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A GENERALIZED NEUMANN SOLUTION FOR THE TWO-PHASE FRACTIONAL LAMÉ-CLAPEYRON-STEFAN PROBLEM

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Abstract. We obtain a generalized Neumann solution for the two-phase fractional Lamé-Clapeyron-Stefan problem for a semi-infinite material with constant boundary and initial conditions. In this problem, the two governing equations and a governing condition for the free boundary include a fractional time derivative in the Caputo sense of order $0 < \alpha \leq 1$. When $\alpha \nearrow 1$ we recover the classical Neumann solution for the two-phase Lamé-Clapeyron-Stefan problem given through the error function.

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1 Introduction

The fractional diffusion equation has been treated by a number of authors (see [10, 20, 15, 17, 22]) and, among the several applications that have been studied, Mainardi [19] studied the application to the theory of linear viscoelasticity.

The free boundary problems for the one-dimensional heat equation are problems linked to the processes of melting and freezing which have a latent heat-type condition at the interface connecting the velocity of the free boundary and the heat flux of the temperatures in both phases. This kind of problems have been widely studied (see [1, 3, 5, 6, 7, 11, 16, 18, 25, 26, 28, 29]). In this paper, we deal with a two-phase Lamé-Clapeyron-Stefan problem for the time fractional diffusion equation, obtained from the standard diffusion equation by replacing the first order time-derivative by a fractional derivative of order $\alpha \in (0,1)$ in the Caputo sense.

We use here the definition introduced by Caputo in 1967 [4], and we will call it *fractional derivative in the Caputo sense*, which is defined by

$$ {}_aD^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau & n-1 < \alpha < n \\ f^{(n)}(t) & \alpha = n \end{cases} $$

where $\alpha > 0$ is the order of derivation, $n \in \mathbb{N}$, Γ is the Gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ and f is a differentiable function up to order n in $[a,b]$.

An interesting physical meaning of the fractional Stefan problems is discussed in [9] and many authors were recently studying this kind of problems (see. [2, ?, 23, 24, 30]). Some applications are linked to the behaviour in simulations of gas in polymer glasses ([12]) or propagation in porous media ([8]). In [14] the classical Lamé-Clapeyron-Stefan problem was studied by using the fractional derivative of order 1/2.

In this paper we consider the following two-phase fractional Lamé-Clapeyron-Stefan Problem

$$ \begin{cases} (i) \quad {}_0D^{\alpha} u_2(x,t) = \lambda_2^2 \dfrac{\partial^2 u_2}{\partial x^2}(x,t) & 0 < x < s(t),\ t > 0,\ 0 < \alpha < 1, \\ (ii) \quad {}_0D^{\alpha} u_1(x,t) = \lambda_1^2 \dfrac{\partial^2 u_1}{\partial x^2}(x,t) & s(t) < x < \infty,\ t > 0,\ 0 < \alpha < 1, \\ (iii) \quad k_1 u_{1x}(s(t),t) - k_2 u_{2x}(s(t),t) = \rho l \, {}_0D^{\alpha} s(t) & t > 0, \\ (iv) \quad u_1(s(t),t) = u_2(s(t),t) = u_m & t > 0, \\ (v) \quad u_1(x,0) = u_1(+\infty,t) = u_i & 0 < x < \infty, \\ (vi) \quad u_2(0,t) = u_0 & t > 0, \\ (vii) \quad s(0) = 0 & \end{cases} \tag{1} $$

where $u_i < u_m < u_0$ and $\lambda_j^2 = \dfrac{k_j}{\rho c_j}$, $j = 1$(solid phase), 2 (liquid phase).

In this problem, the two governing diffusion equations (1-*ii*) and (1-*i*) for u_1 and u_2 respectively, and the governing condition on the free boundary $s(t)$ (1-*iii*) include a fractional time derivative in the Caputo sense of order $0 < \alpha \leq 1$. The goal of this paper is to obtain an explicit solution of the free boundary problem (1), called a generalized Neumann solution with respect to the classical one given in [5], [27], [31]. This explicit solution is obtained through the Wright and Mainardi functions ([21]). In Section 2 a summary of some properties related to these special functions are given which will be useful in the next section. In Section 3 the existence of a generalized Neumann solution is given and an open problem for the uniqueness is posed. Moreover, the classical Neumann solution for the two-phase Lamé-Clapeyron-Stefan problem for a semi-infinite material is well recovered by considering the limit when $\alpha \nearrow 1$.

