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Explicit solution for a Stefan problem with variable latent heat and constant heat flux boundary conditions

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ABSTRACT

In Voller, Swenson and Paola [V.R. Voller, J.B. Swenson, C. Paola, An analytical solution for a Stefan problem with variable latent heat, Int. J. Heat Mass Transfer 47 (2004) 5387–5390], and Lorenzo-Trueba and Voller [J. Lorenzo-Trueba, V.R. Voller, Analytical and numerical solution of a generalized Stefan problem exhibiting two moving boundaries with application to ocean delta formation, J. Math. Anal. Appl. 366 (2010) 538–549], a model associated with the formation of sedimentary ocean deltas is studied through a one-phase Stefan-like problem with variable latent heat. Motivated by these works, we consider a two-phase Stefan problem with variable latent of fusion and initial temperature, and constant heat flux boundary conditions. We obtain the sufficient condition on the data in order to have an explicit solution of a similarity type of the corresponding free boundary problem for a semi-infinite material. Moreover, the explicit solution given in the first quoted paper can be recovered for a particular case by taking a null heat flux condition at the infinity.

1. Introduction

Problems of phase-change are often found in industrial processes [2,8,9,17,23]. In [28] a large bibliography on free boundary problems (particularly regarding the Stefan problem) for the heat and diffusion equations was given.

The movement of sea coasts can be modeled by a generalized one-phase Stefan problem [16,32] where a constant heat flux boundary condition at the fixed face and a linear latent heat of fusion of a semi-infinite material were considered. Recently, similarity solutions for one- or two-phase problem are given in [7,12,13,15]. Other related references on the subject are [19–21,25].

Motivated by the above problems we will consider the following free boundary problem for a semi-infinite material: Find the temperatures $T_1(x,t), T_2(x,t)$ and the moving melt interface $s(t)$ such that they satisfy:

$$T_{1t}(x,t) = \alpha_1 T_{1xx}(x,t), \quad x > s(t),\ t > 0, \tag{1}$$

$$T_{2t}(x,t) = \alpha_2 T_{2xx}(x,t), \quad 0 < x < s(t),\ t > 0, \tag{2}$$

$$T_1\big(s(t),t\big) = 0, \quad t > 0, \tag{3}$$

$$T_2\big(s(t),t\big) = 0, \quad t > 0, \tag{4}$$

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$$k_2 T_{2x}(0,t) = -q_2, \quad t > 0, \tag{5}$$

$$s(0) = 0, \tag{6}$$

$$k_1 T_{1x}\big(s(t),t\big) - k_2 T_{2x}\big(s(t),t\big) = \gamma s(t) s'(t), \quad t > 0, \tag{7}$$

$$k_1 T_{1x}(+\infty,t) = -q_1, \quad t > 0, \tag{8}$$

$$T_1(x,0) = -\frac{q_1}{k_1} x, \quad x > 0, \tag{9}$$

where the solid (liquid) phase is represented by the subscript $i = 1$ ($i = 2$), k_i is the thermal conductivity, c_i is the heat capacity, ρ is the common mass density, $\alpha_i = \sqrt{k_i/\rho c_i}$ is the diffusion coefficient, γx is the variable latent heat of fusion per unity of volume, q_i is the constant heat flux boundary condition, and the melting temperature is zero. On the other hand, $s'(t)$ represents the velocity of the phase-change interface.

There exist explicit solutions to Stefan-like problems in some particular cases, e.g. [1,3–6,10,11,14,18,22,24,26,27,29–31,33].

The goal of this paper is to obtain in Section 2 the sufficient conditions on the data in order to have an explicit solution of a similarity type of the free boundary problem (1)–(9) for a semi-infinite material with variable latent heat of fusion and constant heat flux boundary conditions. This explicit solution is given by (17)–(19). Moreover, the explicit solution given in [32] can be recovered for a particular case of our explicit solution by taking a null heat flux condition at the infinity and a null initial temperature.

2. Explicit solution for phase-change problems with linear latent heat of fusion

The following lemma is useful in order to find solutions for the differential equations (1) and (2).

