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Explicit solution for a two-phase fractional Stefan problem with a heat flux condition at the fixed face

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Abstract A generalized Neumann solution for the two-phase fractional Lamé–Clapeyron–Stefan problem for a semi-infinite material with constant initial temperature and a particular heat flux condition at the fixed face is obtained, when a restriction on data is satisfied. The fractional derivative in the Caputo sense of order $\alpha \in (0, 1)$ respect on the temporal variable is considered in two governing heat equations and in one of the conditions for the free boundary. Furthermore, we find a relationship between this fractional free boundary problem and another one with a constant temperature condition at the fixed face and based on that fact, we obtain an inequality for the coefficient which characterizes the fractional phase–change interface obtained in Roscani and Tarzia (Adv Math Sci Appl 24(2):237–249, 2014). We also recover the restriction on data and the classical Neumann solution, through the error function, for the classical two-phase Lamé–Clapeyron–Stefan problem for the case $\alpha = 1$.

Keywords Caputo fractional derivative · Lamé–Clapeyron–Stefan problem · Neumann solutions · Heat flux boundary condition · Temperature boundary condition

Mathematics Subject Classification 35R11 · 26A33 · 35C05 · 35R35 · 80A22

1 Introduction

In the past decades the fractional diffusion equation has been extensively studied (Eidelman et al. 2004; Luchko et al. 2001; Mainardi 2010; Povstenko 2015; Pskhu 2005, 2009; Sakamoto

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and Yamamoto 2011) and in the recent years some works on fractional free boundary problems (that is, free boundary problems where a fractional derivative is involved) were published (Atkinson 2012; Błasik and Klimek 2015; Ceretani and Tarzia 2017; Junyi and Mingyu 2009; Rajeev and Singh 2017; Roscani and Santillan 2013; Roscani and Tarzia 2014; Tarzia 2015; Voller 2014; Voller et al. 2013). In particular, in Kholpanov et al. (2003), the classical Lamé–Clapeyron–Stefan problem was studied by using the fractional derivative of order $\frac{1}{2}$.

Recall that free boundary problems for the one-dimensional classical heat equation are problems linked to the processes of melting and freezing, which have a latent heat condition at the solid–liquid interface connecting the velocity of the free boundary and the heat fluxes of the two temperatures corresponding to the solid and liquid phases. This kind of problems are known in the literature as Stefan problems or, more precisely, as Lamé–Clapeyron–Stefan problems. We remark that the first work on phase-change problems was done by Lamé and Clapeyron (1831) by studying the solidification of the Earth planet, which has been missing in the scientific literature for more than a century. Next, 60 years later, the phase-change problem was continued by Stefan through several works around year 1890 (Stefan 1889) by studying the melting of the polar ice. For this reason, we call these kind of problems as Lamé–Clapeyron–Stefan problems or simply by Stefan problems.

Nowadays, there exist thousands of papers on the classical Lamé–Clapeyron–Stefan problem, for example the books (Alexiades and Solomon 1993; Cannon 1984; Crank 1984; Elliott and Ockendon 1982; Fasano 2005; Gupta 2003; Lunardini 1991; Rubinstein 1971) and the large bibliography given in Tarzia (2000). Especially, a review on explicit solutions with moving boundaries was given in Tarzia (2011).

In this paper, a generalized Neumann solution for the two-phase fractional Lamé–Clapeyron–Stefan problem for a semi-infinite domain is obtained when a constant initial data and a Neumann boundary condition at the fixed face are considered. Recently, a generalized Neumann solution for the two-phase fractional Lamé–Clapeyron–Stefan problem for a semi-infinite domain with constant initial data and a Dirichlet condition at the fixed face was given in Roscani and Tarzia (2014).

So, the classical time derivative will be replaced by a fractional derivative in the sense of Caputo of order $0 < \alpha < 1$, which is present in the two governing heat equations and in one of the governing conditions for the free boundary. The fractional Caputo derivative is defined in Caputo (1967) as:

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-\tau)^{-\alpha} f'(\tau)\, d\tau, & 0 < \alpha < 1 \\ f'(t), & \alpha = 1 \end{cases} \tag{1}$$

where Γ is the Gamma function defined in $\mathbb{R}^+$ by the following expression:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

It is known that the fractional Caputo derivative verifies that (Kilbas et al. 2006): for every $b \in \mathbb{R}^+$,

$$D^\alpha \text{ is a linear operator in } W^1(0,b) = \{f \in \mathscr{C}^1(0,b] : f' \in L^1(0,b)\}, \tag{2}$$

$$D^\alpha(C) = 0 \quad \text{for every constant } C \in \mathbb{R}, \tag{3}$$

and

$$D^\alpha(t^\beta) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} \quad \text{for every constant } \beta > -1. \tag{4}$$

Now we define the two functions (Wright and Mainardi functions) which are very important in order to obtain the explicit solution given in the following sections.

The Wright function is defined in Wright (1933) as:

$$W(x;\rho;\beta)=\sum_{n=0}^{\infty}\frac{x^n}{n!\Gamma(\rho n+\beta)},\quad x\in\mathbb{R},\quad \rho>-1,\quad \beta\in\mathbb{R} \tag{5}$$

and the Mainardi function, which is a particular case of the Wright functions, is defined in Gorenflo et al. (1999) as:

$$M_\rho(x)=W(-x,-\rho,1-\rho)=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!\Gamma(-\rho n+1-\rho)},\quad x\in\mathbb{R},\ 0<\rho<1. \tag{6}$$