2 The Special Functions Involved

Definition 2.1. *For every $z \in \mathbb{C}$, $\alpha > -1$ and $\beta \in \mathbb{R}$ the Wright function is defined by*

$$W(z;\alpha;\beta) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\alpha k + \beta)}. \tag{2}$$

This function will play a fundamental role in this paper. It is known that the Wright function is an entire function if $\alpha > -1$.

Taking $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}$, we get

$$W\left(-z, -\frac{1}{2}, \frac{1}{2}\right) = M_{1/2}(z) = \frac{1}{\sqrt{\pi}} e^{-z^2/4}. \tag{3}$$

where $M_{1/2}(z)$ is the Mainardi function (see [10]), defined by

$$M_\nu(z) = W(-z, -\nu, 1-\nu) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma\left(-\nu n + 1 - \nu\right)}, \quad z \in \mathbb{C},\ \nu < 1. \tag{4}$$

which is a particular case of the Wright function.

Due to the uniform convergence of the series on compact sets, we have (see [33])

$$\frac{\partial}{\partial z} W(z, \alpha, \beta) = W(z, \alpha, \alpha + \beta). \tag{5}$$

Then, for $x \in \mathbb{R}_0^+$, and taking account that

$$W(-\infty, -\frac{\alpha}{2}, 1) = 0, \qquad \text{if } \alpha \in (0, 2), \tag{6}$$

we have

$$W\left(-x, -\frac{1}{2}, 1\right) = W\left(-x, -\frac{1}{2}, 1\right) - W\left(-\infty, -\frac{1}{2}, 1\right) = \int_{\infty}^{x} \left(\frac{\partial}{\partial x} W\left(-\xi, -\frac{1}{2}, 1\right)\right) d\xi$$

$$= \int_{\infty}^{x} -W\left(-\xi, -\frac{1}{2}, \frac{1}{2}\right) d\xi = \int_{x}^{\infty} W\left(-\xi, -\frac{1}{2}, \frac{1}{2}\right) d\xi = \int_{x}^{\infty} \frac{1}{\sqrt{\pi}} e^{-\xi^2/4} d\xi$$

$$= \frac{2}{\sqrt{\pi}} \int_{x/2}^{\infty} e^{-\xi^2} d\xi = \operatorname{erfc}\left(\frac{x}{2}\right),$$

that is,

$$W\left(-x, -\frac{1}{2}, 1\right) = \operatorname{erfc}\left(\frac{x}{2}\right), \quad 1 - W\left(-x, -\frac{1}{2}, 1\right) = \operatorname{erf}\left(\frac{x}{2}\right). \tag{7}$$

where *erf* and *erfc* are the error and complementary error functions.

The next two propositions were proved in [23].

Lemma 2.2. *If* $0 < \alpha < 1$*, then:*

1. $M_{\alpha/2}(x)$ *is a positive and strictly decreasing positive function in* $\mathbb{R}^+$ *such that* $M_{\alpha/2}(x) < \frac{1}{\Gamma\left(1-\frac{\alpha}{2}\right)}$;
2. $W\left(-x, -\frac{\alpha}{2}, 1\right)$ *is a positive and strictly decreasing function in* $\mathbb{R}^+$ *such that* $0 < W\left(-x, -\frac{\alpha}{2}, 1\right) \leq 1, \ \forall x \in \mathbb{R}_0^+$.

Lemma 2.3. *If* $x \in \mathbb{R}_0^+$ *and* $\alpha \in (0,1)$ *then:*

1. $\lim\limits_{\alpha \nearrow 1} M_{\alpha/2}(x) = M_{1/2}(x) = \frac{e^{-\frac{x^2}{4}}}{\sqrt{\pi}}$;
2. $\lim\limits_{\alpha \nearrow 1} \left[1 - W\left(-x, -\frac{\alpha}{2}, 1\right)\right] = erf\left(\frac{x}{2}\right)$.

