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The similarity method and explicit solutions for the fractional space one-phase Stefan problems

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Abstract
In this paper we obtain self-similarity solutions for a one-phase one-dimensional fractional space Stefan problem in terms of the three parametric Mittag-Leffler function $E_{\alpha,m,l}(z)$. We consider Dirichlet and Neumann conditions at the fixed face, involving Caputo fractional space derivatives of order $0 < \alpha < 1$. We recover the solution for the classical one-phase Stefan problem when the order of the Caputo derivatives approaches one.

Keywords Fractional space Stefan problems · Explicit solution · Similarity method · Caputo derivative

Mathematics Subject Classification 26A33 · 35C06 · 35R11 · 35R35 · 80A22

1 Introduction

This paper deals with a fractional space Stefan problem. More precisely, we consider a phase-change problem where the heat flux is modeled through fractional integrals, and the governed equation is a fractional diffusion equation.

Fractional diffusion equations are a wide scope which could be related to different theories, all of them, converging to the classical diffusion equation which, in a simple

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one dimensional form can be written as

$$u_t(x,t) = u_{xx}(x,t) + F(x,t), \quad x \in \Omega \subset \mathbb{R},\ t \in (0,T). \tag{1.1}$$

Regarding fractional diffusion equations for Caputo and Riemann-Liouvlle derivatives and its applications, a complete view of the state-of-the-art can be found in [18]. A rigorous mathematical analysis is presented in [14, 19] and for applications we refer the reader to [8, 15].

We will work with the following fractional diffusion equation, where a Caputo derivative on the spatial variable is involved

$$u_t(x,t) = \frac{\partial}{\partial x}\, {}_0^C D_x^\alpha u(x,t), \quad x \in \Omega \times (0,T),\ \alpha \in (0,1). \tag{1.2}$$

Recall that, for $\alpha \in (0,1)$, the fractional integral of Riemann-Liouville ${}_aI^{1-\alpha}$ of order α is defined for every summable function f as

$${}_aI^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(p)(x-p)^{\alpha-1}\mathrm{d}p, \quad 0 < \alpha < 1, \tag{1.3}$$

and the Caputo and Riemann-Liouville derivatives of order $\alpha \in (0,1)$ are defined for every $f \in AC[a,b]$ by

$${}_a^C D^\alpha f(x) = {}_aI^{1-\alpha} f'(x) \ \text{ and } \ {}_a^{RL} D^\alpha f(x) = \frac{d}{dx}\, {}_aI^{1-\alpha} f(x), \tag{1.4}$$

respectively.

We will denote by ${}_a^C D_x^\alpha$ and ${}_a^{RL} D_x^\alpha$ to the Caputo and Riemann-Liouville derivatives respect on the spatial variable.

In the whole paper we will consider $\alpha \in (0,1)$, but in general we have such operators defined for arbitrary $\alpha > 0$ (see [4, 9]). Moreover, the subscript x in fractional integral and derivatives will be omitted in the context of one variable functions.

It is worth noting that equation (1.2) is a consolidated model to anomalous diffusion [3, 16, 18] whereas it was proved in [1] that the equation

$$u_t(x,t) = {}_0^C D_x^{\alpha+1} u(x,t)$$

cannot provide a suitable model for anomalous diffusion.

The fractional Stefan problem for the one-dimensional time-fractional diffusion equation has been recently widely studied. Different models are presented in [7, 20] and [31]. A rigorous existence analysis of self-similar solutions was done in [12], and results related to explicit solutions were established in [21–23] and [25]. Moreover, an equivalent formulation for the fractional Stefan condition is given in [24].

Space-fractional Stefan problems have been proposed in [30] and the literature about this topic is currently emerging. Recently, K. Ryszewska provided in [26] the mathematical analysis of a one dimensional, one-phase free boundary problem governed by a space-fractional diffusion equation. In that article, it is proved that the problem to find a pair $\{u, s\}$ verifying that

$$\begin{array}{ll} u_t(x,t) = \frac{\partial}{\partial x}\, {}_0^C D_x^\alpha u(x,t) & 0 < x < s(t),\ 0 < t < T, \\ u_x(0,t) = 0 & 0 < t < T, \\ u(s(t),t) = 0 & 0 < t < T, \\ u(x,0) = u_0(x) & 0 < x < s(0) = b, \\ \dot{s}(t) = -\lim\limits_{x \to s(t)^-} {}_0^C D_x^\alpha u_x(x,t) & 0 < t < T \end{array} \tag{1.5}$$

has a unique solution under suitable regularity on the initial condition and the assumption that b is a positive number.

In this paper two similar problems are treated. Let $Q_{s,T}$ be the parabolic domain, defined as

$$Q_{s,T} = \{(x,t) \colon\ 0 < x < s(t),\ 0 < t < T\}.$$

We consider two instantaneous melting fractional space Stefan problems. The first one addressed with a Dirichlet condition: *Find the pair of functions* $u \colon Q_{s,T} \to \mathbb{R}$ and $s \colon [0,T] \to \mathbb{R}$ with sufficiently regularity such that

$$\begin{array}{ll} u_t(x,t) = \frac{\partial}{\partial x}\, {}_0^C D_x^\alpha u(x,t) & 0 < x < s(t),\ 0 < t < T, \\ u(0,t) = g(t) & 0 < t < T, \\ u(s(t),t) = U_m & 0 < t < T, \\ s(0) = 0, & \\ \dot{s}(t) = -\lim\limits_{x \to s(t)^-} {}_0^C D_x^\alpha u(x,t) & 0 < t < T. \end{array} \tag{1.6}$$

And the second one addressed with a Neumann condition: *Find the pair of functions* $w \colon Q_{s,T} \to \mathbb{R}$ and $r \colon [0,T] \to \mathbb{R}$ with sufficiently regularity such that

$$\begin{array}{ll} w_t(x,t) = \frac{\partial}{\partial x}\, {}_0^C D_x^\alpha w(x,t) & 0 < x < r(t),\ 0 < t < T, \\ {}_0^C D_x^\alpha w(0^+,t) = -h(t) & 0 < t < T, \\ w(r(t),t) = U_m & 0 < t < T, \\ r(0) = 0, & \\ \dot{r}(t) = -\lim\limits_{x \to r(t)^-} {}_0^C D_x^\alpha w(x,t) & 0 < t < T. \end{array} \tag{1.7}$$

Note that a Neumann condition (1.7)$-(ii)$ is different than (1.5)$-(ii)$ and it will be justified in the next section, where the model is presented.

The structure of the paper is the following: Problems (1.6) and (1.7) are derived from physical assumptions in Sect. 2. Then, some useful properties related to the special functions involved in the self-similarity solutions are presented in Sect. 3. In Sect. 4, we apply the similarity method in order to obtain a solution as a function of the three-parameter Mittag-Leffler function $E_{\alpha,m,l}(z)$ and the non-negative property of the function defined in (4.16) is proven. In Sects. 5 and 6 we obtain the unique explicit solutions for the one-phase fractional space Stefan problems (1.6) and (1.7) with a prescribed Dirichlet and fractional heat flux condition at the fixed face $x = 0$, respectively.

2 The mathematical model for instantaneous phase change

Consider an instantaneous phase change problem corresponding to the melting of a semi–infinite slab ($0 \leq x < \infty$) of a material, which is initially at the melting temperature U_m, by imposing a temperature or a heat flux condition at the fixed face $x = 0$. All the thermophysical parameters are considered to be constants.

The notation related to heat conduction with its corresponding physical dimensions are given in the next table:

$$\begin{array}{ccc} [u] & \textit{temperature} & \mathbf{T} \\ [k] & \textit{thermal conductivity} & \frac{\mathbf{m}\,\mathbf{X}}{\mathbf{T}\mathbf{t}^3} \\ [\rho] & \textit{mass density} & \frac{\mathbf{m}}{\mathbf{X}^3} \\ [c] & \textit{specific heat} & \frac{\mathbf{X}^2}{\mathbf{T}\mathbf{t}^2} \\ [d]=\left[\frac{k}{\rho c}\right] & \textit{diffusion coefficient} & \frac{\mathbf{X}^2}{\mathbf{t}} \\ [l] & \textit{latent heat per unit mass} & \frac{\mathbf{X}^2}{\mathbf{t}^2}, \end{array} \tag{2.1}$$

where $\mathbf{T}: temperature$, $\mathbf{t}: time$, $\mathbf{m}: mass$, $\mathbf{X}: position$.

Let $u = u(x,t)$ be the temperature and let $q = q(x,t)$ be the heat flux of the material at position x and time t. Let $x = s(t)$ be the function representing the (unknown) position of the free boundary (phase change interface) at time t such that $s(0) = 0$.