Proposition 1 *Some basic properties of the Wright function are the following:*

1. (Kilbas et al. 2006) *The Wright function* (5) *is a differentiable function for every* $\rho>-1$, $\beta\in\mathbb{R}$ *such that*
$$\frac{\partial W}{\partial x}(x;\rho;\beta)=W(x;\rho;\beta+\rho).$$
2. (Roscani and Santillan 2013) $\lim_{\alpha\to 1^-} W\left(-x;-\frac{\alpha}{2};1\right)=W\left(-x;-\frac{1}{2};1\right)=\operatorname{erfc}\left(\frac{x}{2}\right)$, *and* $\lim_{\alpha\to 1^-} 1-W\left(-x;-\frac{\alpha}{2};1\right)=1-W\left(-x;-\frac{1}{2};1\right)=\operatorname{erf}\left(\frac{x}{2}\right)$.
3. (Pskhu 2005) *For all* $\alpha, c\in\mathbb{R}^+$, $\rho\in(0,1)$, $\beta\in\mathbb{R}$ *we have*
$$D^\alpha(x^{\beta-1}W(-cx^{-\rho},-\rho,\beta))=x^{\beta-\alpha-1}W(-cx^{-\rho},-\rho,\beta-\alpha).$$
4. (Roscani and Santillan 2013) *For every* $\alpha\in(0,1)$, $W(-x,-\frac{\alpha}{2},1)$ *is a positive and strictly decreasing function in* $\mathbb{R}^+$ *such that* $0<W\left(-x,-\frac{\alpha}{2},1\right)<1$.
5. (Wright 1934) *For every* $\alpha\in(0,1)$, *and* $\beta>0$,
$$\lim_{x\to\infty} W\left(-x,-\frac{\alpha}{2},\beta\right)=0. \tag{7}$$

In Tarzia (1981) the following classical phase-change problem was studied:

Problem Find the free boundary $x=s(t)$, and the temperatures $T_s=T_s(x,t)$ and $T_l=T_l(x,t)$ such that the following equations and conditions are satisfied:

$$\begin{array}{lll}
\text{(i)} & T_{s\,t}-\lambda_s^2 T_{s\,xx}=0, & x>s(t),\ t>0,\\
\text{(ii)} & T_{l\,t}-\lambda_l^2 T_{l\,xx}=0, & 0<x<s(t),\ t>0\\
\text{(iii)} & s(0)=0, & \\
\text{(iv)} & T_s(x,0)=T_s(\infty,t)=T_{\mathrm{i}}<T_{\mathrm{m}} & x>0,\ t>0,\\
\text{(v)} & T_s(s(t),t)=T_{\mathrm{m}}, & t>0,\\
\text{(vi)} & T_l(s(t),t)=T_{\mathrm{m}}, & t>0,\\
\text{(vii)} & k_s T_{s\,x}(s(t),t)-k_l T_{l\,x}(s(t),t)=\rho l\dot{s}(t), & t>0,\\
\text{(viii)} & k_l T_{l\,x}(0,t)=-\frac{q_0}{t^{1/2}}, & t>0,
\end{array} \tag{8}$$

where $\lambda_s^2=\frac{k_s}{\rho c_s}$, $\lambda_l^2=\frac{k_l}{\rho c_l}$, k_s, c_s and k_l,c_l are the diffusion, conductivity and specific heat coefficients of the solid and liquid phases, respectively, ρ is the common density of mass, l is the latent heat of fusion by unit of mass, T_{i} is the constant initial temperature, T_{m} is the melting temperature and q_0 is the coefficient which characterizes the heat flux at the fixed face $x=0$.

The explicit solution to problem (8) was obtained in Tarzia (1981) through the error function, when the following restriction is satisfied by data:

$$q_0 > \frac{k_s(T_m - T_i)}{\lambda_s\sqrt{\pi}}. \tag{9}$$

In this paper we consider a "fractional melting problem", of order $0 < \alpha < 1$, for the semi-infinite material $x > 0$ with an initial constant "fractional temperature" and a "fractional heat flux boundary condition" at the face $x = 0$. We will use a Caputo derivative operator, which converges to the classical derivative when α tends to 1. The interesting aspect is that for the limit case ($\alpha = 1$) the results obtained for this generalization coincide with the results of the classical case. So, in view of the analogy that exists between the classical case and the fractional one, we make an abuse of language by using terminologies such as "fractional temperature", "fractional heat equation" or "fractional Stefan condition". These terms go hand in hand with the generalization proposed in the sense of operators and we do not pretend to give them a physical approach.

Although it is known that the Caputo fractional operator is linked to memory effects in materials (Metzler and Klafter 2000; Vazquez 2017) or to processes of anomalous diffusion in non–homogeneous media (Filipovitch et al. 2016), the discussion about fractional Stefan problems and their possible physical interpretations is yet an open problem. An interesting discussion on this topic can be seen in Ceretani (2018).

It is necessary to point that the physical approach associated with this type of operators is of our current interest, mainly because of the mathematical coherence that the results have together with their convergence to the classical known results.

So, the problem to be studied is the following:

Problem Find the fractional free boundary $x = r(t)$, defined for $t > 0$, and the fractional temperature $\Theta = \Theta(x, t)$, defined for $x > 0$, $t > 0$, such that the following equations and conditions are satisfied ($0 < \alpha < 1$):

$$\begin{array}{lll}
\text{(i)} & D_t^\alpha \Theta_s - \lambda_s^2 \Theta_{sxx} = 0, & x > r(t), \quad t > 0, \\
\text{(ii)} & D_t^\alpha \Theta_l - \lambda_l^2 \Theta_{lxx} = 0, & 0 < x < r(t), \quad t > 0 \\
\text{(iii)} & r(0) = 0, & \\
\text{(iv)} & \Theta_s(x,0) = \Theta_s(\infty, t) = T_i < T_m & x > 0, \quad t > 0, \\
\text{(v)} & \Theta_s(r(t), t) = T_m, & t > 0, \\
\text{(vi)} & \Theta_l(r(t), t) = T_m, & t > 0, \\
\text{(vii)} & k_s \Theta_{sx}(r(t), t) - k_l \Theta_{lx}(r(t), t) = \rho l D^\alpha r(t), & t > 0, \\
\text{(viii)} & k_l \Theta_{lx}(0, t) = -\frac{q_0}{t^{\alpha/2}}, & t > 0,
\end{array} \tag{10}$$

Note that the suffix t in the operator D^α denotes that the fractional derivative is taken in the t-variable.