Due to the results in [35], the following assertions are true

$$\lim_{x \to \infty} W\left(-x, -\frac{\alpha}{2}, 1\right) = 0 \quad \text{and} \quad \lim_{x \to \infty} M_{\alpha/2}(x) = 0. \tag{8}$$

Let us work on some problems in the first quadrant. It is known that (see [20])

$$u(x,t) = \int_{-\infty}^{\infty} \frac{t^{-\frac{\alpha}{2}}}{2\lambda} M_{\frac{\alpha}{2}}\left(|x-\xi| \lambda^{-1} t^{-\frac{\alpha}{2}}\right) f(\xi) d\xi \tag{9}$$

is a solution for the fractional diffusion problem

$$\begin{cases} {}_0D^{\alpha} u(x,t) = \lambda^2 \dfrac{\partial^2 u}{\partial x^2}(x,t) & -\infty < x < \infty,\, t > 0,\, 0 < \alpha < 1, \\ u(x,0) = f(x) & -\infty < x < \infty. \end{cases} \tag{10}$$

Using this fact, it is easy to see that

$$v(x,t) = \frac{1}{2\lambda t^{\frac{\alpha}{2}}} \int_0^{\infty} \left[M_{\frac{\alpha}{2}}\left(\frac{|x-\xi|}{\lambda t^{\frac{\alpha}{2}}}\right) - M_{\frac{\alpha}{2}}\left(\frac{x+\xi}{\lambda t^{\frac{\alpha}{2}}}\right)\right] f_0 \, d\xi \tag{11}$$

is a solution for the fractional diffusion problem

$$\begin{cases} {}_0D^\alpha v(x,t) = \lambda^2 \dfrac{\partial^2 v}{\partial x^2}(x,t) & 0 < x < \infty,\ t > 0,\ 0 < \alpha < 1, \\ v(x,0) = f_0 & 0 < x < \infty, \\ v(0,t) = 0 & t > 0. \end{cases} \tag{12}$$

An equivalent expression of (11) is given by

$$\begin{aligned} v(x,t) &= \frac{1}{2\lambda t^{\frac{\alpha}{2}}} \int_0^\infty \left[M_{\frac{\alpha}{2}}\left(\frac{|x-\xi|}{\lambda t^{\frac{\alpha}{2}}}\right) - M_{\frac{\alpha}{2}}\left(\frac{x+\xi}{\lambda t^{\frac{\alpha}{2}}}\right)\right] f_0 d\xi \\ &= \frac{f_0}{2}\left[\int_0^x \frac{1}{\lambda t^{\frac{\alpha}{2}}} M_{\frac{\alpha}{2}}\left(\frac{x-\xi}{\lambda t^{\frac{\alpha}{2}}}\right) d\xi + \int_x^\infty \frac{1}{\lambda t^{\frac{\alpha}{2}}} M_{\frac{\alpha}{2}}\left(\frac{\xi - x}{\lambda t^{\frac{\alpha}{2}}}\right) d\xi - \int_0^\infty \frac{1}{\lambda t^{\frac{\alpha}{2}}} M_{\frac{\alpha}{2}}\left(\frac{x+\xi}{\lambda t^{\frac{\alpha}{2}}}\right) d\xi\right] \\ &= \frac{f_0}{2}\left[-W\left(-\frac{x}{\lambda t^{\frac{\alpha}{2}}}, -\frac{\alpha}{2}, 1\right) + 2 - W\left(-\frac{x}{\lambda t^{\frac{\alpha}{2}}}, -\frac{\alpha}{2}, 1\right)\right] = f_0\left[1 - W\left(-\frac{x}{\lambda t^{\frac{\alpha}{2}}}, -\frac{\alpha}{2}, 1\right)\right]. \end{aligned}$$

Analogously we can check that

$$w(x,t) = g_0\, W\left(-\frac{x}{\lambda t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) \tag{13}$$

is a solution for the fractional diffusion problem

$$\begin{cases} {}_0D^\alpha w(x,t) = \lambda^2 \dfrac{\partial^2 w}{\partial x^2}(x,t) & 0 < x < \infty,\ t > 0,\ 0 < \alpha < 1, \\ w(x,0) = 0 & 0 < x < \infty, \\ w(0,t) = g_0 & t > 0. \end{cases} \tag{14}$$

3 The Two-Phase Fractional Lamé-Clapeyron-Stefan Problem

Hereinafter we will call D^α to the fractional derivative in the Caputo sense of extreme $a = 0$, ${}_0D^\alpha$.