Suppose that, at every time t the heat flux at a position x is a generalized weighted sum of the classical fluxes occurring at every position from the initial position to the current one, where the nearest local fluxes are more relevant than the farthest. That is, we model the heat flux in the slab by the expression

$$q(x,t) = -\nu_\alpha \frac{1}{\Gamma(1-\alpha)} \int_0^x k\, u_x(p,t)(x-p)^{-\alpha} \mathrm{d}p = -\nu_\alpha k\, {}_0I_x^{1-\alpha} \frac{\partial u}{\partial x}(x,t). \tag{2.2}$$

Equation (2.2) can be expressed in terms of Caputo derivatives as follows:

$$q(x,t) = -\nu_\alpha k\, {}_0^C D_x^\alpha u(x,t). \tag{2.3}$$

Note that k is the thermal conductivity whereas the parameter ν_α has been added to preserve the consistency with respect to the units of measure in equation (2.2) such that

$$\lim_{\alpha \nearrow 1} \nu_\alpha = 1. \tag{2.4}$$

From the units of measure given in (2.1), we have that $[q] = \frac{\mathbf{m}}{\mathbf{t}^3}$. Then

$$[\nu_\alpha]\left[I_x^{1-\alpha} k u_x(x,t)\right] = [\nu_\alpha]\left[\frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{k u_x(p,t)}{(x-p)^\alpha}\mathrm{d}p\right] = [\nu_\alpha]\,\frac{\mathbf{m}}{\mathbf{t}^3}\mathbf{m}^{1-\alpha} \tag{2.5}$$

and $[\nu_\alpha] = \mathbf{m}^{\alpha-1}$.

Now, let us derive the two governing equations of the problem. From the first principle of the thermodynamics, we have that

$$\rho c \frac{\partial u}{\partial t}(x,t) = -\frac{\partial q}{\partial x}(x,t). \tag{2.6}$$

Then, by replacing (2.3) in the continuity equation (2.6), the governing equation (now with all the physical parameters) becomes

$$\rho c \frac{\partial u}{\partial t}(x,t) = \nu_\alpha k \frac{\partial}{\partial x}\, {}_0^C D_x^\alpha u(x,t) \tag{2.7}$$

which in terms of the fractional diffusivity constant, defined by

$$\lambda_\alpha = \nu_\alpha \lambda, \qquad \lambda = \frac{k}{\rho c},$$

is expressed as

$$\frac{\partial u}{\partial t}(x,t) = \lambda_\alpha \frac{\partial}{\partial x}\, {}_0^C D_x^\alpha u(x,t). \tag{2.8}$$

With respect to the interface, we are considering a sharp model where the solid phase is at constant temperature equal to U_m. Then the Rankine–Hugonoit condition gives

$$[\![\mathbf{q}]\!]_l^s = -\rho l \dot{s}(t), \tag{2.9}$$

where the double brackets represents the difference between the limits of the fluxes from the solid phase and the liquid phase and l is the latent heat of fusion by unit of mass. The fractional Stefan condition then, is obtained from (2.3) and (2.9) and it is given by

$$\rho l \dot{s}(t) = -\nu_\alpha k \lim_{x \to s(t)^-} ({}_0^C D_x^\alpha u)(x,t), \quad t \in (0,T), \tag{2.10}$$

which, for simplicity, will be written as

$$\rho l \dot{s}(t) = -\nu_\alpha k\, {}_0^C D_x^\alpha u(s(t),t), \quad t \in (0,T). \tag{2.11}$$

Then, by supposing that the melting temperature is given by $u(s(t), t) = U_m$, we can address the problem with a Dirichlet type condition

$$u(0, t) = g(t) \tag{2.12}$$

or by considering a Neumann boundary condition at $x = 0$ which, according to (2.3), is given by

$$\lim_{x \to 0^+} {}_0^C D_x^\alpha u(x, t) = -h(t), \tag{2.13}$$

where $g(t) \geq U_m$ for every t and h is a positive function conforming to the melting model considered.

Thus, the one-dimensional fractional space one-phase free-boundary problems for Dirichlet and Neumann conditions at $x = 0$ are given respectively by the following expressions:

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t} u(x, t) - \lambda_\alpha \frac{\partial}{\partial x} {}_0^C D_x^\alpha u(x, t) = 0, & 0 < x < s(t),\ 0 < t < T, \\ (ii) & u(0, t) = g(t), & 0 < t < T, \\ (iii) & u(s(t), t) = U_m, & 0 < t < T, \\ (iv) & s(0) = 0, & \\ (v) & \rho l \dot{s}(t) = -\nu_\alpha k ({}_0^C D_x^\alpha u)(s(t), t), & 0 < t < T, \end{array} \tag{2.14}$$

and

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t} w(x, t) - \lambda_\alpha \frac{\partial}{\partial x} {}_0^C D_x^\alpha w(x, t) = 0, & 0 < x < s(t),\ 0 < t < T, \\ (ii) & \lim\limits_{x \to 0^+} {}_0^C D_x^\alpha w(x, t) = -h(t), & 0 < t < T, \\ (iii) & w(s(t), t) = U_m, & 0 < t < T, \\ (iv) & s(0) = 0, & \\ (v) & \rho l \dot{s}(t) = -\nu_\alpha k ({}_0^C D_x^\alpha w)(s(t), t), & 0 < t < T. \end{array} \tag{2.15}$$

In [30] the quasi-stationary case was solved. There, it was shown that the pair

$$u(x, t) = 1 - \frac{x^\alpha}{[\Gamma(2+\alpha)]^{\frac{\alpha}{1+\alpha}}\, t^{\frac{\alpha}{1+\alpha}}}, \quad s(t) = [\Gamma(2+\alpha)]^{\frac{1}{1+\alpha}}\, t^{\frac{1}{1+\alpha}} \tag{2.16}$$

is a solution to problem

$$\begin{array}{lll} (i) & \frac{\partial}{\partial x} {}_0^C D_x^\alpha u(x, t) = 0, & 0 < x < s(t),\ 0 < t < T, \\ (ii) & u(0, t) = 1, & 0 < t < T, \\ (iii) & u(s(t), t) = 0, & 0 < t < T, \\ (iv) & s(0) = 0, & \\ (v) & \dot{s}(t) = -({}_0^C D_x^\alpha u)(s(t), t), & 0 < t < T. \end{array} \tag{2.17}$$

3 Some basics of the fractional calculus involved in this model

Proposition 1 *The following properties involving the fractional integrals and derivatives of order* $\alpha \in (0,1)$ *hold:*

1. *The fractional Riemann–Liouville derivative is a left inverse operator of the fractional Riemann–Liouville integral of the same order* $\alpha \in (0,1)$. *If* $f \in L^1(a,b)$, *then*

$$ {}_a^{RL}D^{\alpha}\, {}_aI^{\alpha} f(x) = f(x) \quad a.e.in \ (a,b). $$

2. *The fractional Riemann–Liouville integral, in general, is not a left inverse operator of the fractional Riemann–Liouville derivative.*
   In particular, if f *is such that* ${}_aI^{1-\alpha}f \in AC[0,b]$, *we have* ${}_aI^{\alpha}({}_a^{RL}D^{\alpha}f)(x) = f(x) - \dfrac{{}_aI^{1-\alpha}f(a^+)}{\Gamma(\alpha)(x-a)^{1-\alpha}}$ *for every* $x \in [a,b]$.
3. *If there exist some function* $\phi \in L^1(a,b)$ *such that* $f = {}_aI^{\alpha}\phi$, *then*

$$ {}_aI^{\alpha}\, {}_a^{RL}D^{\alpha} f(x) = f(x) \quad \forall x \in [a,b]. $$

4. *If* $f \in AC[a,b]$, *then*

$$ {}_a^{RL}D^{\alpha} f(x) = \frac{f(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha} + {}_a^{C}D^{\alpha} f(x) \quad a.e.in \ (a,b). \tag{3.1} $$

5. *For every* $f \in AC[a,b]$ *such that* ${}_aI^{1-\alpha}f' \in AC[a,b]$ *it holds that*

$$ \frac{d}{dx}\, {}_a^{C}D^{\alpha} f(x) = {}_a^{RL}D^{\alpha}(f')(x), \qquad a.e. \ in \ (a,b). $$

Proof See Chapter 2 of [4] for 1, 2 and 3, Chapter 3 of [4] for 4, and [9,Theorem 2.1] for 5. □

Proposition 2 *The following limits hold:*

1. *If we set* ${}_aI^0 = Id$, *the identity operator, then for every* $f \in L^1(a,b)$,

$$ \lim_{\alpha \searrow 0} {}_aI^{\alpha} f(x) = {}_aI^{0} f(x) = f(x), \qquad a.e. \ in \ (a,b). \tag{3.2} $$

2. *For every* $f \in AC[a,b]$,

$$ \lim_{\alpha \nearrow 1} {}_a^{RL}D^{\alpha} f(x) = f'(x), \quad and \quad \lim_{\alpha \nearrow 1} {}_a^{C}D^{\alpha} f(x) = f'(x) \qquad a.e. \ in \ (a,b). $$

Proof 1. See [27,Th. 2.7].

2. We note that $f' \in L^1(a,b)$ and ${}_a^C D^\alpha f(x) = {}_a I^{1-\alpha} f'(x)$. Then, by (3.2)

$$\lim_{\alpha\nearrow 1} {}_a^C D^\alpha f(x) = \lim_{\alpha\nearrow 1} {}_a I^{1-\alpha} f'(x) = f'(x), \qquad a.e. \text{ in } (a,b).$$

Then, using (3.1), we deduce

$$\lim_{\alpha\nearrow 1} {}_a^{RL} D^\alpha f(x) = \lim_{\alpha\nearrow 1} \left(\frac{f(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha} + {}_a^C D^\alpha f(x) \right) = f'(x),$$

where the limits hold $a.e.$ in (a,b).

