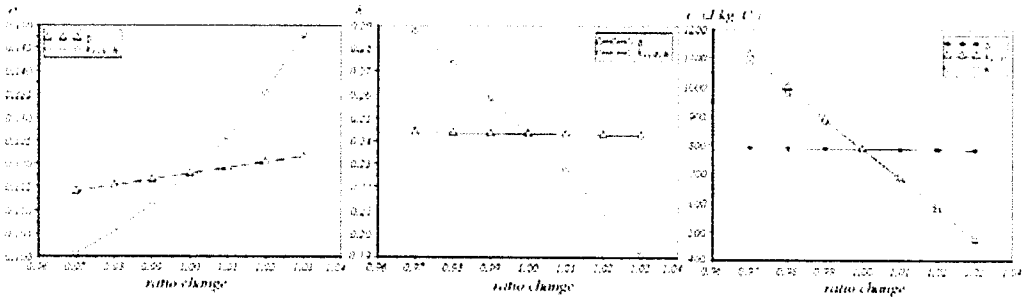

Figure 5. Parametric change of $\{\delta, \lambda, c\}$ versus the relative change of $\{\beta, k_o, \rho, h\}$ in Case 1 of the free boundary problem.

5.2. Determination of coefficients through a moving boundary problem

We analyzed the relationship between the solution and the different parameters. Table 4 shows the right and left normalized sensitivities, in each case, taking $\varepsilon = 0.01$ in Cases 3, 5, 6 and 7, and $\varepsilon = 0.001$ in Cases 1, 2, 8, 9 and 10.

Table 4. Left and right normalized sensitivities in the ten cases of moving boundary problems

Case N°	Unknown	σ		β		k_o		ρ		c		h	
1	δ	−3491.7	3579.7	-	-	-	-	−3491.7	3579.7	−196.2	195.8	−3291	3381.1
	β	−3372.5	3759.1	-	-	-	-	−3372.5	3759.1	−198.5	199.2	−3189.4	3542
	k_o	52.6	−52.7	-	-	-	-	53.6	−53.7	3	−3	50.6	−50.7
2	δ	−62.9	62.8	-	-	62.8	−62.8	-	-	1.6	−1.6	−1.6	1.6
	β	−63.6	63.6	-	-	63.7	−63.5	-	-	1.1	−1.1	−1.1	1.1
	ρ	0.9	−0.9	-	-	−0.01	−0.01	-	-	0.05	−0.05	0.9	−0.9
3	δ	−61.9	60.5	-	-	63	−62.7	1.7	−1.7	1.7	−1.7	-	-
	β	−62.4	62.4	-	-	64.3	−63	1.3	−1.2	1.3	−1.2	-	-
	h	1	−1	-	-	0.02	−0.02	1	−1	0.06	−0.06	-	-
4	δ	−90.4	92.1	-	-	62.3	−62.3	−28.4	29.4	-	-	−28.4	29.4
	β	−84.2	84.7	-	-	63.3	−63.2	−20.9	21.4	-	-	−20.9	21.4
	c	17.4	−17.3	-	-	0.3	−0.3	17.8	−17.6	-	-	16.8	−16.6
5	δ	0	0	−0.9	0.9	-	-	-	-	0.4	−0.4	−0.4	0.4
	k_o	−1	1	0.01	−0.01	-	-	-	-	0.01	−0.01	−0.01	0.01
	ρ	1	−1	$-2.9 * 10^{-4}$	$2.8 * 10^{-4}$	-	-	-	-	0.05	−0.05	0.9	−0.9
6	δ	0.4	−0.4	−0.9	0.9	-	-	0.4	−0.4	0.4	−0.4	-	-