Let us return to problem (1). Taking into account the previous section and the method developed in [23], the following explicit solution is obtained.

Theorem 3.1. *An explicit solution for the two-phase Lamé-Clapeyron-Stefan problem (1) is given by*

$$\begin{cases} u_2(x,t) = u_0 - (u_0 - u_m)\dfrac{1 - W\left(-\frac{x}{\lambda_2 t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)}{1 - W\left(-\xi\lambda, -\frac{\alpha}{2}, 1\right)} \\ u_1(x,t) = u_i + (u_m - u_i)\dfrac{W\left(-\frac{x}{\lambda_1 t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)}{W\left(-\xi, -\frac{\alpha}{2}, 1\right)} \\ s(t) = \xi\lambda_1 t^{\alpha/2} \end{cases} \tag{15}$$

where ξ is a solution to the equation

$$F(x) = \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} x, \; x > 0 \tag{16}$$

and the function $F : \mathbb{R}_0^+ \to \mathbb{R}$ is defined by

$$F(x) = \frac{k_2(u_0 - u_m)}{\rho l \lambda_1 \lambda_2} F_1(\lambda x) - \frac{k_1(u_m - u_i)}{\rho l \lambda_1^2} F_2(x) \tag{17}$$

with

$$F_1(x) = \frac{M_{\alpha/2}(x)}{1 - W\left(-x, -\frac{\alpha}{2}, 1\right)}, \; F_2(x) = \frac{M_{\alpha/2}(x)}{W\left(-x, -\frac{\alpha}{2}, 1\right)}, \; \lambda = \frac{\lambda_1}{\lambda_2} > 0. \tag{18}$$

Proof. The following solution is proposed

$$\begin{cases} u_2(x,t) = A + B\left[1 - W\left(-\frac{x}{\lambda_2 t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right] \\ u_1(x,t) = C + D\left[1 - W\left(-\frac{x}{\lambda_1 t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right] \\ s(t) = \xi \lambda_1 t^{\alpha/2} \end{cases} \tag{19}$$

where A, B, C, D and $\xi > 0$ must be determined.
According with the results in the previous section and the linearity of the fractional derivative D^α, functions u_2 and u_1 are solutions of the fractional diffusion equations (1-i) and (1-ii) respectively.
From conditions (1-iv) and (1-vi) we have,

$$u_2(0,t) = A + B\left[1 - W\left(0, -\frac{\alpha}{2}, 1\right)\right] = u_0 \tag{20}$$

$$u_2(s(t),t) = u_0 + B\left[1 - W\left(-\xi\frac{\lambda_1}{\lambda_2}, -\frac{\alpha}{2}, 1\right)\right] = u_m. \tag{21}$$

and therefore we obtain:

$$A = u_0, \quad \text{and} \quad B = -\frac{u_0 - u_m}{1 - W\left(-\xi\lambda, -\frac{\alpha}{2}, 1\right)}. \tag{22}$$

So,

$$u_2(x,t) = u_0 - (u_0 - u_m)\frac{1 - W\left(-\frac{x}{\lambda_2 t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)}{1 - W\left(-\xi\lambda, -\frac{\alpha}{2}, 1\right)} < u_0, \tag{23}$$

or equivalently

$$u_2(x,t) = u_m + (u_0 - u_m)\frac{W\left(-\frac{x}{\lambda_2 t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) - W\left(-\xi\lambda, -\frac{\alpha}{2}, 1\right)}{1 - W\left(-\xi\lambda, -\frac{\alpha}{2}, 1\right)}. \tag{24}$$

Taking into account the results in Proposition 2.2, (23) and (24) it is easy to see that

$$u_m < u_2(x,t) < u_0, \quad 0 < x < s(t), \; t > 0. \tag{25}$$

From conditions (1-v) and (1-iv) we have,

$$u_1(x,0) = C + D\left[1 - W\left(-\infty, -\frac{\alpha}{2}, 1\right)\right] = C + D = u_i, \tag{26}$$

$$u_1(s(t),t) = C + D\left[1 - W\left(-\xi, -\frac{\alpha}{2}, 1\right)\right] = u_m, \tag{27}$$

and therefore we get:

$$C = u_i + \frac{u_m - u_i}{W\left(-\xi, -\frac{\alpha}{2}, 1\right)}, \quad D = -\frac{u_m - u_i}{W\left(-\xi, -\frac{\alpha}{2}, 1\right)}. \tag{28}$$