□

The fractional integrals and derivatives of a power function are well known, as

$$ {}_a I^\alpha \left((x-a)^\beta\right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(x-a)^{\beta+\alpha}, \quad \text{for every} \quad \beta > -1, \tag{3.3}$$

and

$$ {}_a^{RL} D^\alpha \left((x-a)^\beta\right) = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha} & \text{if } \beta > -1, \ \beta \neq \alpha - 1, \\ 0 & \text{if } \beta > -1, \ \beta = \alpha - 1. \end{cases} \tag{3.4}$$

Fractional integrals and derivatives are evaluated also for most of the special functions related to fractional order models, considered as particular cases of the Wright generalized hypergeometric functions, see [11]. Especially, in [10] one can find Riemann-Liouville integrals and derivatives of the Mittag-Leffler type function with 3 parameters, introduced by Kilbas and Saigo, that are presented below.

Definition 1 Let $\alpha > 0, m > 0$, and l such that $\alpha(jm+l) \neq -1, -2, -3, \ldots$ $(j = 0, 1, 2, \ldots)$. The three-parametric Mittag-Leffler function $E_{\alpha,m,l}(z)$ is defined by

$$E_{\alpha,m,l}(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \text{with} \quad c_0 = 1, \quad c_n = \prod_{j=0}^{n-1} \frac{\Gamma(\alpha(jm+l)+1)}{\Gamma(\alpha(jm+l+1)+1)}, \quad n \in \mathbb{N}. \tag{3.5}$$

Remark 1 In [10,Th. 1], it is prove that for every $\alpha > 0$, $m > 0$, and l such that $\alpha(jm+l) \neq -1, -2, -3, \ldots \quad (j = 0, 1, 2, \ldots)$, the function $E_{\alpha,m,l}(z)$ is an entire function of the variable z. The proof is based on the next asymptotic behavior [5,1.18(4)]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + O\left(\frac{1}{z}\right)\right] \quad (|z| \to \infty, \ |arg(z+a)| < \pi), \tag{3.6}$$

from where

$$\frac{c_{n+1}}{c_n} = \frac{\Gamma(\alpha(nm+l)+1)}{\Gamma(\alpha(nm+l+1)+1)} \sim (\alpha m n)^{-\alpha} \to 0 \quad (n \to \infty). \tag{3.7}$$

Hence, the power series (3.5) converges in the whole complex plane, and therefore $E_{\alpha,m,l}(z)$ is an entire function.

Remark 2 In particular $E_{1,1,0}(z) = e^z$ and we recover the classical Mittag-Leffler function for $m = 1$ and $l = 0$ $E_{\alpha,1,0}(z) = E_\alpha(z)$. Also, a two parametric Mittag–Leffler function is recovered for the case $E_{\alpha,1,l}(z) = \Gamma(\alpha l + 1) E_{\alpha,\alpha l+1}(z)$. Finally, a special case of our interest is $E_{1,2,1}\left(-\frac{z^2}{2}\right) = e^{-\left(\frac{z}{2}\right)^2}$. For the theory and details on the classical Mittag-Leffler functions with 1 and 2 parameters, see [6].

We will focus on the function $\sigma_\alpha(z) = z^{\alpha-1}E_{\alpha,1+\frac{1}{\alpha},1}\left(-\frac{z^{1+\alpha}}{1+\alpha}\right)$ defined in $\mathbb{R}^+$, which will take part in the explicit solutions that will be presented in the next section.

By applying Theorem 4 in [10] to function σ_α for the particular case

$$l = 1, \quad m = 1 + \frac{1}{\alpha}, \quad a = -\frac{1}{1+\alpha},$$

it yields that

$$\left({}_0^{RL}D_x^\alpha\left[p^{\alpha-1}E_{\alpha,1+\frac{1}{\alpha},1}\left(-\frac{p^{1+\alpha}}{1+\alpha}\right)\right]\right)(x) = -\frac{1}{1+\alpha}x^\alpha E_{\alpha,1+\frac{1}{\alpha},1}\left(-\frac{x^{1+\alpha}}{1+\alpha}\right) \tag{3.8}$$

for every $x \in \mathbb{R}^+$.

Besides, the next interesting convergence statement holds.

Proposition 3 *Let* $f_\alpha(x) := \int_0^x w^{\alpha-1}E_{\alpha,1+\frac{1}{\alpha},1}\left(-\frac{w^{1+\alpha}}{1+\alpha}\right)\mathrm{d}w$, *for* $\alpha \in (0,1)$. *Then we have*

$$\lim_{\alpha\nearrow 1} f_\alpha(x) = \sqrt{\pi}\, f\left(\frac{x}{2}\right) \quad \textit{for every } x \in \mathbb{R}_0^+,$$

where $f(x) := \mathrm{erf}(x)$ *is the error function defined in* $\mathbb{R}_0^+$ *by the expression* $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}\mathrm{d}s$.

Proof Note that

$$\begin{aligned}\sigma_\alpha(x) &= x^{\alpha-1}E_{\alpha,1+\frac{1}{\alpha},1}\left(-\frac{x^{1+\alpha}}{1+\alpha}\right) = \sum_{n=0}^{\infty} c_n(-1)^n\frac{x^{(n+1)(1+\alpha)-2}}{(1+\alpha)^n}\\ &= x^{\alpha-1} - \sum_{n=0}^{\infty} c_{n+1}(-1)^n\frac{x^{n(1+\alpha)+2\alpha}}{(1+\alpha)^{n+1}}.\end{aligned} \tag{3.9}$$

We will show that the series above is uniformly absolutely convergent. In fact, let us denote

$$a_n = c_{n+1}(-1)^n \frac{M^{n(1+\alpha)+2\alpha}}{(1+\alpha)^{n+1}},$$

for arbitrary $M > 0$. Then, by (3.7) we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{c_{n+1}M^{1+\alpha}}{c_n(1+\alpha)} \to 0, \quad n \to \infty.$$

Hence, integrating term by term in the series in (3.9), the following expression to f_α holds

$$\begin{aligned} f_\alpha(x) &= \frac{x^\alpha}{\alpha} - \sum_{n=0}^{\infty} c_{n+1}(-1)^n \frac{x^{(n+1)(1+\alpha)+\alpha}}{[(n+1)(1+\alpha)+\alpha](1+\alpha)^{n+1}} \\ &= \frac{x^\alpha}{\alpha} + \sum_{n=1}^{\infty} c_n(-1)^n \frac{x^{n(1+\alpha)+\alpha}}{[n(1+\alpha)+\alpha](1+\alpha)^n} \\ &= x^\alpha \sum_{n=0}^{\infty} \frac{c_n}{[(n+1)(1+\alpha)-1]} \left(-\frac{x^{1+\alpha}}{1+\alpha}\right)^n . \end{aligned} \tag{3.10}$$

We can see that the series $\sum\limits_{n=0}^{\infty} \frac{c_n}{[(n+1)(1+\alpha)-1]} \left(-\frac{x^{1+\alpha}}{1+\alpha}\right)^n$ is uniformly absolutely convergent. Hence, taking into account that

$$\lim_{\alpha \nearrow 1} c_n = \lim_{\alpha \nearrow 1} \prod_{j=0}^{n-1} \frac{\Gamma((j+1)(1+\alpha))}{\Gamma((j+1)(1+\alpha)+\alpha)} = \frac{1}{2^n n!} \tag{3.11}$$

we have

$$\begin{aligned} \lim_{\alpha \nearrow 1} f_\alpha(x) &= x \sum_{n=0}^{\infty} \frac{1}{2^n n![2(n+1)-1]} \left(-\frac{x^2}{2}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!(2n+1)} x^{2n+1} \\ &= \sqrt{\pi}\, \mathrm{erf}\left(\frac{x}{2}\right). \end{aligned} \tag{3.12}$$

$\square$

4 The self-similar solution in terms of the Mittag-Leffler function and its properties

The aim of this section is to obtain an exact solution to problems (2.14) and (2.15). For simplicity, all the thermophysical parameters will be considered as constants equals to one.

First, we will look for a self-similar solution through the method of similarity variables [2, 17, 29]. Suppose that $u = u(x,t)$ is a solution to the space fractional diffusion equation (1.2) and let the function u_λ be defined by

$$u_\lambda(x,t) = u\left(\frac{x}{\lambda}, \frac{t}{\lambda^b}\right), \tag{4.1}$$

for $b \in \mathbb{R}$ and $\lambda > 0$.