	k_o	−0.9	0.9	0.01	−0.01	-	-	0.02	−0.01	0.02	−0.01	-	-
	h	1	−1	$-3*10^{-4}$	$3*10^{-4}$	-	-	1.1	−1	0.05	−0.05	-	-
7	δ	−7.2	9.3	−0.9	0.9	-	-	−7.2	9.3	-	-	−7.2	9.3
	k_o	−1.3	1.3	0.01	−0.01	-	-	−0.3	0.3	-	-	−0.3	0.3
	c	18.1	−17.7	−0.005	0.005	-	-	18.1	−17.7	-	-	16.9	−16.8
8	δ	25.4	−24.2	−1.3	1.3	−24.2	25.4	-	-	0	0	-	-
	ρ	−50	51.4	0.7	−0.7	52.5	−50.9	-	-	1	−1	-	-
	h	56.9	−52.7	−0.8	0.8	−52.8	56.9	-	-	−1	1	-	-
9	δ	25.4	−24.2	−1.3	1.3	−24.2	25.4	-	-	-	-	0	0
	ρ	4	-4	−0.04	0.04	−3	3	-	-	-	-	1	−1
	c	−53.9	55.7	0.8	−0.8	55.7	−53.8	-	-	-	-	−1	1
10	δ	25.4	−24.2	−1.3	1.3	−24.2	25.4	0	0	-	-	-	-
	c	−50	51.4	0.7	−0.7	52.5	−50.9	1	−1	-	-	-	-
	h	4	−4	−0.04	0.04	−3	3	−1	1	-	-	-	-

5.2.1. Case 1

In this case, we can observe that the solution behaves in the same way when we modify σ or ρ. Figure 6 shows the relationship between the parameter solutions $\{\delta, \beta \text{ and } k_o\}$ and the given parameters $\{\sigma, \rho, c \text{ and } h\}$. The parameters δ, β and k_o are more sensitive to changes in σ or ρ.

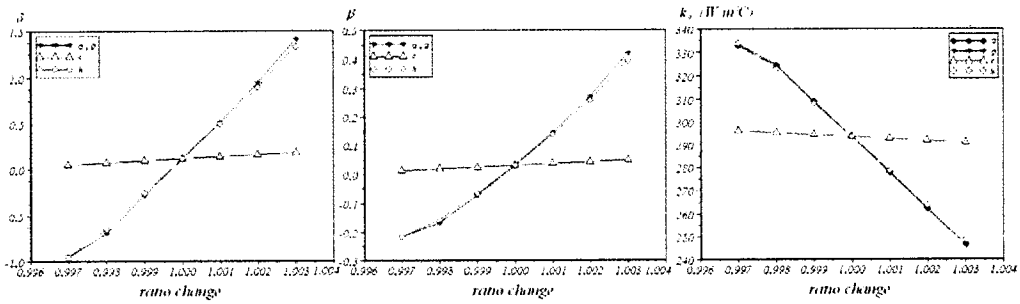

Figure 6. Parametric change of $\{\delta, \beta, k_o\}$ versus the relative change of $\{\sigma, \rho, c, h\}$ in Case 1 of the moving boundary problem.

5.2.2. Case 2

Figure 7 shows the relationship between the parameter solutions $\{\delta, \beta \text{ and } \rho\}$ and the given parameters $\{\sigma, k_o, c \text{ and } h\}$. The parameters δ and β are more sensitive to changes in σ or k_o, the parameter ρ is more sensitive to changes in σ.

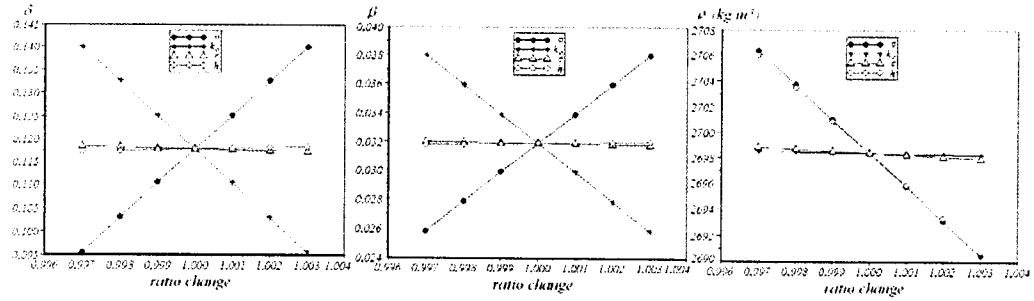

Figure 7. Parametric change of $\{\delta, \beta, \rho\}$ versus the relative change of $\{\sigma, k_o, c, h\}$ in Case 2 of the moving boundary problem.

5.2.3. Case 3

We can observe that the parameters δ and β behave in the same way when we modify c or ρ. Figure 8 shows the relationship between the parameter solutions $\{\delta, \beta$ and $h\}$ and the given parameters $\{\sigma, k_o, \rho$ and $c\}$. The parameters δ and β are more sensitive to changes in σ or k_o, and the parameter h is more sensitive to changes in σ.