In Sect. 2, a necessary condition for the coefficient $q_0 > 0$, which characterizes the fractional heat flux boundary condition at the face $x = 0$, is obtained in order to have an instantaneous two-phase fractional Lamé–Clapeyron–Stefan problem.

In Sect. 2.1, we give a sufficient condition for the coefficient $q_0 > 0$ (which coincides with the necessary condition for q given in Sect. 2) in order to obtain a generalized Neumann solution for the two-phase fractional Lamé–Clapeyron–Stefan problem (10) for a semi-infinite material with a constant initial condition and a fractional heat flux boundary condition at the fixed face $x = 0$. This solution is given as a function of the Wright and Mainardi functions.

Moreover, when $\alpha = 1$, we recover the Neumann solution, through the error function, for the classical two-phase Lamé–Clapeyron–Stefan problem given in Tarzia (1981), when an inequality for the coefficient that characterizes the heat flux boundary condition is satisfied.

In Sect. 2.2, we consider two two-phase fractional Lamé–Clapeyron–Stefan problems having a fractional heat flux and a fractional temperature boundary conditions on the fixed face $x = 0$, respectively, and a possible equivalence between them is analyzed.

In Sect. 2.3, an inequality for the coefficient which characterizes the free boundary of the two-phase fractional Lamé–Clapeyron–Stefan problem with a fractional temperature boundary condition given recently in Roscani and Tarzia (2014), is also obtained.

In Sect. 3, we recover the results obtained in Roscani and Santillan (2013) for the one-phase fractional Lamé–Clapeyron–Stefan problem as a particular case of the present work (see Sects. 2.1, 2.2).

2 Necessary condition to obtain an instantaneous two-phase fractional Stefan problem with a heat flux boundary condition at the fixed face

In order to obtain a necessary condition for data to have an instantaneous phase-change process for problem (10) we consider the following fractional heat conduction problem of order $0 < \alpha < 1$ for the solid phase in the first quadrant with an initial constant temperature and a heat flux boundary condition at $x = 0$:

$$\begin{array}{lll} \text{(i)} & D_t^\alpha \Theta - \lambda_s^2 \Theta_{xx} = 0, & x > 0,\ t > 0, \\ \text{(ii)} & \Theta(x,0) = T_{\rm i}, & x > 0, \\ \text{(iii)} & k_s \Theta_x(0,t) = -\frac{q_0}{t^{\alpha/2}}, & t > 0. \end{array} \tag{11}$$

Lemma 1 *We have:*

1. *The solution of the fractional heat problem* (11) *is given by*

$$\Theta(x,t) = T_{\rm i} + \frac{q_0 \lambda_s \Gamma(1-\alpha/2)}{k_S} W\left(-\frac{x}{\lambda_s t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right), \quad x > 0,\ t > 0. \tag{12}$$

2. *If the coefficient q_0 satisfies the inequalities*

$$0 < q_0 \le \frac{k_s (T_{\rm m} - T_{\rm i})}{\lambda_s \Gamma(1-\alpha/2)}, \tag{13}$$

then problem (10) *is only a fractional heat conduction problem for the initial solid phase. By the contrary, if*

$$q_0 > \frac{k_s (T_{\rm m} - T_{\rm i})}{\lambda_s \Gamma(1-\alpha/2)}, \tag{14}$$

then (14) *is a necessary condition for data which ensures an instantaneous fractional phase-change problem* (10).

Proof 1. From Proposition 1, items 1 and 3, and properties (2) and (3), we can state that

$$\Theta(x,t) = a + b\left[1 - W\left(-\frac{x}{\lambda_s t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right], \quad x > 0, \quad t > 0, \tag{15}$$

is a solution to the fractional diffusion equation (11–*i*), where a and b are two constants to be determined.

Taking the derivative of (15) with respect to x, by using Proposition 1 item 1, we get

$$\Theta_x(x,t) = \frac{b}{\lambda_s t^{\alpha/2}} M_{\alpha/2}\left(\frac{x}{\lambda_s t^{\alpha/2}}\right). \tag{16}$$

From conditions (11-ii) and (11-iii), and being $M_{\alpha/2}(0) = \frac{1}{\Gamma(1-\alpha/2)}$ and $W\left(0, -\frac{\alpha}{2}, 1\right) = 1$, we obtain that

$$a = T_{\mathrm{i}} + \frac{q_0 \lambda_s}{k_s}\Gamma(1-\alpha/2), \quad b = -\frac{q_0\lambda_s \Gamma(1-\alpha/2)}{k_S}, \tag{17}$$

that is, we obtain the expression (12) as a solution to problem (11).

2. From Proposition 1 items 4 and 5, it results that function (12)

$$\Theta(x,t) = T_{\mathrm{i}} + \frac{q_0\lambda_s\Gamma(1-\alpha/2)}{k_S} W\left(-\frac{x}{\lambda_s t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)$$

is a decreasing function in the variable x for every $t \in \mathbb{R}^+$ such that $\Theta(\infty, t) = T_{\mathrm{i}}$ is a constant for all $t > 0$. Therefore, problem (10) has an instantaneous fractional phase-change problem if and only if the constant temperature at the boundary $x = 0$ is greater than the melting temperature T_{m}, that is if and only if

$$T_{\mathrm{i}} + \frac{q_0\lambda_s}{k_s}\Gamma(1-\alpha/2) > T_{\mathrm{m}},$$

which is equivalent to have that inequality (14) holds. □

Remark 1 When $\alpha = 1$, the inequality (14) is given by (9) because $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, which was first established in Tarzia (1981).