Proposition 4 *A function $u = u(x,t)$ is a solution to (1.2) in $[0,d] \times (0,T)$ if and only if $u_\lambda = u_\lambda(x,t)$ is a solution to (1.2) in $\left[0, \frac{d}{\lambda}\right] \times \left(0, \frac{T}{\lambda^{1+\alpha}}\right)$ for $b = 1+\alpha$, for all $\lambda > 0$.*

Proof Let us define the function $u_\lambda(x,t) := u(\bar{x}, \bar{t})$ where $x = \lambda\bar{x}$ and $t = \lambda^b \bar{t}$. It is straightforward to see that

$$\frac{\partial}{\partial t} u_\lambda(x,t) = u_{\bar{t}}(\bar{x},\bar{t}) \frac{1}{\lambda^b}, \tag{4.2}$$

$$\frac{\partial}{\partial x} u_\lambda(x,t) = u_{\bar{x}}(\bar{x},\bar{t}) \frac{1}{\lambda}, \tag{4.3}$$

$${}_0^C D_x^\alpha u_\lambda(x,t) = \lambda^{-\alpha}\, {}_0^C D_{\bar{x}}^\alpha u(\bar{x},\bar{t}), \tag{4.4}$$

and

$$\frac{\partial}{\partial x}\, {}_0^C D_x^\alpha u_\lambda(x,t) = \lambda^{-\alpha-1} \frac{\partial}{\partial \bar{x}}\, {}_0^C D_{\bar{x}}^\alpha u(\bar{x},\bar{t}). \tag{4.5}$$

Then, from (4.2), (4.3) and (4.5) it follows that

$$\frac{\partial}{\partial t} u_\lambda(x,t) - \frac{\partial}{\partial x}\, {}_0^C D_x^\alpha u_\lambda(x,t) = \lambda^{-b} u_{\bar{t}}(\bar{x},\bar{t}) - \lambda^{-\alpha-1} \frac{\partial}{\partial \bar{x}}\, {}_0^C D_{\bar{x}}^\alpha u(\bar{x},\bar{t}). \tag{4.6}$$

From equality (4.6) our thesis holds. □

The scaling in the previous result indicates that the ratio $\frac{x}{t^{\frac{1}{1+\alpha}}}$ plays an important role in equation (1.2). This fact suggests us to search for a solution $u(x,t) = \theta\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right)$. Thus we define the one variable function

$$\theta(z) := u(x,t), \tag{4.7}$$

where z is the similarity variable defined as

$$z := \frac{x}{t^{\frac{1}{1+\alpha}}}. \tag{4.8}$$

Now, we apply the chain rule in order to obtain an ordinary fractional differential equation for the function $\theta = \theta(z)$. Note that

$$\frac{\partial}{\partial t} u(x,t) = -\frac{z}{(1+\alpha)t}\theta'(z). \tag{4.9}$$

Also, by making the substitution $w = \frac{p}{t^{\frac{1}{1+\alpha}}}$, it follows that

$$\begin{aligned}\frac{\partial}{\partial x}\left({}_0^C D_x^\alpha u(x,t)\right) &= \frac{\partial}{\partial x}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^z \frac{\theta'(w)}{t^{\frac{\alpha}{1+\alpha}}(z-w)^\alpha}\mathrm{d}w\right)\\ &= \frac{1}{t}\frac{\partial}{\partial z}\,{}_0^C D_z^\alpha \theta(z).\end{aligned} \tag{4.10}$$

From (4.9) and (4.10), we deduce

$$0 = \frac{\partial}{\partial t}u(x,t) - \frac{\partial}{\partial x}\,{}_0^C D_x^\alpha u(x,t) = -\frac{1}{t}\left[\frac{z}{1+\alpha}\theta'(z) + \frac{\partial}{\partial z}\,{}_0^C D_z^\alpha \theta(z)\right], \tag{4.11}$$

and then,

$$\frac{z}{1+\alpha}\theta'(z) + \frac{\partial}{\partial z}\,{}_0^C D_z^\alpha \theta(z) = 0. \tag{4.12}$$

Reciprocally, if θ is a solution to (4.12), we can go back over previous calculations and obtain that u is a solution of (1.2). More precisely:

Proposition 5 *The function u is a self-similar solution to equation (1.2) if and only if the function θ defined by (4.7) with the similarity variable (4.8) is a solution to equation (4.12).*

Now, we seek for a solution to (4.12). Making the substitution $\sigma(z) = \theta'(z)$, and using Proposition 1 (part 5), we convert (4.12) into the next equation

$$\frac{z}{1+\alpha}\sigma(z) + {}^{RL}D_z^\alpha(\sigma)(z) = 0. \tag{4.13}$$

From [9,Example 4.3] we know that a solution to (4.13) is given by

$$\sigma(z) = z^{\alpha-1}E_{\alpha,1+\frac{1}{\alpha},1}\left(-\frac{z^{1+\alpha}}{1+\alpha}\right) = \sum_{n=0}^{\infty} c_n(-1)^n \frac{z^{(n+1)(1+\alpha)-2}}{(1+\alpha)^n}, \tag{4.14}$$

where c_n is given in (3.5).

Hence,

$$\theta(z) = A + B\int_0^z w^{\alpha-1} E_{\alpha,1+\frac{1}{\alpha},1}\left(-\frac{w^{1+\alpha}}{1+\alpha}\right)\mathrm{d}w \\ = A + B\sum_{n=0}^{\infty}\frac{c_n(-1)^n}{(1+\alpha)^n}\frac{z^{(n+1)(1+\alpha)-1}}{(n+1)(1+\alpha)-1} \tag{4.15}$$

is a solution to equation (4.12) for arbitrary real constants A and B.

Remark 3 Note that the unique continuous solution to (4.13) at $z = 0$ is the null function, that is, the solution such that $\theta'(0^+) = 0$. But, adressing the problem with initial conditions in terms of fractional integrals, we obtain solutions with a singularity at $z = 0$ that verify the requirest initial condition.

Remark 4 We can say that θ is an absolutely continuous function, since $\theta(z) = \theta(0) + \int_0^z \theta'(w)\mathrm{d}w$. Therefore, by Proposition 1, $\frac{\partial}{\partial z}\,{}_0^C D_z^{\alpha}\theta(z) = {}^{RL}D_z^{\alpha}(\theta')(z)$.

Hereinafter we denote by

$$\sigma_\alpha(w) := w^{\alpha-1}E_{\alpha,1+\frac{1}{\alpha},1}\left(-\frac{w^{1+\alpha}}{1+\alpha}\right). \tag{4.16}$$

Proposition 6 *For every $A, B \in \mathbb{R}$, the function $u\colon \mathbb{R}_0^+ \times (0, T) \to \mathbb{R}$ such that*

$$u(x,t) = A + B\int_0^{x/t^{\frac{1}{1+\alpha}}} \sigma_\alpha(w)\mathrm{d}w \tag{4.17}$$

is a solution to the space-fractional diffusion equation (1.2).

Proof The proof is a direct consequence from the chain rule, property (5) of Proposition 1 and expression (3.8). □

Remark 5 It is also interesting the series approach in the aim to prove that (4.17) is a solution of (1.2). First, note that

$$u(x,t) = A + B\sum_{n=0}^{\infty}\frac{c_n(-1)^n}{(1+\alpha)^n}\frac{x^{(n+1)(1+\alpha)-1}}{[(n+1)(1+\alpha)-1]t^{\frac{(n+1)(1+\alpha)-1}{1+\alpha}}},$$

where the series in right side is absolutely convergent over compact sets in $\mathbb{R}_0^+ \times (0, T)$. Then, we can interchange ${}_0^C D_x^{\alpha}$ and partial derivatives with the series, obtaining that

$${}_0^C D_x^{\alpha}u(x,t) = B\sum_{n=0}^{\infty}\frac{c_n(-1)^n}{(1+\alpha)^n}\frac{\Gamma((n+1)(1+\alpha)-1)}{\Gamma((n+1)(1+\alpha)-\alpha)}\frac{x^{n(1+\alpha)}}{t^{\frac{(n+1)(1+\alpha)-1}{1+\alpha}}}, \tag{4.18}$$

$$\frac{\partial}{\partial x}\,{}_0^C D_x^\alpha u(x,t) = B\sum_{n=1}^{\infty}\frac{c_n(-1)^n n}{(1+\alpha)^{n-1}}\frac{\Gamma((n+1)(1+\alpha)-1)}{\Gamma((n+1)(1+\alpha)-\alpha)}\frac{x^{n(1+\alpha)-1}}{t^{\frac{(n+1)(1+\alpha)-1}{1+\alpha}}}, \tag{4.19}$$

and

$$u_t(x,t) = B\sum_{n=1}^{\infty}\frac{c_{n-1}(-1)^n}{(1+\alpha)^n}\frac{x^{n(1+\alpha)-1}}{t^{\frac{(n+1)(1+\alpha)-1}{1+\alpha}}}. \tag{4.20}$$

Then, if we denote $C_{\alpha,n} := \frac{c_{n-1}}{(1+\alpha)} - \frac{c_n n\Gamma((n+1)(1+\alpha)-1)}{\Gamma((n+1)(1+\alpha)-\alpha)}$, we have

$$\begin{aligned}
C_{\alpha,n} &= \frac{1}{(1+\alpha)}\prod_{j=1}^{n-2}\frac{\Gamma((j+1)(1+\alpha))}{\Gamma((j+1)(1+\alpha)+\alpha)} \\
&\quad -\prod_{j=1}^{n-1}\frac{\Gamma((j+1)(1+\alpha))}{\Gamma((j+1)(1+\alpha)+\alpha)}\frac{n\Gamma((n+1)(1+\alpha)-1)}{\Gamma((n+1)(1+\alpha)-\alpha)} \\
&= \frac{1}{(1+\alpha)}\prod_{j=1}^{n-2}\frac{\Gamma((j+1)(1+\alpha))}{\Gamma((j+1)(1+\alpha)+\alpha)} \\
&\quad \left[1-\frac{n(1+\alpha)\Gamma(n(1+\alpha))}{\Gamma(n(1+\alpha)+\alpha)}\frac{\Gamma((n+1)(1+\alpha)-1)}{\Gamma((n+1)(1+\alpha)-\alpha)}\right] \\
&= \frac{1}{(1+\alpha)}\prod_{j=1}^{n-2}\frac{\Gamma((j+1)(1+\alpha))}{\Gamma((j+1)(1+\alpha)+\alpha)} \\
&\quad \left[1-\frac{\Gamma(n(1+\alpha)+1)}{\Gamma(n(1+\alpha)+\alpha)}\frac{\Gamma(n(1+\alpha)+\alpha)}{\Gamma(n(1+\alpha)+1)}\right] \\
&= \frac{1}{(1+\alpha)}\prod_{j=1}^{n-2}\frac{\Gamma((j+1)(1+\alpha))}{\Gamma((j+1)(1+\alpha)+\alpha)}[1-1] = 0, \quad \forall n\in\mathbb{N}.
\end{aligned} \tag{4.21}$$

The result (4.21) holds for every $n\in\mathbb{N}$, hence function u is a solution to (1.2).