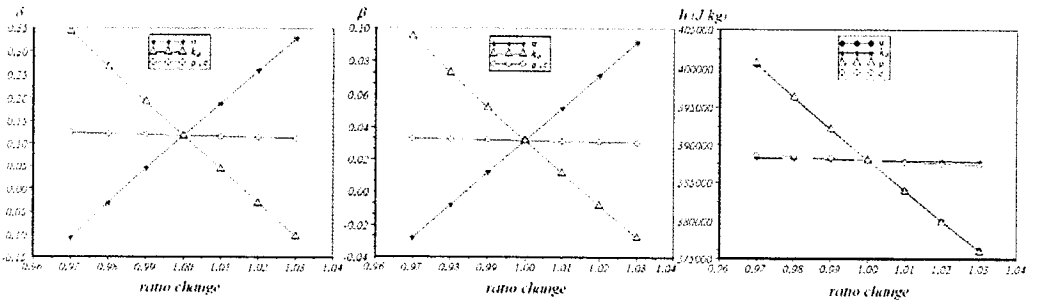

Figure 8. Parametric change of $\{\delta, \beta, h\}$ versus the relative change of $\{\sigma, k_o, \rho, c\}$ in Case 3 of the moving boundary problem.

5.2.4. Case 4

In this case, we can observe that the parameters δ and β behave in the same way when we modify h or ρ. The restriction (R2) is verify until we increase a 0.75% the parameters σ, ρ and h. Figure 9 shows the relationship between the parameter solutions $\{\delta, \beta$ and $c\}$ and the given parameters $\{\sigma, k_o, \rho$ and $h\}$. The parameter δ is more sensitive to changes in σ; the parameter β is more sensitive to changes in k_o and σ; and the parameter c is more sensitive to changes in ρ.

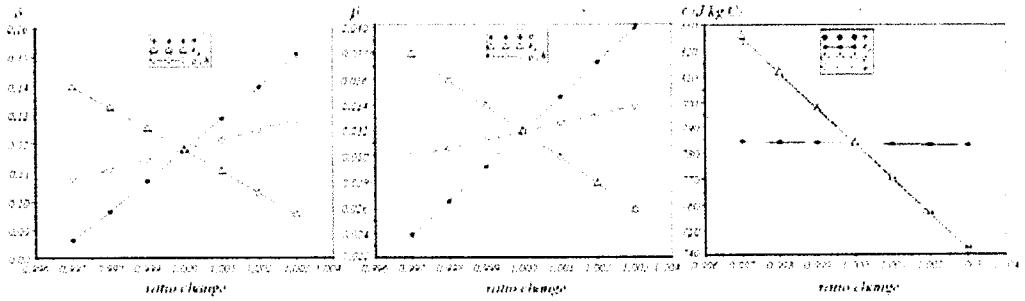

Figure 9. Parametric change of $\{\delta, \beta, c\}$ versus the relative change of $\{\sigma, k_o, \rho, h\}$ in Case 4 of the moving boundary problem.

5.2.5. Case 5

In this case, we can observe that the parameters δ, λ and k_o behave in the opposite way when we modify c or h. Another remark is that in the system (S8) there is no intervention of σ. Therefore, the parameters δ and λ remain constant when we modify the values of σ. Figure 10 shows the relationship between the parameter solutions $\{\delta, k_o$ and $\rho\}$ and the given parameters $\{\sigma, \beta, c$ and $h\}$. The parameter δ is more sensitive to changes in β; the parameter k_o is more sensitive to changes in σ; and the parameter ρ is more sensitive to changes in σ and h.

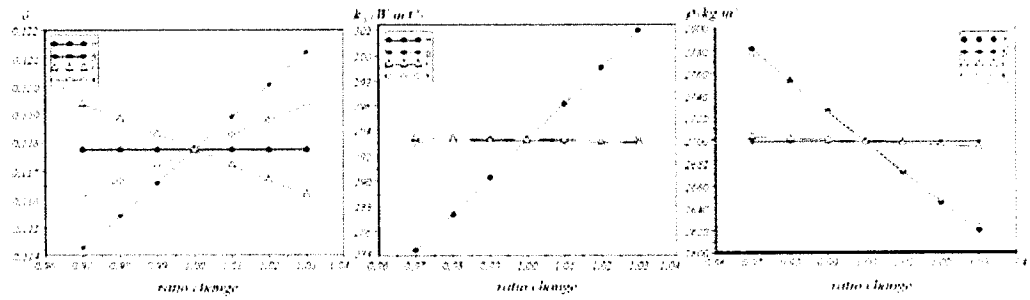

Figure 10. Parametric change of $\{\delta, k_o, \rho\}$ versus the relative change of $\{\sigma, \beta, c, h\}$ in Case 5 of the moving boundary problem.