Accordingly,

$$u_1(x,t) = u_i + (u_m - u_i)\frac{W\left(-\frac{x}{\lambda_1 t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)}{W\left(-\xi, -\frac{\alpha}{2}, 1\right)}, \tag{29}$$

or equivalently

$$u_1(x,t) = u_m - (u_m - u_i)\left[1 - \frac{W\left(-\frac{x}{\lambda_1 t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)}{W\left(-\xi, -\frac{\alpha}{2}, 1\right)}\right]. \tag{30}$$

Taking into account Proposition 2.2, (29) and (30) we obtain

$$u_i < u_1(x,t) < u_m, \quad x > s(t) = \xi\lambda_1 t^{\alpha/2},\ t > 0. \tag{31}$$

In order to determine $\xi > 0$, let us work with the "fractional Lamé-Clapeyron-Stefan condition" (1-iii). From (2) and (5) we have

$$u_{2x}(x,t) = \frac{B}{\lambda_2 t^{\alpha/2}} M_{\alpha/2}\left(\frac{x}{\lambda_2 t^{\alpha/2}}\right), \quad u_{1x}(x,t) = \frac{D}{\lambda_1 t^{\alpha/2}} M_{\alpha/2}\left(\frac{x}{\lambda_1 t^{\alpha/2}}\right),$$

which evaluated on $(s(t),t)$, gives

$$u_{2x}(s(t),t) = \frac{B}{\lambda_2 t^{\alpha/2}} M_{\alpha/2}(\lambda\xi), \quad u_{1x}(s(t),t) = \frac{D}{\lambda_1 t^{\alpha/2}} M_{\alpha/2}(\xi). \tag{32}$$

Taking into account that ([22])

$$D^\alpha(t^\beta) = \frac{\Gamma(\beta+1)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha} \quad \text{if } \beta > -1,$$

it results that

$$D^\alpha s(t) = D^\alpha(\xi\lambda_1 t^{\alpha/2}) = \lambda_1\xi D^\alpha(t^{\alpha/2}) = \lambda_1\xi\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} t^{-\alpha/2}. \tag{33}$$

Replacing (32) and (33) in the fractional condition (1-iii), we get for the unknown coefficient $\xi > 0$ the following equation:

$$k_1 u_{1x}(s(t),t) - k_2 u_{2x}(s(t),t) = \rho l\, D^\alpha s(t) \Leftrightarrow$$

$$-k_1\frac{u_m-u_i}{1-W\left(-\xi,-\frac{\alpha}{2},1\right)}\frac{1}{\lambda_1 t^{\alpha/2}}M_{\alpha/2}(\xi)+k_2\frac{u_0-u_m}{1-W\left(-\lambda\xi,-\frac{\alpha}{2},1\right)}\frac{1}{\lambda_2 t^{\alpha/2}}M_{\alpha/2}(\lambda\xi)$$

$$=\rho l\lambda_1\xi\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}t^{-\alpha/2}\Leftrightarrow$$

$$\frac{k_2(u_0-u_m)}{\lambda_2}\frac{M_{\alpha/2}(\lambda\xi)}{1-W\left(-\lambda\xi,-\frac{\alpha}{2},1\right)}-\frac{k_1(u_m-u_i)}{\lambda_1}\frac{M_{\alpha/2}(\xi)}{W\left(-\xi,-\frac{\alpha}{2},1\right)}=\xi\rho l\lambda_1\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}\Leftrightarrow$$

$$\frac{k_2(u_0-u_m)}{\rho l\lambda_1\lambda_2}F_1(\lambda\xi)-\frac{k_1(u_m-u_i)}{\rho l\lambda_1^2}F_2(\xi)=\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}\xi\Leftrightarrow$$

$$\Leftrightarrow F(\xi)=\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}\xi\,; \tag{34}$$

that is, the equation (16) holds, where F, F_1 and F_2 where defined in (17) and (18) respectively.