Proposition 7 *If u is the selfsimilar solution given in (4.17), then we have*

$$ {}_0^C D_x^\alpha u(x,t) = B\left(\frac{\Gamma(\alpha)}{t^{\frac{\alpha}{1+\alpha}}} - \sum_{n=1}^{\infty}\frac{c_{n-1}(-1)^{n-1}}{n(1+\alpha)^{n+1}}\frac{x^{n(1+\alpha)}}{t^{n+1-\frac{1}{1+\alpha}}}\right) \tag{4.22}$$

or equivalently,

$$-\,{}_0^C D_x^\alpha u(x,t) = -B\Gamma(\alpha)t^{-\frac{\alpha}{1+\alpha}} + \frac{Bt^{-\frac{\alpha}{1+\alpha}}}{1+\alpha}\int_0^{x/t^{\frac{1}{1+\alpha}}} w^\alpha E_{\alpha,1+\frac{1}{\alpha},1}\left(-\frac{w^{1+\alpha}}{1+\alpha}\right)\mathrm{d}w. \tag{4.23}$$

Proof Using the fact that

$$
\begin{aligned}
c_n &= c_{n-1} \frac{\Gamma(n(1+\alpha))}{\Gamma(n(1+\alpha)+\alpha)} \\
&= c_{n-1} \frac{\Gamma(n(1+\alpha)+1)}{\Gamma(n(1+\alpha)+\alpha) n(1+\alpha)} \\
&= \frac{c_{n-1}}{n(1+\alpha)} \frac{\Gamma((n+1)(1+\alpha)-\alpha)}{\Gamma((n+1)(1+\alpha)-1)}
\end{aligned}
\tag{4.24}
$$

and replacing the above expression in (4.18) we get

$$
\begin{aligned}
-{}_0^C D_x^\alpha u(x,t) &= -\frac{B\Gamma(\alpha)}{t^{\frac{\alpha}{1+\alpha}}} - B \sum_{n=1}^{\infty} \frac{c_n (-1)^n}{(1+\alpha)^n} \frac{\Gamma((n+1)(1+\alpha)-1)}{\Gamma((n+1)(1+\alpha)-\alpha)} \frac{x^{n(1+\alpha)}}{t^{\frac{(n+1)(1+\alpha)-1}{1+\alpha}}} \\
&= -B\Gamma(\alpha) t^{-\frac{\alpha}{1+\alpha}} + B \sum_{n=1}^{\infty} \frac{c_{n-1}(-1)^{n-1}}{n(1+\alpha)^{n+1}} \frac{x^{n(1+\alpha)}}{t^{n+1-\frac{1}{1+\alpha}}}.
\end{aligned}
\tag{4.25}
$$

Then

$$
\begin{aligned}
-{}_0^C D_x^\alpha u(x,t) &= -B\Gamma(\alpha) t^{-\frac{\alpha}{1+\alpha}} + B \sum_{n=1}^{\infty} \frac{c_{n-1}(-1)^{n-1}}{n(1+\alpha)^{n+1}} \frac{x^{n(1+\alpha)}}{t^{n+1-\frac{1}{1+\alpha}}} \\
&= -B\Gamma(\alpha) t^{-\frac{\alpha}{1+\alpha}} \\
&\quad + \frac{Bt^{-\frac{\alpha}{1+\alpha}}}{1+\alpha} \int_0^{x/t^{\frac{1}{1+\alpha}}} \frac{\mathrm{d}}{\mathrm{d}w} \left(\sum_{n=1}^{\infty} \frac{c_{n-1}(-1)^{n-1}}{n(1+\alpha)^n} w^{n(1+\alpha)} \right) \mathrm{d}w \\
&= -B\Gamma(\alpha) t^{-\frac{\alpha}{1+\alpha}} \\
&\quad + \frac{Bt^{-\frac{\alpha}{1+\alpha}}}{1+\alpha} \int_0^{x/t^{\frac{1}{1+\alpha}}} \sum_{n=0}^{\infty} \frac{c_n(-1)^n}{(1+\alpha)^n} w^{(n+1)(1+\alpha)-1} \mathrm{d}w \\
&= -B\Gamma(\alpha) t^{-\frac{\alpha}{1+\alpha}} \\
&\quad + \frac{Bt^{-\frac{\alpha}{1+\alpha}}}{1+\alpha} \int_0^{x/t^{\frac{1}{1+\alpha}}} w^\alpha E_{\alpha,1+\frac{1}{\alpha},1} \left(-\frac{w^{1+\alpha}}{1+\alpha} \right) \mathrm{d}w.
\end{aligned}
\tag{4.26}
$$

□

The last aim of this section is to prove that the kernel of the selfsimilar solution given in (4.16) is non-negative in $\mathbb{R}^+$ and the proof will be supported in a weak extremum principle for the space fractional diffusion equation

$$
u_t - \frac{\partial}{\partial x} {}_0^C D_x^\alpha u = f \tag{4.27}
$$

in the region

$$Q_{s,T} = \{(x,t) : 0 < x < s(t), 0 < t < T\}, \tag{4.28}$$

for a given function $s : [0, T] \to \mathbb{R}$ such that $s \in C[0, T]$, $s(0) = b > 0$ and there exists $M > 0 \,/\, 0 < \dot{s}(t) \leq M$ for every $t \in [0, T]$. We define the parabolic boundary of $Q_{s,T}$ by

$$\partial\gamma_{s,T} = \gamma_1 \cup \gamma_2 \cup \gamma_3,$$

where $\gamma_1 = \{(0,t) : 0 \leq t \leq T\}$, $\gamma_2 = \{(x,0) : 0 \leq x \leq s(0) = b\}$ and $\gamma_3 = \{(s(t),t) : 0 \leq t \leq T\}$.

The next weak extremum principle was stated in [26] and we recall it below for the benefit of the reader.

Theorem 1 *Let u be a solution to (4.27) in the region $Q_{s,T}$ defined in (4.28), such that u has the following regularity: $u \in C(\overline{Q_{s,T}})$, $u_t \in C(Q_{s,T})$ and for every $t \in (0,T)$, for every $0 < \eta < \omega < s(t)$ we have $u(\cdot,t) \in W^{2,\frac{1}{1-\beta}}(\eta,\omega) = \{f \in L^{\frac{1}{1-\beta}} : D^k f \in L^{\frac{1}{1-\beta}},\ 0 \leq |k| \leq 2\}$ for some $\beta \in (\alpha, 1]$. Let $\partial\gamma_{s,T}$ be its parabolic boundary. Then,*

1. *If $f \leq 0$, then u attains its maximum on $\partial\gamma_{s,T}$,*
2. *If $f \geq 0$, then u attains its minimum on $\partial\gamma_{s,T}$.*

In the next proposition we prove that the function σ_α which is the kernel of the selfsimilarity solution is a non-negative function in $\mathbb{R}^+$. The idea of the proof is to proceed by contradiction. We suppose first that there exists $z_0 > 0$ such that $\sigma_\alpha(z_0) < 0$ and construct a solution $u^\varepsilon(x,t) = u(x, t+\varepsilon)$, in terms of σ_α, to a moving-boundary problem, for arbitrary $\varepsilon \in (0,1)$, where u is of the form (4.17). Then, we obtain that $u^\varepsilon \geq 0$ by using Theorem 1. After that, we define the function $w_\kappa^\varepsilon = u^\varepsilon + \kappa v$, for $\kappa > 0$ a constant to be determined, such that w_κ^ε verifies the assumptions in Theorem 1 and we finally get a contradiction by choosing a suitable k.

Proposition 8 *Let $\alpha \in (0,1)$. Then the function σ_α defined in (4.16) is a non-negative function in $\mathbb{R}^+$.*

Proof We know that $\sigma_\alpha(0^+) = +\infty$ and that $\sigma_\alpha \in C^1(\mathbb{R}^+)$. Suppose that there exists $z_0 > 0$ such that $\sigma_\alpha(z_0) < 0$. Then we can affirm that there exists a "first value" $c > 0$ for which $\sigma_\alpha(c) = 0$. Also, from [10,Th. 1] we know that the complex variable Mittag-Leffler function $E_{\alpha,1+\frac{1}{\alpha},1}(z)$ is an entire function, then it has isolated roots and we can choose a sufficiently small $\delta > 0$ such that $\sigma_\alpha(z) \geq 0$ for $z \in (0, c]$, $\sigma_\alpha(z) < 0$ for $z \in (c, c+\delta]$ and

$$\int_0^{c+\delta} \sigma_\alpha(w)\mathrm{d}w > 0. \tag{4.29}$$

Now, let $0 < \varepsilon < 1$ be, and consider the functions $s^{\varepsilon,\delta}$ and u^ε defined by

$$s^{\varepsilon,\delta}(t) = (c+\delta)(t+\varepsilon)^{\frac{1}{1+\alpha}}, \quad t \geq 0, \quad \text{for } t \in (0,T) \tag{4.30}$$

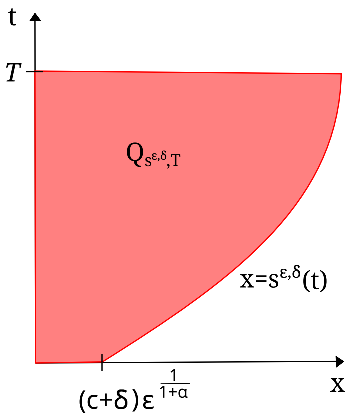

Fig. 1 Region $Q_{s^{\varepsilon,\delta},T}$ and its parabolic boundary

and

$$u^{\varepsilon}(x,t)=u(x,t+\varepsilon)=\frac{\displaystyle\int_0^{x/(t+\varepsilon)^{1/(1+\alpha)}}\sigma_\alpha(w)\mathrm{d}w}{\displaystyle\int_0^{c+\delta}\sigma_\alpha(w)\mathrm{d}w}\quad\text{for } 0<x<s^{\varepsilon,\delta}(t),\, 0<t<T,$$

where u is the function defined in (4.17) for $A=0$ and $B=\dfrac{1}{\displaystyle\int_0^{c+\delta}\sigma_\alpha(w)\mathrm{d}w}$.