Lemma 1.

a) *Let*

$$T(x,t) = 2\sqrt{t}\,\eta(\xi), \quad \text{with } \xi = \frac{x}{2\sqrt{t}}. \tag{10}$$

Then $T = T(x,t)$ is a solution of the heat equation $T_t(x,t) = \alpha T_{xx}(x,t)$ ($\alpha > 0$) if and only if $\eta = \eta(\xi)$ is solution of the ordinary differential equation

$$\frac{\alpha}{2}\eta''(\xi) + \xi\eta'(\xi) - \eta(\xi) = 0. \tag{11}$$

b) *The solution of the ordinary differential equation* (11) *is given by*:

$$\eta(\xi) = -c_1\left[e^{-\frac{\xi^2}{\alpha}} + \sqrt{\frac{\pi}{\alpha}}\,\xi\, \mathrm{erf}\left(\frac{\xi}{\sqrt{\alpha}}\right)\right] + c_2\xi \tag{12}$$

where c_1 and c_2 are arbitrary real constants.

Proof. a) We have:

$$T_x(x,t) = \eta'(\xi), \qquad T_{xx}(x,t) = \frac{1}{2\sqrt{\mathrm{t}}}\eta''(\xi), \qquad T_t(x,t) = \frac{1}{\sqrt{\mathrm{t}}}\big(\eta(\xi) - \xi\eta'(\xi)\big),$$

and therefore, we get the following equivalence:

$$\begin{aligned} T_t(x,t) = \alpha T_{xx}(x,t) \quad &\Leftrightarrow \quad \frac{1}{\sqrt{\mathrm{t}}}\big(\eta(\xi) - \xi\eta'(\xi)\big) = \frac{\alpha}{2\sqrt{\mathrm{t}}}\eta''(\xi) \\ &\Leftrightarrow \quad \frac{\alpha}{2}\eta''(\xi) + \xi\eta'(\xi) - \eta(\xi) = 0. \end{aligned} \tag{13}$$

b) Taking into account that $\eta(\xi) = \xi$ is a solution of the ordinary differential equation (11) we define $\eta(\xi) = \xi\mu(\xi)$. Derivating with respect to ξ, we have

$$\eta'(\xi) = \mu(\xi) + \xi\mu'(\xi), \qquad \eta''(\xi) = \xi\mu''(\xi) + 2\mu'(\xi), \tag{14}$$

and replacing (14) in the ordinary differential equation (11) we obtain

$$\frac{\alpha}{2}\xi\mu''(\xi) + \big(\alpha + \xi^2\big)\mu'(\xi) = 0. \tag{15}$$

This ordinary differential equation can be solved by the method of separation of variables, i.e., $\mu'(\xi) = \frac{c_1}{\xi^2} e^{-\frac{\xi^2}{\nu}}$, where c_1 is an arbitrary real constant. By a new integration we obtain that $\mu(\xi)$ is given by:

$$\mu(\xi) = -c_1 \left[\frac{e^{-\frac{\xi^2}{\alpha}}}{\xi} + \sqrt{\frac{\pi}{\alpha}}\, erf\left(\frac{\xi}{\sqrt{\alpha}} \right) \right] + c_2 \tag{16}$$

where c_2 is another arbitrary real constant. Multiplying (16) by ξ we obtain the solution $\eta(\xi)$ given by (12). □

Now, we can give our main result:

Theorem. *If $0 < q_1 < q_2$ then there exists a unique solution of the two-phase Stefan problem* (1)–(9) *which is given by*:

$$T_1(x,t) = -c_{11} \left[2\sqrt{t} e^{-\frac{x^2}{4t\alpha_1}} + \sqrt{\frac{\pi}{\alpha_1}} x\, erf\left(\frac{x}{2\sqrt{\alpha_1 t}} \right) \right] + c_{12} x, \quad \text{for } x > s(t),\ t > 0, \tag{17}$$

$$T_2(x,t) = -c_{21} \left[2\sqrt{t} e^{-\frac{x^2}{4t\alpha_2}} + \sqrt{\frac{\pi}{\alpha_2}} x\, erf\left(\frac{x}{2\sqrt{\alpha_2 t}} \right) \right] + c_{22} x, \quad \text{for } 0 < x < s(t),\ t > 0, \tag{18}$$