In order to guarantee the existence of a solution of the equation (16), we will study the behavior of the functions F, F_1 and F_2. From Proposition 2.2 and (8), it results that

$$F_1 \text{ is a positive decreasing function, } F_1(0^+)=\infty, \text{ and } F_1(+\infty)=0 \tag{35}$$

and

$$F_2 \text{ is a positive function and } F_2(0)=\frac{1}{\Gamma(1-\alpha/2)}. \tag{36}$$

Let us prove that

$$F_2(+\infty)=+\infty. \tag{37}$$

In [34, 35] the asymptotic expansion for $x\to\infty$ of the Wright function was studied, and an interesting summary of these results can be founded in [32], from where we can say that if $\alpha\in(0,1)$ we have

$M_{\alpha/2}(x)=\left(\frac{\alpha}{2}x\right)^{-\frac{1-\alpha}{2-\alpha}}\exp\left\{\left(1-\frac{2}{\alpha}\right)\left(\frac{\alpha}{2}x\right)^{\frac{1}{1-\alpha/2}}\right\}\left[a_0+\mathcal{O}\left(\left(\frac{\alpha}{2}x\right)^{-\frac{1}{1-\alpha/2}}\right)\right],\quad a_0=\frac{1}{\sqrt{2\pi(1-\alpha/2)}}$

Therefore

$$M_{\alpha/2}(x)\sim b(\alpha)x^{-\frac{1-\alpha}{2-\alpha}}\exp\left\{-c(\alpha)x^{\frac{1}{1-\alpha/2}}\right\} \tag{38}$$

where $b(\alpha)=\frac{1}{\sqrt{2\pi(1-\alpha/2)}}\left(\frac{\alpha}{2}\right)^{-\frac{1-\alpha}{2-\alpha}}>0$ and $c(\alpha)=\frac{2-\alpha}{2}\left(\frac{\alpha}{2}\right)^{\frac{1}{1-\alpha/2}}>0$.

On the other hand

$$W\left(-x,-\frac{-\alpha}{2},1\right)=\left(\frac{\alpha}{2}x\right)^{-\frac{1}{2-\alpha}}\exp\left\{\left(1-\frac{2}{\alpha}\right)\left(\frac{\alpha}{2}x\right)^{\frac{1}{1-\alpha/2}}\right\}\left[a_0+\mathcal{O}\left(\left(\frac{\alpha}{2}x\right)^{-\frac{1}{1-\alpha/2}}\right)\right],$$

therefore

$$W\left(-x,-\frac{-\alpha}{2},1\right)\sim d(\alpha)x^{-\frac{1}{2-\alpha}}\exp\left\{-c(\alpha)x^{\frac{1}{1-\alpha/2}}\right\} \tag{39}$$

where $d(\alpha)=\frac{1}{\sqrt{2\pi(1-\alpha/2)}}\left(\frac{\alpha}{2}\right)^{-\frac{1}{2-\alpha}}>0$.

From (38) and (39), we have

$$F_2(x) \sim \left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2-\alpha}} x^{\frac{\alpha}{2-\alpha}}, \quad \text{as } x \to \infty \tag{40}$$

and then (37) holds.

Now, from Proposition 2.2 and properties (36) and (37), we can ensure that

$$F \text{ is a continuous function, } F(0^+) = +\infty \quad \text{and} \quad F(+\infty) = -\infty. \tag{41}$$

Therefore, there exists at least one $\xi > 0$ which is solution of the equation (16). Finally, we are able to state that (15) is a solution to the free boundary problem (1).

■

Remark 3.2. *We will denote (15)-(16) as the generalized Neumann solution of the two-phase fractional Lamé-Clapeyron-Stefan problem (1).*

Theorem 3.3. *The limit when $\alpha \nearrow 1$ of the generalized Neumann solution (15)-(16) is the classical Neumann solution for the two-phase Lamé-Clapeyron-Stefan problem.*

Proof. We denote u_1^α, u_2^α and s_α as the functions defined in (15), and ξ_α the solution of the equation (16) for each $0 < \alpha < 1$. Now, we analyze the convergence of (15) when $\alpha \nearrow 1$. Applying Proposition 2.3 we obtain

$$\lim_{\alpha \nearrow 1} u_1^\alpha(x,t) = u_i + (u_m - u_i) \lim_{\alpha \nearrow 1} \frac{W\left(-\frac{x}{\lambda_1 t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)}{W\left(-\xi, -\frac{\alpha}{2}, 1\right)} = u_i + (u_m - u_i) \frac{\operatorname{erfc}\left(\frac{x}{2\lambda_1\sqrt{t}}\right)}{\operatorname{erfc}\left(\frac{\xi}{2}\right)} \tag{42}$$