2.1 Sufficient condition to obtain an instantaneous two-phase-fractional Stefan problem with a heat flux boundary condition at the fixed face

In this section, we study a two-phase Lamé–Clapeyron–Stefan problem for the time fractional diffusion equation, of order $0 < \alpha < 1$, with an initial constant temperature and a heat flux boundary condition at the face $x = 0$ given by the differential equations and initial and boundary conditions given in problem (10). Taking into account the result in the previous Section and the method developed in Roscani and Tarzia (2014), an explicit solution to problem (10) can be obtained. In fact, we have the following result:

Proposition 2 *Let $T_{\mathrm{i}} < T_{\mathrm{m}}$ be. If the coefficient q_0 satisfies the inequality* (14) *then there exists an instantaneous fractional phase-change (melting) process and the problem* (10) *has the generalized Neumann explicit solution given by:*

$$\Theta_l(x,t) = T_{\mathrm{m}} + \frac{q_0\lambda_l\Gamma(1-\alpha/2)}{k_l}\left[W\left(-\frac{x}{\lambda_l t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) - W\left(-\lambda_l\mu_\alpha, -\frac{\alpha}{2}, 1\right)\right], \tag{18}$$

$$\Theta_s(x,t) = T_{\mathrm{i}} + (T_{\mathrm{m}} - T_{\mathrm{i}})\frac{W\left(-\frac{x}{\lambda_s t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)}{W\left(-\mu_\alpha, -\frac{\alpha}{2}, 1\right)}, \tag{19}$$

$$r(t) = \mu_\alpha \lambda_s t^{\alpha/2}, \tag{20}$$

where the coefficient $\mu_\alpha > 0$ *is a solution of the following equation:*

$$G_\alpha(x) = \frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)}x, \quad x > 0 \tag{21}$$

with

$$G_\alpha(x) = \frac{q_0\lambda_l\Gamma(1-\alpha/2)}{\rho l \lambda_s}M_{\alpha/2}(\lambda x) - \frac{k_s(T_{\mathrm{m}}-T_{\mathrm{i}})}{\rho l \lambda_s^2}F_{2\alpha}(x), \tag{22}$$

$$F_{2\alpha}(x) = \frac{M_{\alpha/2}(x)}{W\left(-x,-\frac{\alpha}{2},1\right)}, \tag{23}$$

and

$$\lambda = \frac{\lambda_s}{\lambda_l} > 0. \tag{24}$$

Proof Following Roscani and Tarzia (2014), we propose the following solution:

$$\Theta_l(x,t) = A + B\left[1 - W\left(-\frac{x}{\lambda_l t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right], \tag{25}$$

$$\Theta_s(x,t) = C + D\left[1 - W\left(-\frac{x}{\lambda_s t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right], \tag{26}$$

$$r(t) = \mu\lambda_s t^{\alpha/2}, \tag{27}$$

where the coefficients A, B, C, D and μ are constants and must be determined. According to the results in the previous section and the linearity of the fractional derivative, functions Θ_s and Θ_l are solutions of the fractional diffusion equations (10-i) and (10-ii), respectively. Starting from conditions (10-vi) and (10-viii), we obtain the following system of two equations:

$$T_{\mathrm{m}} = \Theta_l(r(t),t) = A + B\left[1 - W\left(\mu\lambda, -\frac{\alpha}{2}, 1\right)\right] \tag{28}$$

$$-\frac{q_0}{t^{\alpha/2}} = k_l\Theta_{l_\infty}(0,t) = \frac{Bk_l}{\lambda_l t^{\alpha/2}}M_{\alpha/2}(0), \tag{29}$$

from which we obtain:

$$A = T_{\mathrm{m}} + \frac{q_0\lambda_l\Gamma\left(1-\frac{\alpha}{2}\right)}{k_l}\left[1 - W\left(-\mu\lambda_l, -\frac{\alpha}{2}, 1\right)\right], \quad B = \frac{q_0\lambda_l\Gamma\left(1-\frac{\alpha}{2}\right)}{k_l}. \tag{30}$$

Then, the fractional temperature of the liquid phase is given by (18).

From conditions (10-iv) and (10-v) we have the system of equations:

$$T_{\mathrm{i}} = \Theta_s(x,0) = C + D, \tag{31}$$

$$T_{\mathrm{m}} = \Theta_s(r(t),t) = C + D\left[1 - W\left(-\mu, -\frac{\alpha}{2}, 1\right)\right], \tag{32}$$

and then we have:

$$C = T_{\mathrm{i}} + \frac{T_{\mathrm{m}}-T_{\mathrm{i}}}{W\left(-\mu,-\frac{\alpha}{2},1\right)}, \quad D = -\frac{T_{\mathrm{m}}-T_{\mathrm{i}}}{W\left(-\mu,-\frac{\alpha}{2},1\right)}. \tag{33}$$

Therefore, the fractional temperature of the solid phase is given by (19).

In order to determine the coefficient $\mu > 0$ we must consider the fractional Lamé–Clapeyron–Stefan condition (10-vii) which, taking into account Proposition 1 and (4), gives us the Eq. (21).

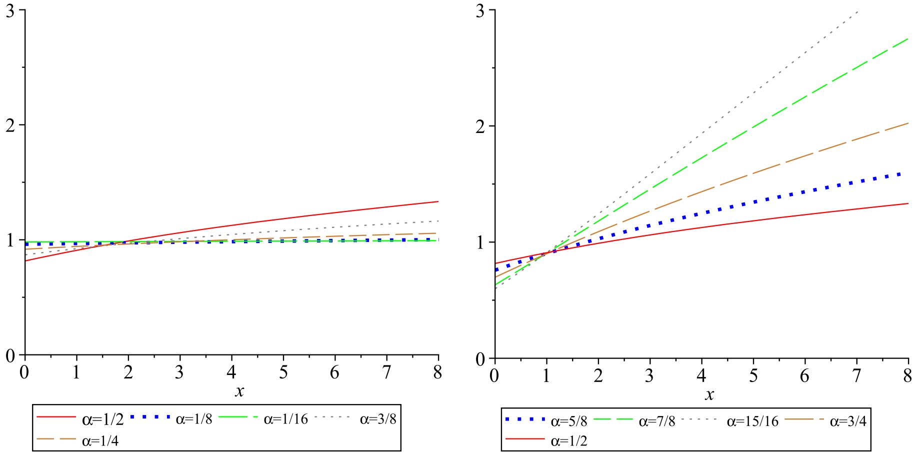
It was proved in Roscani and Tarzia (2014) that $F_{2\alpha}(+\infty) = +\infty$; then the function $G_\alpha = G_\alpha(x)$, defined by (43), has the following properties:

$$G_\alpha(0^+) = \frac{q_0 \lambda_l}{\rho l \lambda_s} - \frac{k_s(T_\mathrm{m} - T_\mathrm{i})}{\rho l \lambda_s^2 \Gamma\left(1 - \frac{\alpha}{2}\right)}, \quad G_\alpha(+\infty) = -\infty. \tag{34}$$

From the continuity of G_α [due to Proposition 1 item (4) and (34)], it yields that equation (21) has a solution $\mu_\alpha > 0$ if $G_\alpha(0^+) > 0$ which is verified under condition (14). Then, the solution $\{(18)$–$(20)\}$ holds.

Remark 2 The solution of the Eq. (21) will be unique if we can prove that function G_α is a strictly decreasing function in $\mathbb{R}^+$, or equivalently if we can prove that function $F_{2\alpha}$ is an increasing function in $\mathbb{R}^+$ (taking into account that function $M_{\alpha/2}$ is a decreasing function).

Function $F_{2\alpha}(x) = \frac{M_{\alpha/2}(x)}{W(-x;-\frac{\alpha}{2},1)}$ is a continuous positive function, which is a quotient of two decreasing functions. Some graphics are presented below:



(a) $F_{2\alpha}$ for $\alpha = 1/16, 1/8, 1/4, 3/8$ and $1/2$. **(b)** $F_{2\alpha}$ for $\alpha = 1/2, 5/8, 7/8, 3/4$ and $15/16$.

We can see in the graphics that $F_{2\alpha}$ is an increasing function for every chosen parameter. We wonder if this situation is true for every $\alpha \in (0, 1)$. A simple computation gives that $F_{2\alpha}$ is an increasing function if and only if

$$\left[M_{\alpha/2}(x)\right]^2 - W\left(-x; -\frac{\alpha}{2}, 1\right) \cdot W\left(-x; -\frac{\alpha}{2}, 1 - \alpha\right) > 0. \tag{35}$$

We have proved in Roscani and Tarzia (2017) that for every $x > 0$,

$$\Gamma(1-\alpha) W\left(-x; -\frac{\alpha}{2}, 1-\alpha\right) > \Gamma\left(1 - \frac{\alpha}{2}\right) M_{\alpha/2}(x) > W\left(-x; -\frac{\alpha}{2}, 1\right) > 0, \tag{36}$$

but this is not a sufficient condition to prove (35).

Also, the inequality (35) is a Turán-type inequality for Wright functions of parameter $-\frac{\alpha}{2} \in (-1, 0)$. An analogue result for Wright functions with positive parameter was proved in Mehretz (2017), that is, it was proved that

$$[W(x; \alpha, \beta + \alpha)]^2 - W(x; \alpha, \beta) W(x; \alpha, \beta + 2\alpha) \geq 0, \quad \forall\, x > 0, \quad \alpha > 0, \quad \beta > 0.$$

So we state the following conjecture:

Conjecture 1 *The function* $F_{2\alpha}: \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $F_{2\alpha}(x) = \frac{M_{\alpha/2}(x)}{W(-x;-\frac{\alpha}{2},1)}$ *is an increasing function with* $F_{2\alpha}(0^+) = \frac{1}{\Gamma(1-\frac{\alpha}{2})}$ *and* $F_{2\alpha}(+\infty) = +\infty$.

Theorem 1 *Let* $T_{\rm i} < T_{\rm m}$ *be. If the coefficient* q_0 *satisfies the inequality* (14)*, under the assumption of Conjecture 1, then* { (18)–(20)} is the unique generalized Nuemann similarity-solution to the free boundary problem (10), where μ_α is the unique solution to Eq. (21).

Remark 3 If (x,t) is in the liquid face $(0 < x < s(t), t > 0)$, then $0 < x < \lambda_s \mu t^{\alpha/2}$, or equivalently $0 < \frac{x}{t^{\alpha/2}} < \lambda_s \mu$. Multiplying by λ_l gives $0 < \frac{x}{\lambda_l t^{\alpha/2}} < \lambda\mu$. Then, from Proposition 1 item 4 it follows that $W\left(-\frac{x}{\lambda_l t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) - W\left(-\lambda_l \mu_\alpha, -\frac{\alpha}{2}, 1\right) > 0$ and, therefore, the explicit temperature of the liquid phase corresponding to problem (10) satisfies the following inequality:

$$\begin{aligned}\Theta_l(x,t) &= T_{\rm m} + \frac{q_0 \lambda_l \Gamma(1-\alpha/2)}{k_l}\left[W\left(-\frac{x}{\lambda_l t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) - W\left(-\lambda_l \mu_\alpha, -\frac{\alpha}{2}, 1\right)\right] \\ &> T_{\rm m}, \quad 0 < x < r(t), \quad t > 0 \end{aligned} \tag{37}$$

Analogously the explicit temperature of the solid phase corresponding to problem (10) satisfies the following inequality:

$$T_{\rm i} < \Theta_s(x,t) < T_{\rm m}, \quad x > r(t), \quad t > 0. \tag{38}$$

Proposition 3 *Let* $T_{\rm i} < T_{\rm m}$ *be. By considering* $\alpha = 1$ *in Proposition 2, we recover the classical Neumann explicit solution and the inequality* (9) *for the coefficient which characterized the heat flux at* $x = 0$ *obtained in* Tarzia (1981).