5.2.6. Case 6

We can observe that the parameters δ and λ behave in the same way when we modify σ, c or ρ. Instead, the parameter k_o behaves in the same way when we modify c or ρ (but not σ). On the other hand, the parameter h behaves in the same way when we modify σ or ρ (but not c). Figure 11 shows the relationship between

the parameter solutions $\{\delta, k_o \text{ and } h\}$ and the given parameters $\{\sigma, \beta, \rho \text{ and } c\}$. The parameter δ is more sensitive to changes in β, the parameter h is more sensitive to changes in ρ and σ, and the parameter k_o is more sensitive to changes in σ.

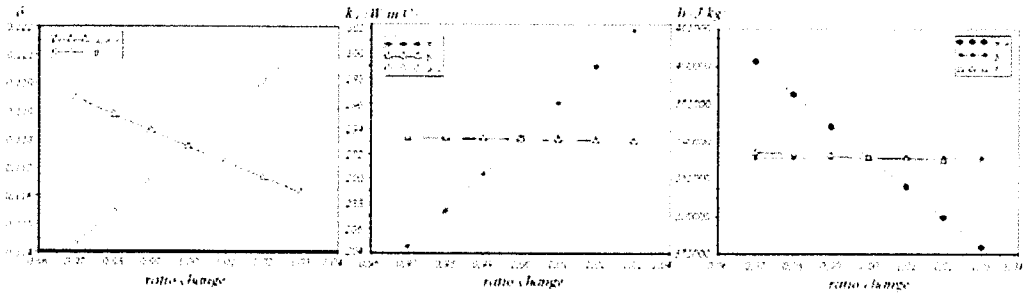

Figure 11. Parametric change of $\{\delta, k_o, h\}$ versus the relative change of $\{\sigma, \beta, \rho, c\}$ in Case 6 of the moving boundary problem.

5.2.7. Case 7

In this case, we can observe that the parameters δ and λ behave in the same way when we modify σ, h or ρ. Instead, the parameter c behaves in the same way when we modify σ or ρ (but not h). On the other hand, the parameter k_o behaves in the same way when we modify h or ρ (but not σ). The restriction (R2) is limited to a 0.75% increase of the parameters σ, ρ and h, as in Case 4. Figure 12 shows the relationship between the parameter solutions $\{\delta, k_o \text{ and } c\}$ and the given parameters $\{\sigma, \beta, \rho \text{ and } h\}$. The parameter δ is more sensitive to changes in σ, h and ρ; the parameter c is more sensitive to changes in ρ and σ; the parameter k_o is more sensitive to changes in σ.

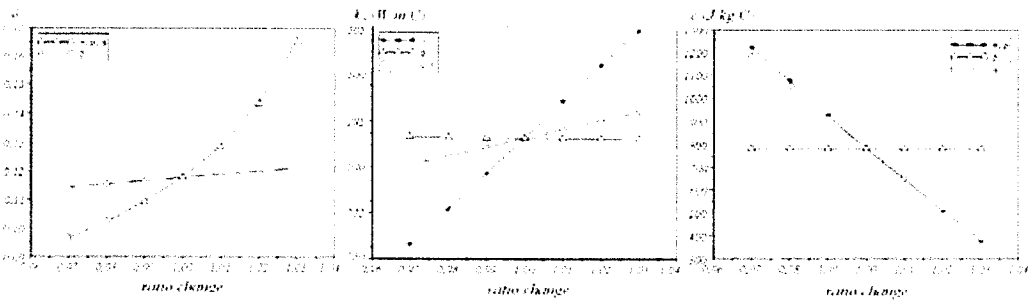

Figure 12. Parametric change of $\{\delta, k_o, c\}$ versus the relative change of $\{\sigma, \beta, \rho, h\}$ in Case 7 of the moving boundary problem.