$$s(t) = 2\lambda\sqrt{\alpha_1 t}, \quad t > 0, \tag{19}$$

where the constants c_{ij} $(i, j = 1, 2)$ are given by

$$c_{11} = -\frac{q_1}{k_1} \frac{\lambda\sqrt{\alpha_1}}{(e^{-\lambda^2} - \lambda\sqrt{\pi}\, erfc(\lambda))}, \tag{20}$$

$$c_{12} = -\frac{q_1}{k_1} \left[1 + \frac{\lambda\sqrt{\pi}}{(e^{-\lambda^2} - \lambda\sqrt{\pi}\, erfc(\lambda))} \right], \tag{21}$$

$$c_{21} = -\frac{q_2}{k_2} \frac{\lambda\sqrt{\alpha_1}}{(e^{-(\alpha_{12}\lambda)^2} + \lambda\alpha_{12}\sqrt{\pi}\, erf(\lambda\alpha_{12}))}, \tag{22}$$

$$c_{22} = -\frac{q_2}{k_2} \tag{23}$$

and the dimensionless coefficient λ is the unique positive solution of the equation:

$$q_1 \frac{1}{1 - \sqrt{\pi}\lambda e^{\lambda^2} erfc(\lambda)} + 2\alpha_1\gamma\lambda^2 = q_2 \frac{1}{1 + \sqrt{\pi}(\alpha_{12}\lambda) e^{(\alpha_{12}\lambda)^2} erf(\alpha_{12}\lambda)}, \quad \lambda > 0 \tag{24}$$

with α_{12} being a dimensionless parameter defined by:

$$\alpha_{12} = \sqrt{\frac{\alpha_1}{\alpha_2}} > 0. \tag{25}$$

Proof. Setting $\eta_i(\xi) = \frac{T_i(x,t)}{2\sqrt{t}}$ for $i = 1, 2$, and using the previous Lemma 1, the heat equations (1) and (2) have the following solution:

$$\eta_1(\xi) = -c_{11} \left[e^{-\frac{\xi^2}{\alpha_1}} + \sqrt{\frac{\pi}{\alpha_1}} \xi\, erf\left(\frac{\xi}{\sqrt{\alpha_1}} \right) \right] + c_{12}\xi, \quad \text{if } \xi \in (\lambda, +\infty), \tag{26}$$

$$\eta_2(\xi) = -c_{21} \left[e^{-\frac{\xi^2}{\alpha_2}} + \sqrt{\frac{\pi}{\alpha_2}} \xi\, erf\left(\frac{\xi}{\sqrt{\alpha_2}} \right) \right] + c_{22}\xi, \quad \text{if } \xi \in (0, \lambda), \tag{27}$$

where $c_{11}, c_{12}, c_{21}, c_{22}$ are unknown constants to be determined. Conditions (3) and (4) are satisfied if and only if

$$\eta_1(\lambda\sqrt{\alpha_1}) = 0, \qquad \eta_2(\lambda\sqrt{\alpha_1}) = 0. \tag{28}$$

Condition (5) is equivalent to

$$c_{22} = -\frac{q_2}{k_2}. \tag{29}$$

As $T_{1_x}(x,t) = \eta_1'(\xi)$, by using expression (26) in the condition (8), we have:

$$\begin{aligned} \lim_{x\to+\infty} T_{1x}(x,t) &= \lim_{\xi\to+\infty} \eta_1'(\xi) = \lim_{\xi\to+\infty} \left[-c_{11}\sqrt{\frac{\pi}{\alpha_1}}\, erf\left(\frac{\xi}{\sqrt{\alpha_1}} \right) + c_{12} \right] \\ &= -c_{11}\sqrt{\frac{\pi}{\alpha_1}} + c_{12} = -\frac{q_1}{k_1}. \end{aligned} \tag{30}$$

From conditions (28)–(30) we can obtain $c_{11}, c_{12}, c_{21}, c_{22}$ as functions of λ, arriving to expressions (20)–(23).