Proof As it was said in Remark 1, the inequality (9) is recovered because $\Gamma(1/2) = \sqrt{\pi}$. By the other side,

$$\Theta_l(x,t) = T_{\rm m} + \frac{q_0 \lambda_l \Gamma(1-\alpha/2)}{k_l}\left[W\left(-\frac{x}{\lambda_l t^{1/2}}, -\frac{1}{2}, 1\right) - W\left(-\lambda_l \mu_1, -\frac{\alpha}{2}, 1\right)\right], \tag{39}$$

$$\Theta_s(x,t) = T_{\rm i} + (T_{\rm m} - T_{\rm i}) \frac{W\left(-\frac{x}{\lambda_2 t^{1/2}}, -\frac{1}{2}, 1\right)}{W\left(-\mu_1, -\frac{1}{2}, 1\right)}, \tag{40}$$

$$r(t) = \mu_1 \lambda_s t^{1/2}, \tag{41}$$

where the coefficient $\mu = \mu_1 > 0$ is the solution of equation:

$$G_1(x) = \frac{\Gamma(3/2)}{\Gamma(1/2)} x, \quad x > 0 \tag{42}$$

with

$$G_1(x) = \frac{q_0 \lambda_l \Gamma(1/2)}{\rho l \lambda_s} M_{1/2}(\lambda x) - \frac{k_s (T_{\rm m} - T_{\rm i})}{\rho l \lambda_s^2} F_2(x), \tag{43}$$

$$F_2(x) = \frac{M_{1/2}(x)}{W\left(-x, -\frac{1}{2}, 1\right)}, \tag{44}$$

and

$$\lambda = \frac{\lambda_s}{\lambda_l} > 0. \tag{45}$$

Taking into account that $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$, $M_{1/2}(x) = e^{-(x/2)^2}$ and that $W\left(-x, -\frac{1}{2}, 1\right) = \operatorname{erfc}\left(\frac{x}{2}\right)$ (see Roscani and Santillan 2013), it results that

$$\Theta_s^1(x,t) = T_\mathrm{i} + (T_\mathrm{m} - T_\mathrm{i}) \frac{\operatorname{erfc}\left(\frac{x}{\lambda_s t^{\alpha/2}}\right)}{\operatorname{erfc}\left(\frac{\mu_1}{2}\right)}, \tag{46}$$

$$\Theta_l^1(x,t) = T_\mathrm{m} + \frac{q_0 \lambda_l \sqrt{\pi}}{k_l}\left[\operatorname{erfc}\left(\frac{x}{\lambda_l t^{1/2}}\right) - \operatorname{erfc}\left(\frac{\lambda \mu_1}{2}\right)\right], \tag{47}$$

$$r_1(t) = \mu_1 \lambda_s \sqrt{t}, \tag{48}$$

where $\mu_1 > 0$ is the solution of the equation:

$$\frac{q_0}{\rho l \lambda_s} \exp\left(-\frac{\lambda^2 x^2}{4}\right) - \frac{k_s (T_\mathrm{m} - T_\mathrm{i})}{\rho l \lambda_s^2 \sqrt{\pi}} \frac{\exp\left(-\frac{x^2}{4}\right)}{\operatorname{erfc}\left(\frac{x}{2}\right)} = \frac{x}{2}, \quad x > 0 \tag{49}$$

or equivalently, $\frac{\mu_1}{2}$ is a solution of the equation:

$$\frac{q_0}{\rho l \lambda_s} \exp\left(-\lambda^2 x^2\right) - \frac{k_s (T_\mathrm{m} - T_\mathrm{i})}{\rho l \lambda_s^2 \sqrt{\pi}} \frac{\exp\left(-x^2\right)}{\operatorname{erfc}(x)} = x, \quad x > 0. \tag{50}$$

Therefore, the tender $\left\{\Theta_s^1(x,t), \Theta_l^1(x,t), r_1(t)\right\}$, where $\mu_1/2$ is the solution of the Eq. (50), is the solution of the problem (8) given in Tarzia (1981).

Theorem 2 *Let $T_\mathrm{i} < T_\mathrm{m}$ be. If the coefficient q_0 satisfies the inequality (14) and the Conjecture 1 holds, then the similarity-solution to the problem (10) converges to the similarity-solution to the classical Lamé–Clapeyron–Stefan problem (8) when $\alpha \to 1^-$.*

2.2 Two-phase fractional Stefan problems with a heat flux and a temperature boundary condition at the fixed face admitting the same similarity solution

Let $T_\mathrm{i} < T_\mathrm{m}$ be. If the coefficient q_0 satisfies the inequality (14), then the solution of the problem (10) is given by (18)–(20) where μ_α is a solution of the Eq. (21). In this case, we can compute the liquid temperature Θ_l at the fixed face $x = 0$, which is given by:

$$\Theta_l(0^+, t) = T_\mathrm{m} + \frac{q_0 \lambda_l \Gamma(1-\alpha/2)}{k_l}\left[1 - W\left(-\lambda \mu_\alpha; -\frac{\alpha}{2}, 1\right)\right] > T_\mathrm{m}, \quad \forall t > 0. \tag{51}$$

Since this temperature is greater than T_m the melting temperature and it is constant for all positive time, we can consider the following fractional free boundary problem:

Problem Find the free boundary $x = s(t)$, defined for $t > 0$, and the temperature $T = T(x,t)$, defined for $x > 0, t > 0$ such that the following equations and conditions are satisfied ($0 < \alpha < 1$):

$$\begin{array}{lll}
\text{(i)} & D_t^\alpha T_s - \lambda_s^2 T_{s\,xx} = 0, & x > s(t), \quad t > 0, \\
\text{(ii)} & D_t^\alpha T_l - \lambda_l^2 T_{l\,xx} = 0, & 0 < x < s(t), \quad t > 0 \\
\text{(iii)} & s(0) = 0, & \\
\text{(iv)} & T_s(x,0) = T_s(+\infty, t) = T_\mathrm{i} < T_\mathrm{m} & x > 0, \quad t > 0, \\
\text{(v)} & T_s(s(t),t) = T_\mathrm{m}, & t > 0, \\
\text{(vi)} & T_l(s(t),t) = T_\mathrm{m}, & t > 0, \\
\text{(vii)} & k_s T_{s\,x}(s(t),t) - k_l T_{l\,x}(s(t),t) = \rho l D_t^\alpha s(t), & t > 0, \\
\text{(viii)} & T(0,t) = T_0, & t > 0,
\end{array} \tag{52}$$

where the imposed temperature T_0 at the fixed face $x = 0$ is greater than the melting temperature, that is $T_0 > T_{\rm m}$. The problem (52) was recently solved in Roscani and Tarzia (2014) and the solution is given by:

$$T_{\rm m} < T_l(x,t) = T_{\rm m} + (T_0 - T_{\rm m}) \frac{W\left(-\frac{x}{\lambda_l t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) - W\left(-\lambda \xi_\alpha, -\frac{\alpha}{2}, 1\right)}{1 - W\left(-\lambda \xi_\alpha, -\frac{\alpha}{2}, 1\right)}$$

$$= T_0 - (T_0 - T_{\rm m}) \frac{1 - W\left(-\frac{x}{\lambda_l t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)}{1 - W\left(-\lambda \xi_\alpha, -\frac{\alpha}{2}, 1\right)}, \quad 0 < x < s(t), \quad t > 0 \tag{53}$$

$$T_{\rm i} < T_s(x,t) = T_{\rm i} + (T_{\rm m} - T_{\rm i}) \frac{W\left(-\frac{x}{\lambda_s t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)}{W\left(-\xi_\alpha, -\frac{\alpha}{2}, 1\right)}$$

$$= T_{\rm m} - (T_{\rm m} - T_{\rm i}) \left[1 - \frac{W\left(-\frac{x}{\lambda_s t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)}{W\left(-\xi_\alpha; \frac{\alpha}{2}, 1\right)} \right] < T_{\rm m}, \quad x > s(t), \quad t > 0 \tag{54}$$

$$s(t) = \xi_\alpha \lambda_s t^{\alpha/2}, \tag{55}$$

where the coefficient $\xi = \xi_\alpha > 0$ is a solution of the following equation:

$$F_\alpha(x) = \frac{\Gamma\left(1 + \frac{\alpha}{2}\right)}{\Gamma\left(1 - \frac{\alpha}{2}\right)} x, \quad x > 0 \tag{56}$$

with

$$F_\alpha(x) = \frac{k_l (T_0 - T_{\rm m})}{\rho l \lambda_s \lambda_l} F_{1\alpha}(\lambda x) - \frac{k_s (T_{\rm m} - T_{\rm i})}{\rho l \lambda_s^2} F_{2\alpha}(x), \tag{57}$$

$$F_{1\alpha}(x) = \frac{M_{\alpha/2}(x)}{1 - W\left(-x, -\frac{\alpha}{2}, 1\right)}, \tag{58}$$

and $F_{2\alpha}$ was defined in (44).

Proposition 4 *Let $T_{\rm i} < T_{\rm m}$ be. If the coefficient q_0 satisfies the inequality (14) then both free boundary problems (10) and (52) with data T_0 given by*

$$T_0 = T_{\rm m} + \frac{q_0 \lambda_l \Gamma(1 - \alpha/2)}{k_l} \left[1 - W\left(-\lambda \mu_a l; -\frac{\alpha}{2}, 1\right) \right] \tag{59}$$

admit the same similarity solutions.

Proof Let $T_{\rm i} < T_{\rm m}$ be. If the coefficient q_0 satisfies the inequality (14) then the solution of the free boundary problem (10) is given by (18)–(20), where the coefficient μ_α is a solution of Eq. (21). In this case, the temperature at the fixed face $x = 0$ is given by (51) and, therefore, we can now consider the free boundary problem (52) with data (52-vii) at the fixed face

$x = 0$, where T_0 is defined by (51). Note that

$$F_\alpha(\mu_\alpha) = \frac{k_l(T_0 - T_{\rm m})}{\rho l \lambda_s \lambda_l} F_{1\alpha}(\lambda\mu_\alpha) - \frac{k_s(T_{\rm m} - T_{\rm i})}{\rho l \lambda_s^2} F_{2\alpha}(\mu_\alpha)$$
$$= \frac{q_0 \Gamma(1-\alpha/2)}{\rho l \lambda_s}\left[1 - W\left(-\lambda\mu_\alpha; -\frac{\alpha}{2}, 1\right)\right] F_{1\alpha}(\lambda\mu_\alpha) - \frac{k_s(T_{\rm m} - T_{\rm i})}{\rho l \lambda_s^2} F_{2\alpha}(\mu_\alpha)$$
$$= \frac{q_0 \Gamma(1-\alpha/2)}{\rho l \lambda_s} M_{\alpha/2}(\lambda\mu_\alpha) - \frac{k_s(T_{\rm m} - T_{\rm i})}{\rho l \lambda_s^2} F_{2\alpha}(\mu_\alpha) = G_\alpha(\mu_\alpha) \tag{60}$$

Then, we can affirm that μ_α is a solution to (21) if and only if μ_α is a solution to (56).

Therefore, we have solutions given by (53)–(55) and (18)–(20) to problems (52) and (10), respectively, where the coefficient $\xi_\alpha = \mu_\alpha$.

Clearly, for every $0 < \alpha < 1$, it results that $r(t) = s(t)$ for all $t > 0$. Moreover $T_s(x,t) = \Theta_s(x,t)$ and $T_l(x,t) = \Theta_l(x,t)$, and the thesis holds.