5.2.8. Cases 8, 9 and 10

In all these three cases, the system of equations that the parameters δ and λ must verify is the same ((S11) in our cases). The only two parameters that appear in these equations are k_o and σ; neither h, c or ρ are involved in (S11). This means that δ and λ maintain constant when we modify c (in Case 8), h (in Case 9) or ρ (in Case 10). Figures 13, 14 and 15 show the relationship between the parameter solutions and the other given parameters, in Cases 8, 9 and 10, respectively. The parameter δ is more sensitive to changes in σ and k_o; and the most influential parameters are σ and k_o, for the rest of the given parameters.

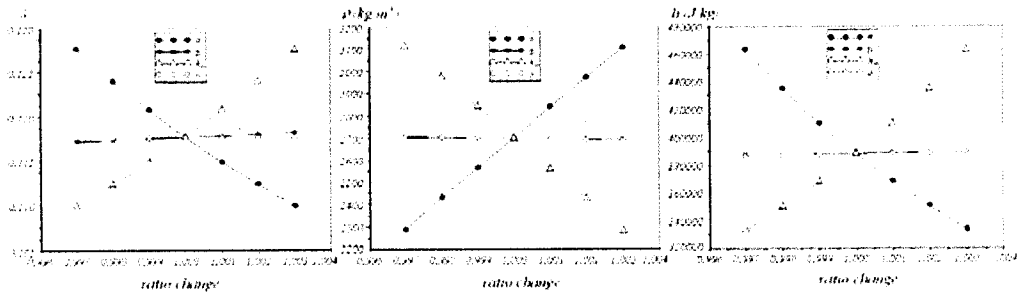

Figure 13. Parametric change of $\{\delta, \rho, h\}$ versus the relative change of $\{\sigma, \beta, k_o, c\}$ in Case 8 of the moving boundary problem.

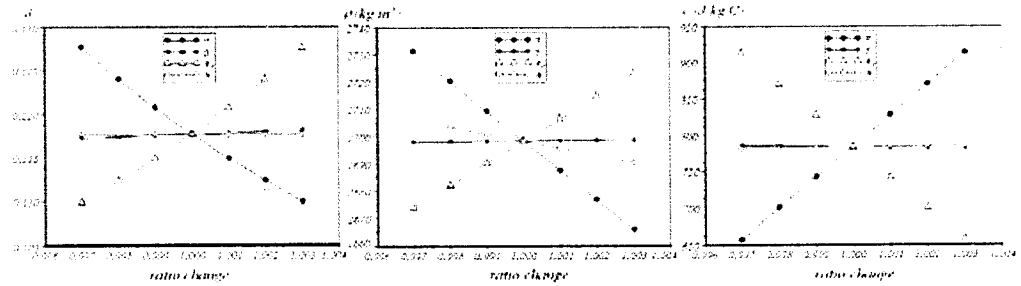

Figure 14. Parametric change of $\{\delta, \rho, c\}$ versus the relative change of $\{\sigma, \beta, k_o, h\}$ in Case 9 of the moving boundary problem.

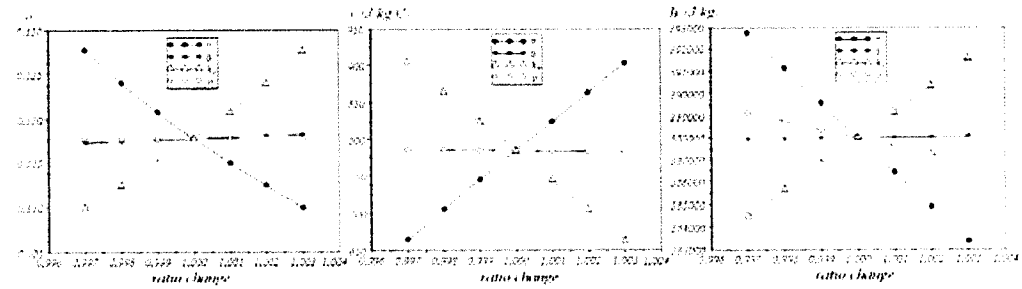

Figure 15. Parametric change of $\{\delta, c, h\}$ versus the relative change of $\{\sigma, \beta, k_o, \rho\}$ in Case 10 of the moving boundary problem.

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