Then, if we define the region $Q_{s^{\varepsilon,\delta},T}=\{(x,t):0<x<s^{\varepsilon,\delta}(t),0<t<T\}$, and its parabolic boundary $\partial\gamma_{s^{\varepsilon,\delta},T}=\gamma_1\cup\gamma_2\cup\gamma_3$, where $\gamma_1=\{(0,t):0\le t\le T\}$, $\gamma_2=\{(x,0):0\le x\le s^{\varepsilon,\delta}(0)=(c+\delta)\varepsilon^{\frac{1}{1+\alpha}}\}$ and $\gamma_3=\{(s^{\varepsilon,\delta}(t),t):0\le t\le T\}$ (see Figure 1), it results that u^{ε} is a solution to the moving-boundary problem

$$\begin{array}{lll}
(i) & u_t-\frac{\partial}{\partial x}\,{}_0^C D_x^\alpha u(x,t)=0, & 0<x<s^{\varepsilon,\delta}(t),\ 0<t<T,\\
(ii) & u(0,t)=0, & 0<t<T,\\
(iii) & u(s^{\varepsilon,\delta}(t),t)=1, & 0<t<T,\\
(iv) & u(x,0)=\dfrac{\displaystyle\int_0^{x/\varepsilon^{1/(1+\alpha)}}\sigma_\alpha(w)\mathrm{d}w}{\displaystyle\int_0^{c+\delta}\sigma_\alpha(w)\mathrm{d}w}\ge 0, & 0\le x\le s^{\varepsilon,\delta}(0)=(c+\delta)\varepsilon^{\frac{1}{1+\alpha}}>0.
\end{array}\tag{4.31}$$

Moreover,

$$
\begin{aligned}
u^{\varepsilon}(x,t) &= \frac{1}{\displaystyle\int_0^{c+\delta}\sigma_\alpha(w)\mathrm{d}w}\sum_{n=0}^{\infty}\frac{c_n(-1)^n}{(1+\alpha)^n[(n+1)(1+\alpha)-1]}\frac{x^{(n+1)(1+\alpha)-1}}{(t+\varepsilon)^{n+1-\frac{1}{1+\alpha}}} \\
&= \frac{\left(\frac{x}{(t+\varepsilon)^{\frac{1}{1+\alpha}}}\right)^{\alpha}}{\displaystyle\int_0^{c+\delta}\sigma_\alpha(w)\mathrm{d}w}\sum_{n=0}^{\infty}\frac{c_n(-1)^n}{(1+\alpha)^n[(n+1)(1+\alpha)-1]}\left(\frac{x^{1+\alpha}}{t+\varepsilon}\right)^n .
\end{aligned}
\tag{4.32}
$$

The power series in (4.32) is absolutely convergent on $\overline{Q_{s^{\varepsilon,\delta},T}}$. Then, $u^{\varepsilon} \in C(\overline{Q_{s^{\varepsilon,\delta},T}})$ and $u^{\varepsilon} \in C^{\infty}(Q_{s^{\varepsilon,\delta},T})$. In particular, $u^{\varepsilon} \in C^2(Q_{s^{\varepsilon,\delta},T})$, and for every $0<\delta<\omega<s^{\varepsilon,\delta}(t)$ and an arbitrary $\beta\in(\alpha,1)$, we have $u^{\varepsilon}(\cdot,t)\in W^{2,\frac{1}{1-\beta}}(\delta,\omega)$.

Thus, by Theorem 1 (or [26,Lemma 6]), it results that u^{ε} attains its minimum and its maximum at the parabolic boundary $\partial\gamma_{s^{\varepsilon,\delta},T}$. Hence it easily straightforward that $u^{\varepsilon}(x,t)\geq 0$ for all $(x,t)\in Q_{s^{\varepsilon,\delta},T}$.

Next, we analyze the behavior of u^{ε} at the parabolic boundary $\partial\gamma_{s^{\varepsilon,\delta},T}$. Let $f_t(x) := u^{\varepsilon}(x,t)$. Thus,

$$
f_t'(x) = \frac{\sigma_\alpha\left(\frac{x}{(t+\varepsilon)^{1/(1+\alpha)}}\right)}{(t+\varepsilon)^{1/(1+\alpha)}\displaystyle\int_0^{c+\delta}\sigma_\alpha(w)\mathrm{d}w},
$$

being $f_t'(x)\geq 0$ in $(0,c(t+\varepsilon)^{1/(1+\alpha)})$ and $f_t'(x)<0$ in $(c(t+\varepsilon)^{1/(1+\alpha)},(c+\delta)(t+\varepsilon)^{1/(1+\alpha)})$. We conclude then that f_t attains its maximum over $[0,(c+\delta)(t+\varepsilon)^{1/(1+\alpha)}]=[0,s^{\varepsilon,\delta}(t)]$ at the point $x=c(t+\varepsilon)^{1/(1+\alpha)}$. Moreover, $f_t(c(t+\varepsilon)^{1/(1+\alpha)})>f_t(s^{\varepsilon,\delta}(t))$.

Besides, if we denote by $\xi_t=c(t+\varepsilon)^{1/1+\alpha}$, for every $t\in[0,T]$, we can state that

$$
f_t(\xi_t)=u^{\varepsilon}(\xi_t,t)=\frac{\displaystyle\int_0^{c}\sigma_\alpha(w)\mathrm{d}w}{\displaystyle\int_0^{c+\delta}\sigma_\alpha(w)\mathrm{d}w}=u^{\varepsilon}(\xi_{t'},t')=f_{t'}(\xi_{t'}),\quad \forall t,t'\in[0,T],
$$

and then, u^{ε} attains its maximum on $\overline{Q_{s^{\varepsilon,\delta},T}}$ at every point (ξ_t,t), for all $t\in[0,T]$. In particular, u^{ε} attains its maximum at $(\xi_0,0)\in\partial\gamma_{s^{\varepsilon,\delta},T}$. Denote by $A:=u^{\varepsilon}(\xi_t,t)$, for every $t\in[0,T]$. Note that $A=u^{\varepsilon}(\xi_t,t)>u^{\varepsilon}((s^{\varepsilon,\delta}(t),t)=1,\forall t\leq T$.

Let us consider now the function $v(x,t)=\frac{x^{1+\alpha}}{1+\alpha}+\Gamma(1+\alpha)t$ and define

$$
w_\kappa^{\varepsilon}(x,t)=u^{\varepsilon}(x,t)+\kappa v(x,t), \tag{4.33}
$$

where the constant κ will be specified latter. Observe that v and w_κ^ε are both solutions to (4.31)$-(i)$, for every $\kappa > 0$, and w_κ^ε verifies the hypothesis of Theorem 1. Then,

$$\max_{Q_{s^{\varepsilon,\delta},T}} w_\kappa^\varepsilon = \max_{\partial\gamma_{s^{\varepsilon,\delta},T}} w_\kappa^\varepsilon. \tag{4.34}$$

Finally, let us make some computations in order to evaluate w_κ^ε at the parabolic boundary:

$$\begin{aligned}\max_{x\in[0,s^{\varepsilon,\delta}(0)]} w_\kappa^\varepsilon(x,0) &\leq \max_{x\in[0,s^{\varepsilon,\delta}(0)]} u^\varepsilon(x,0) + \kappa \max_{x\in[0,s^{\varepsilon,\delta}(0)]} v(x,0)\\ &= u^\varepsilon(\xi_0,0) + \kappa v(s^{\varepsilon,\delta}(0),0)\\ &= A + \kappa\frac{(c+\delta)^{1+\alpha}\varepsilon}{1+\alpha}.\end{aligned} \tag{4.35}$$

Also, we have that

$$\begin{aligned}\max_{x\in[0,s^{\varepsilon,\delta}(t)]} w_\kappa^\varepsilon(x,t) &\geq w_\kappa^\varepsilon(\xi_t,t)\\ &= A + \kappa\left[\frac{c^{1+\alpha}(t+\varepsilon)}{1+\alpha} + \Gamma(1+\alpha)t\right].\end{aligned} \tag{4.36}$$

Then, taking $t_0 > \varepsilon\frac{(c+\delta)^{1+\alpha}-c^{1+\alpha}}{c^{1+\alpha}+\Gamma(2+\alpha)} > 0$, it holds that

$$\frac{c^{1+\alpha}(t_0+\varepsilon)}{1+\alpha} + \Gamma(1+\alpha)t_0 > \frac{(c+\delta)^{1+\alpha}\varepsilon}{1+\alpha},$$

and therefore, taking into account (4.35) and (4.36) we get:

$$\max_{x\in[0,s^{\varepsilon,\delta}(t_0)]} w_\kappa^\varepsilon(x,t_0) > \max_{x\in[0,s^{\varepsilon,\delta}(0)]} w_\kappa^\varepsilon(x,0),$$

and we conclude that w_γ^ε does not attains its maximum at γ_2.