Theorem 3 *Let $T_{\rm i} < T_{\rm m}$ be. If the coefficient q_0 satisfies the inequality (14), under the assumption of Conjecture 1, then the free boundary problem (10) is equivalent to the free boundary problem (52), in the sense of similarity solutions, with data T_0 given by:*

$$T_0 = T_{\rm m} + \frac{q_0 \lambda_l \Gamma(1-\alpha/2)}{k_l}\left[1 - W\left(-\lambda\mu_\alpha; -\frac{\alpha}{2}, 1\right)\right]. \tag{61}$$

Proof If the Conjecture 1 is true, then Eq. (56) admits a unique positive solution.

2.3 Inequality for the coefficient which characterizes the free boundary for the two-phase fractional Stefan problem with a temperature boundary condition at the fixed face

Now, we consider problem (52) with data $T_{\rm i} < T_{\rm m} < T_0$, whose solution given by (53)–(56) has been recently obtained in Roscani and Tarzia (2014).

Theorem 4 *The coefficient ξ_α which characterizes the phase–change interface (55) of the free boundary problem (52) verifies the inequality*

$$1 - W\left(-\frac{\lambda_s}{\lambda_l}\xi_\alpha; -\frac{\alpha}{2}, 1\right) < \frac{T_0 - T_{\rm m}}{T_{\rm m} - T_{\rm i}} \frac{k_l \lambda_s}{k_s \lambda_l}. \tag{62}$$

Proof If we consider the solution (53)–(56) of the free boundary problem (52) where the coefficient $\xi_\alpha > 0$ is a solution of the Eq. (56) for data $T_0 > T_{\rm m}$, then, by taking into account Proposition 1, we have that the corresponding coefficient q_0 (which characterizes the heat flux boundary condition (10-viii) on the fixed face $x = 0$) is given by:

$$q_0 = \frac{T_0 - T_{\rm m}}{1 - W\left(-\lambda\mu_\alpha; -\frac{\alpha}{2}, 1\right)} \frac{k_l}{\lambda_l \Gamma(1-\alpha/2)}. \tag{63}$$

and then we can compute the coefficient q_0. Therefore, the inequality (14) for q_0 is transformed in the inequality (62) for the coefficient ξ_α defined in Roscani and Tarzia (2014), and, therefore, the result holds.

Remark 4 If we consider $\alpha = 1$ in the inequality (62) we obtain the inequality

$$\operatorname{erf}\left(\frac{\lambda_s}{\lambda_l}\frac{\mu_1}{2}\right) < \frac{T_0 - T_{\rm m}}{T_{\rm m} - T_{\rm i}} \frac{k_l \lambda_s}{k_s \lambda_l} \tag{64}$$

given in Tarzia (1981) for the Neumann solution for the classical two-phase Stefan problem.

3 The one-phase fractional Stefan problem

In Roscani and Santillan (2013), the following two one-phase fractional Lamé–Clapeyron–Stefan problems were studied:

$$\begin{array}{rll} \text{(i)} & D_t^\alpha \Theta - \lambda^2 \Theta_{xx} = 0, & 0 < x < r(t), \quad t > 0 \\ \text{(ii)} & r(0) = 0, & \\ \text{(iii)} & \Theta(r(t), t) = T_\mathrm{m}, & t > 0, \\ \text{(iv)} & -k\Theta_x(r(t), t) = \rho l D_t^\alpha r(t), & t > 0, \\ \text{(v)} & k\Theta_x(0, t) = -\frac{q_0}{t^{\alpha/2}}, & t > 0, \end{array} \tag{65}$$

and

$$\begin{array}{rll} \text{(i)} & D_t^\alpha T - \lambda^2 T_{xx} = 0, & 0 < x < s(t), \quad t > 0 \\ \text{(ii)} & s(0) = 0, & \\ \text{(iii)} & T(s(t), t) = T_\mathrm{m}, & t > 0, \\ \text{(iv)} & -kT_x(s(t), t) = \rho l D_t^\alpha s(t), & t > 0, \\ \text{(v)} & kT_x(0, t) = T_0, & t > 0, \end{array} \tag{66}$$

where $\lambda^2 = \frac{k}{\rho c}$. These two problems can be considered as particular cases of the free boundary problems (10) and (52), respectively.

Corollary 1 *The results given in* Roscani and Santillan (2013) *for the one-phase fractional Stefan problems* (65) *and* (66) *can be recovered by taking* $T_\mathrm{i} = T_\mathrm{m}$ *in the free boundary problems* (10) *and* (52)*, respectively.*

Proof It is sufficient to observe that the inequality (14) is automatically verified if we take $T_\mathrm{i} = T_\mathrm{m}$ because $q_0 > 0$. Then the two free boundary problems (65) and (66) are equivalents respect on similarity solutions.

4 Conclusions

- We have obtained a generalized Neumann solution for the two-phase fractional Lamé–Clapeyron–Stefan problem for a semi-infinite material with a constant initial condition and a heat flux boundary condition on the fixed face $x = 0$, when a restriction on data is satisfied. The explicit solution is given through the Wright and Mainardi functions.
- When $\alpha = 1$, we have recovered the Neumann solution through the error function for the corresponding classical two-phase Lamé–Clapeyron–Stefan problem given in Tarzia (1981). We also recover the inequality for the corresponding coefficient that characterizes the heat flux boundary condition at $x = 0$.
- We have proposed a conjecture, from which it can be proved the equivalence between the two-phase fractional Lamé–Clapeyron–Stefan problems with a heat flux and a temperature boundary conditions on the fixed face $x = 0$ for similarity solutions. Moreover, an inequality for the coefficient which characterizes the free boundary given in Roscani and Tarzia (2014) was obtained.
- We have recovered the results obtained in Roscani and Santillan (2013) for the one-phase fractional Lamé–Clapeyron–Stefan problem as a particular case of the present work by taking $T_\mathrm{i} = T_\mathrm{m}$.

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