From condition (7), we deduce that the unknown coefficient λ must satisfy Eq. (24). Taking into account that $T_{ix}(s(t), t) = \eta_i'(\lambda\sqrt{\alpha_1})$ for $i = 1, 2$, the condition (7) is satisfied if and only if:

$$k_1\eta_1'(\lambda\sqrt{\alpha_1}) - k_2\eta_2'(\lambda\sqrt{\alpha_1}) = 2\alpha_1\gamma\lambda^2$$
$$\Leftrightarrow \quad -c_{11}k_1\sqrt{\frac{\pi}{\alpha_1}}\,erf(\lambda) + c_{12}k_1 + c_{21}k_2\sqrt{\frac{\pi}{\alpha_2}}\,erf(\alpha_{12}\lambda) - c_{22}k_2 = 2\alpha_1\gamma\lambda^2. \tag{31}$$

Replacing the constants $c_{11}, c_{12}, c_{21}, c_{22}$ for the expressions given by (20)–(23) in Eq. (31), we arrive to Eq. (24) which is equivalent to the following equation:

$$R_1(\lambda) = R_2(\alpha_{12}\lambda), \quad \lambda > 0. \tag{32}$$

where the functions R_1 and R_2 are defined by:

$$R_1(x) = 2\alpha_1\gamma x^2 + q_1\frac{1}{1 - \sqrt{\pi}x\exp(x^2)\,erfc(x)}, \quad x > 0, \tag{33}$$

$$R_2(x) = q_2\frac{1}{1 + \sqrt{\pi}x\exp(x^2)\,erf(x)}, \quad x > 0 \tag{34}$$

which have the following properties:

$$R_1(0) = q_1, \qquad R_1(+\infty) = +\infty, \qquad R_1'(x) > 0, \quad \forall x > 0, \tag{35}$$

$$R_2(0) = q_2, \qquad R_2(+\infty) = 0, \qquad R_2'(x) < 0, \quad \forall x > 0. \tag{36}$$

Therefore, R_1 is a strictly increasing function and R_2 is a strictly decreasing function and by using the properties (35) and (36) we obtain that if $q_1 < q_2$ then there exists a unique solution of Eq. (32); then the explicit solution to the free boundary problem (1)–(9) is given by (17)–(19). □

Remark 1. In our problem, the heat flux boundary condition (8) is equivalent to assume that the initial temperature of the solid phase is a linear function of the spatial variable x.

The solution of the free boundary problem solved in [32] can be recovered as a particular solution given in the previous theorem as follows:

Lemma 2. *The solution of the free boundary problem* (1)–(9) *for the particular case* $q_1 = 0$ (*null heat flux condition at* $x = +\infty$ *and null initial temperature*) *and* $q_2 > 0$ *is given by*:

$$T_2(x, t) = \frac{q_2}{k_2}s(t)\left[\frac{Q_2(\lambda\alpha_{12})}{Q_2(\frac{x}{2\sqrt{\alpha_2 t}})} - \frac{x}{s(t)}\right], \quad 0 < x < s(t),\ t > 0, \tag{37}$$

$$T_1(x, t) = 0, \quad x > s(t),\ t > 0, \tag{38}$$

$$s(t) = 2\lambda\sqrt{\alpha_1 t}, \quad t > 0 \tag{39}$$

where the dimensionless coefficient $\lambda > 0$ *is the unique solution of the equation*:

$$2\gamma\alpha_1 P_1(\lambda) = q_2 P_2(\alpha_{12}\lambda), \quad \lambda > 0 \tag{40}$$

where

$$P_1(x) = x^2, \qquad P_2(x) = \frac{1}{1 + \sqrt{\pi}x\exp(x^2)\,erf(x)}, \qquad Q_2(x) = P_2(x)\exp(x^2). \tag{41}$$

Proof. It is sufficient to compute the solution (17)–(24) for the particular case $q_1 = 0$. □

Remark 2. The particular solution given by Lemma 2 is exactly the solution given in [32] by taking into account that λ, ν and $\bar{q}$ in [32] are given by $\lambda\sqrt{\alpha_1}, \alpha_2$ and q_2 respectively with $\rho c_2 = 1$ in our solution (37)–(40).

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