On the other hand, $v(\cdot,t)$ is a strictly increasing function for every fixed t. Then, w_κ^ε does not attains its maximum at γ_1.

Finally, asking κ to verify that $\kappa < \frac{(A-1)(\alpha+1)}{[(c+\delta)^{1+\alpha}-c^{1+\alpha}](T+\varepsilon)}$, we can affirm that

$$\begin{aligned}w_\kappa^\varepsilon(s^{\varepsilon,\delta}(t),t) &= 1 + \kappa\left[\frac{(c+\delta)^{1+\alpha}(t+\varepsilon)}{1+\alpha} + \Gamma(1+\alpha)t\right]\\ &< A + \kappa\left[\frac{c^{1+\alpha}(t+\varepsilon)}{1+\alpha} + \Gamma(1+\alpha)t\right] = w_\kappa^\varepsilon(\xi_t,t), \quad \forall t \leq T,\end{aligned} \tag{4.37}$$

from where we claim that w_κ^ε does not attains its maximum at γ_3.

Therefore, w_κ^ε does not attains its maximum at the parabolic boundary $\partial\gamma_{s^{\varepsilon,\delta},T}$, which contradicts the equality (4.34).

This contradiction comes from assuming that there exists $z_0 > 0$ such that $\sigma_\alpha(z_0) < 0$. Thus,

$$\sigma_\alpha(z) \geq 0, \quad \forall z > 0, \tag{4.38}$$

and our thesis holds. □

Corollary 1 *The three parametric Mittag-Leffler function involved in the kernel of the self-similar solution (4.17) verifies that*

$$E_{\alpha,1+\frac{1}{\alpha},1}\left(-\frac{x^{1+\alpha}}{1+\alpha}\right) \geq 0 \;\; \textit{for all} \;\; x > 0. \tag{4.39}$$

5 Explicit solution for the fractional space one-phase Stefan problem with a Dirichlet condition at the fixed face

Let us return to problem (2.14) for a constant Dirichlet boundary data $g \equiv U_0$ and melting temperature U_m such that $U_0 > U_m$, given by the following free boundary problem:

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t}u(x,t) - \frac{\partial}{\partial x}\,{}_0^C D_x^\alpha u(x,t) = 0, & 0 < x < s(t), 0 < t < T, \\ (ii) & u(0,t) = U_0 > U_m, & 0 < t < T, \\ (iii) & u(s(t),t) = U_m, & 0 < t < T, \\ (iv) & s(0) = 0, & \\ (v) & \dot{s}(t) = -({}_0^C D_x^\alpha u)(s(t),t), & 0 < t < T. \end{array} \tag{5.1}$$

Let u be defined by (4.17). From (5.1)$-(ii)$, we deduce that $A = U_0$. Now, from condition (5.1)$-(iii)$, we have

$$u(s(t),t) = U_0 + B\int_0^{s(t)/t^{1/(1+\alpha)}} \sigma_\alpha(w)\mathrm{d}w = U_m. \tag{5.2}$$

Note that (5.2) must be verified for all $t \in (0,T)$, then the free boundary s must be proportional to $t^{1/1+\alpha}$, that is to say

$$s(t) = \xi t^{\frac{1}{1+\alpha}}, \quad \text{for some} \;\; \xi \in \mathbb{R}^+, \;\; t \in (0,T), \tag{5.3}$$

which satisfies (5.1)$-(iv)$. Replacing (5.3) in (5.2) yields that

$$B = \frac{-(U_0 - U_m)}{\int_0^\xi \sigma_\alpha(w)\mathrm{d}w}, \tag{5.4}$$

where we have used inequality (4.38), the fact that σ_α is positive in a neighborhood of 0 and that $\xi > 0$.

Replacing (4.23) on (5.1)(v), and deriving (5.3), we have

$$\begin{aligned}\frac{\xi}{1+\alpha}t^{-\frac{\alpha}{1+\alpha}} &= -B\Gamma(\alpha)t^{-\frac{\alpha}{1+\alpha}} + B\frac{t^{-\frac{\alpha}{1+\alpha}}}{1+\alpha}\int_0^{\xi} w\sigma_\alpha(w)\mathrm{d}w \\ &= \frac{Bt^{-\frac{\alpha}{1+\alpha}}}{1+\alpha}\left(-\Gamma(\alpha)(1+\alpha)+\int_0^{\xi} w\sigma_\alpha(w)\mathrm{d}w\right).\end{aligned} \tag{5.5}$$

Then, combining (5.4) and (5.5), we have the following condition

$$\xi = \frac{(U_0-U_m)\left(\Gamma(\alpha)(1+\alpha)-\displaystyle\int_0^{\xi} w\sigma_\alpha(w)\mathrm{d}w\right)}{\displaystyle\int_0^{\xi}\sigma_\alpha(w)\mathrm{d}w}. \tag{5.6}$$

Therefore, we seek for a positive number ξ which verifies the following equation

$$x = H(x), \quad x > 0, \tag{5.7}$$

where the function $H\colon \mathbb{R}_0^+ \to \mathbb{R}$ is defined by the expression:

$$H(x) = \frac{(U_0-U_m)\left(\Gamma(\alpha)(1+\alpha)-\displaystyle\int_0^{x} w\sigma_\alpha(w)\mathrm{d}w\right)}{\displaystyle\int_0^{x}\sigma_\alpha(w)\mathrm{d}w}. \tag{5.8}$$

Or equivalently, we seek for a positive root of $p(x) = x - H(x)$. Observe that

$$\lim_{x\searrow 0}\int_0^{x} w\sigma_\alpha(w)\mathrm{d}w = 0^+ \quad \text{and} \quad \lim_{x\searrow 0}\int_0^{x}\sigma_\alpha(w)\mathrm{d}w = 0^+.$$

Then $p(0^+) = -\infty$ because $(U_0-U_m)\Gamma(\alpha)(1+\alpha) > 0$. Moreover, by Proposition 8,

$$H'(x) = \frac{-(U_0-U_m)\sigma_\alpha(x)\left[\Gamma(\alpha)(1+\alpha)+\displaystyle\int_0^{x}(x-w)\sigma_\alpha(w)\mathrm{d}w\right]}{\left(\displaystyle\int_0^{x}\sigma_\alpha(w)\mathrm{d}w\right)^2} \le 0, \tag{5.9}$$

from where $p'(x) \ge 1$. Thus, we can affirm that there exists a unique $\xi > 0$ such that $p(\xi) = 0$.

From the preceding analysis, the next theorem follows.

Theorem 2 *An explicit solution for the fractional space one-phase Stefan problem* (5.1) *is given by*

$$u_\alpha(x,t) = U_0 - \frac{(U_0 - U_m)}{\displaystyle\int_0^{\xi_\alpha} \sigma_\alpha(w)\mathrm{d}w} \int_0^{x/t^{1/(1+\alpha)}} \sigma_\alpha(w)\mathrm{d}w, \tag{5.10}$$

$$s_\alpha(t) = \xi_\alpha t^{\frac{1}{1+\alpha}}, \quad t \in (0,T), \tag{5.11}$$

where $\xi_\alpha \in \mathbb{R}^+$ *is the unique solution to the equation*

$$H_\alpha(x) = x, \quad x > 0, \tag{5.12}$$

and the function H_α *is defined by* (5.8).

Remark 6 If we take $\alpha = 1$ in (5.1), we recover the classical Lamé-Clapeyron-Stefan problem

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = 0, & 0 < x < s(t),\ 0 < t < T, \\ (ii) & u(0,t) = U_0 > U_m, & 0 < t < T, \\ (iii) & u(s(t),t) = U_m, & 0 < t < T, \\ (iv) & s(0) = 0, & \\ (v) & \dot{s}(t) = -\frac{\partial}{\partial x}u(s(t),t), & 0 < t < T. \end{array} \tag{5.13}$$

By Remark 2 we know that

$$\sigma_1(w) = w^0 E_{1,2,1}\left(-\frac{w^2}{2}\right) = e^{-(\frac{w}{2})^2}. \tag{5.14}$$

Then, the pair

$$\begin{aligned} u_1(x,t) &= U_0 - \frac{(U_0 - U_m)}{\displaystyle\int_0^{\xi_1} \sigma_1(w)\mathrm{d}w} \int_0^{x/t^{1/2}} \sigma_1(w)\mathrm{d}w \\ &= U_0 - \frac{(U_0 - U_m)}{erf\left(\frac{\xi_1}{2}\right)} erf\left(\frac{x}{2\sqrt{t}}\right), \end{aligned} \tag{5.15}$$

$$s_1(t) = \xi_1 t^{\frac{1}{2}}, \quad t \in (0,T), \tag{5.16}$$

is a solution to (5.13) where $\xi_1 \in \mathbb{R}^+$ is the unique solution to the equation

$$H_1(x) = x, \quad x > 0, \tag{5.17}$$

with

$$H_1(x) = \frac{(U_0 - U_m)\left(2 - \int_0^x we^{-w^2/4}\mathrm{d}w\right)}{2\frac{\sqrt{\pi}}{2}erf\left(\frac{x}{2}\right)} = \frac{U_0 - U_m}{\sqrt{\pi}}\frac{2e^{-\left(\frac{x}{2}\right)^2}}{erf\left(\frac{x}{2}\right)}.$$

That is, we have recovered the classical Lamé-Clapeyron-Stefan solution to problem (5.13) given in [13].

6 Explicit solution for the fractional space one-phase Stefan problem with a Neumann condition at the fixed face

Now, we consider the problem (2.15) for a heat flux boundary data given by $h(t) = g_0 t^{-\frac{\alpha}{1+\alpha}}$ and melting temperature g_m such that $g_0 > g_m$,

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t}v(x,t) - \frac{\partial}{\partial x}\, {}_0^C D_x^\alpha v(x,t) = 0, & 0 < x < s(t),\ 0 < t < T, \\ (ii) & \lim\limits_{x\to 0^+} {}_0^C D_x^\alpha v(x,t) = -g_0 t^{-\frac{\alpha}{1+\alpha}}, & 0 < t < T, \\ (iii) & v(s(t),t) = g_m, & 0 < t < T, \\ (iv) & s(0) = 0, & \\ (v) & \dot{s}(t) = -({}_0^C D_x^\alpha v)(s(t),t), & 0 < t < T. \end{array} \tag{6.1}$$

Let v be defined by (4.17). From (6.1)$-(ii)$ and (4.22), we deduce that $B = -\frac{g_0}{\Gamma(\alpha)}$, because

$$\lim_{x\to 0^+} {}_0^C D_x^\alpha v(x,t) = \lim_{x\to 0^+} B\left(\frac{\Gamma(\alpha)}{t^{\frac{\alpha}{1+\alpha}}} - \sum_{n=1}^{\infty}\frac{c_{n-1}(-1)^{n-1}}{n(1+\alpha)^{n+1}}\frac{x^{n(1+\alpha)}}{t^{n+1-\frac{1}{1+\alpha}}}\right) = \frac{B\Gamma(\alpha)}{t^{\frac{\alpha}{1+\alpha}}}. \tag{6.2}$$

From condition (6.1)$-(iii)$, we have

$$v(s(t),t) = A - \frac{g_0}{\Gamma(\alpha)}\int_0^{s(t)/t^{1/(1+\alpha)}} \sigma_\alpha(w)\mathrm{d}w = g_m, \tag{6.3}$$

and from where, we will ask again the free boundary s to be proportional to $t^{1/1+\alpha}$,

$$s(t) = \eta t^{\frac{1}{1+\alpha}}, \quad \text{for some} \quad \eta \in \mathbb{R}, \quad t \in (0,T). \tag{6.4}$$

Then

$$A = g_m + \frac{g_0}{\Gamma(\alpha)}\int_0^{\eta} \sigma_\alpha(w)\mathrm{d}w. \tag{6.5}$$

From conditions (6.1)(v), (4.23) and (6.4), we have

$$\begin{aligned}\frac{\eta}{1+\alpha}t^{-\frac{\alpha}{1+\alpha}} &= g_0 t^{-\frac{\alpha}{1+\alpha}} - \frac{g_0}{\Gamma(\alpha)}\frac{t^{-\frac{\alpha}{1+\alpha}}}{1+\alpha}\int_0^{\eta} w\sigma_\alpha(w)\mathrm{d}w \\ &= g_0 t^{-\frac{\alpha}{1+\alpha}}\left(1-\frac{1}{(1+\alpha)\Gamma(\alpha)}\int_0^{\eta} w\sigma_\alpha(w)\mathrm{d}w\right),\end{aligned} \tag{6.6}$$

or equivalently,

$$\eta = g_0\left((1+\alpha)-\frac{1}{\Gamma(\alpha)}\int_0^{\eta} w\sigma_\alpha(w)\mathrm{d}w\right). \tag{6.7}$$

Therefore, η should verify the following equation

$$x = G(x), \quad x > 0, \tag{6.8}$$

where the function G is defined in $\mathbb{R}_0^+$ by the expression

$$G(x) = g_0\left((1+\alpha)-\frac{1}{\Gamma(\alpha)}\int_0^{x} w\sigma_\alpha(w)\mathrm{d}w\right). \tag{6.9}$$

Observe that G is continuous in $[0, +\infty)$. From Proposition 8, it easily follows that G is an decreasing function. Moreover,

$$G(0) = g_0(1+\alpha) > 0.$$

From the preceding analysis, we conclude that there exists a unique $\eta \in \mathbb{R}^+$ such that $\eta = G(\eta)$, and the next theorem follows.

Theorem 3 *An explicit solution for the space-fractional Stefan problem* (6.1) *is given by*

$$\begin{aligned}v_\alpha(x,t) &= g_m + \frac{g_0}{\Gamma(\alpha)}\int_0^{\eta_\alpha} w^{\alpha-1}E_{\alpha,1+\frac{1}{\alpha},1}\left(-\frac{w^{1+\alpha}}{1+\alpha}\right)\mathrm{d}w \\ &\quad -\frac{g_0}{\Gamma(\alpha)}\int_0^{x/t^{1/(1+\alpha)}} w^{\alpha-1}E_{\alpha,1+\frac{1}{\alpha},1}\left(-\frac{w^{1+\alpha}}{1+\alpha}\right)\mathrm{d}w \\ &= g_m + \frac{g_0}{\Gamma(\alpha)}\int_{x/t^{1/(1+\alpha)}}^{\eta_\alpha} w^{\alpha-1}E_{\alpha,1+\frac{1}{\alpha},1}\left(-\frac{w^{1+\alpha}}{1+\alpha}\right)\mathrm{d}w,\end{aligned} \tag{6.10}$$

$$s_\alpha(t) = \eta_\alpha t^{\frac{1}{1+\alpha}}, \quad t \in (0,T), \tag{6.11}$$

where $\eta_\alpha \in \mathbb{R}^+$ *is the unique solution to the equation*

$$G_\alpha(x) = x, \quad x > 0,$$

and the function G_α *is defined by* (6.9)

Remark 7 For $\alpha = 1$ in (6.1), we have

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t} v(x,t) - \frac{\partial^2}{\partial x^2} v(x,t) = 0, & 0 < x < s(t),\ 0 < t < T, \\ (ii) & \lim\limits_{x \to 0^+} \frac{\partial}{\partial x} v(x,t) = -g_0 t^{-\frac{1}{2}}, & 0 < t < T, \\ (iii) & v(s(t),t) = g_m, & 0 < t < T, \\ (iv) & s(0) = 0, & \\ (v) & \dot{s}(t) = -\frac{\partial}{\partial x} v(s(t),t), & 0 < t < T, \end{array} \tag{6.12}$$

and

$$\sigma_1(w) = w^0 E_{1,2,1}\left(-\frac{w^2}{2}\right) = e^{-\left(\frac{w}{2}\right)^2}. \tag{6.13}$$

Then, the pair

$$\begin{aligned} v_1(x,t) &= g_m + g_0 \int_{x/t^{\frac{1}{2}}}^{\eta_1} w^0 E_{1,2,1}\left(-\frac{w^2}{2}\right) \mathrm{d}w \\ &= g_m + g_0\sqrt{\pi}\left[erf\left(\frac{\eta_1}{2}\right) - erf\left(\frac{x}{2\sqrt{t}}\right)\right], \end{aligned} \tag{6.14}$$

$$s_1(t) = \eta_1 t^{\frac{1}{2}}, \quad t \in (0,T) \tag{6.15}$$

is a solution to the classical Lamé-Clapeyron-Stefan problem (6.12), where $\eta_1 \in \mathbb{R}^+$ is the unique solution to the equation

$$G_1(x) = x, \quad x > 0, \tag{6.16}$$

with

$$G_1(x) = g_0\left(2 - \int_0^x w\sigma_1(w)\mathrm{d}w\right) = 2g_0 e^{-\left(\frac{x}{2}\right)^2},$$

as stated in [28, 29].

Remark 8 For the solution (6.10) to the space-fractional Stefan problem (6.1) a classical Neuman condition of the form $v_x(0^+,t) = g(t)$ cannot be considered. In fact, observe that

$$v_x(x,t) = -g_0 \frac{x^{\alpha-1}}{t^{\frac{\alpha}{1+\alpha}}} \sum_{k=0}^{\infty} c_k \left(\frac{-x^{1+\alpha}}{(1+\alpha)t}\right)^k, \tag{6.17}$$

and $c_k < 2$ for all k. Then, the series in the right hand of (6.17) is convergent for $x < 1$. Moreover, for $x = 0$, the series is equal to 1. Hence, since $\alpha - 1 < 0$, we conclude that

$$\lim_{x \to 0^+} v_x(x,t) = -\infty.$$

7 Conclusions

We obtained exact self-similarity solutions for a one-phase one-dimensional fractional space Stefan problem in terms of the three parametric Mittag-Leffler function $E_{\alpha,m,l}(z)$. We considered Dirichlet and Neumann boundary conditions at the fixed face, involving Caputo fractional space derivatives of order $0 < \alpha < 1$. In both cases, the free boundary term is proportional to $t^{\frac{1}{1+\alpha}}$. Finally, we recover the solution for the classical one-phase Stefan problem when the order of the Caputo derivatives approaches one.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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