$$\lim_{\alpha \nearrow 1} u_2^\alpha(x,t) = u_0 - (u_0 - u_m) \lim_{\alpha \nearrow 1} \frac{1 - W\left(-\frac{x}{\lambda_2 t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)}{1 - W\left(-\xi\lambda, -\frac{\alpha}{2}, 1\right)} = u_0 - (u_0 - u_m) \frac{\operatorname{erf}\left(\frac{x}{2\lambda_2\sqrt{t}}\right)}{\operatorname{erf}\left(\frac{\xi\lambda}{2}\right)} \tag{43}$$

$$\lim_{\alpha \nearrow 1} s_\alpha(t) = \lim_{\alpha \nearrow 1} \xi_\alpha \lambda_1 t^{\alpha/2} = \xi_1 \lambda_1 \sqrt{t} = 2\mu\lambda_1\sqrt{t} \tag{44}$$

where $\mu = \frac{\xi_1}{2}$ is a solution to the equation

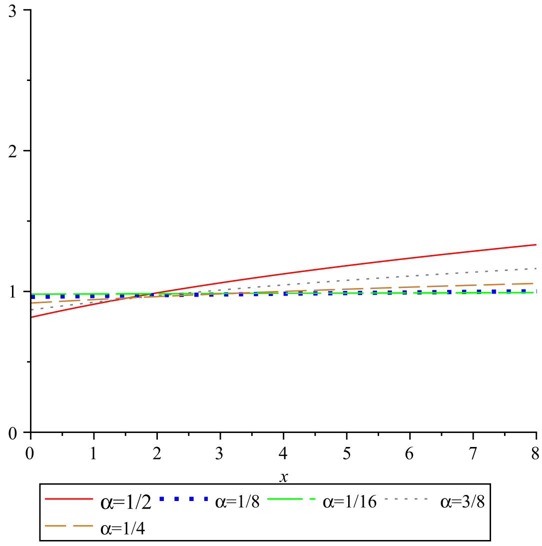
$$\frac{k_2(u_0 - u_m)}{\rho l \lambda_1 \lambda_2} \frac{\exp\{-\lambda^2\mu^2\}}{\sqrt{\pi}\operatorname{erf}(\lambda\mu)} - \frac{k_1(u_m - u_i)}{\rho l \lambda_1^2} \frac{\exp\{-\mu^2\}}{\sqrt{\pi}\operatorname{erfc}(\mu)} = \mu, \quad \mu > 0. \tag{45}$$

The expressions (42)-(45) give us the classical Neumann solution, given in [5, 27, 31], for the two-phase Lamé-Clapeyron-Stefan problem defined by the following equations, and constant boundary and initial conditions:

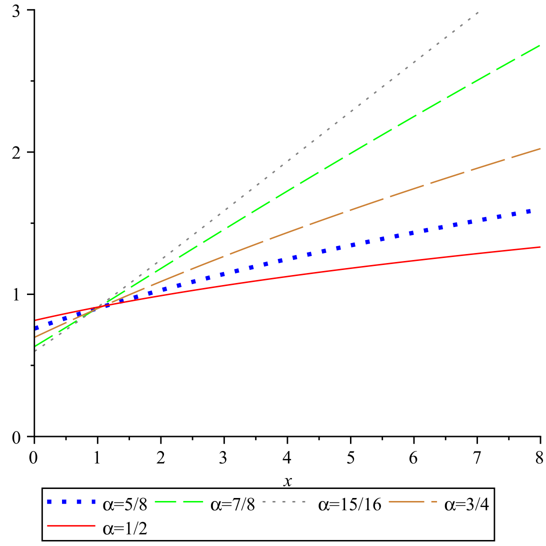
$$\begin{cases} \frac{\partial u_2}{\partial t}(x,t) = \lambda_2^2 \frac{\partial^2 u_2}{\partial x^2}(x,t) & 0 < x < s(t),\ t > 0, \\ \frac{\partial u_1}{\partial t}(x,t) = \lambda_1^2 \frac{\partial^2 u_1}{\partial x^2}(x,t) & s(t) < x < \infty,\ t > 0, \\ k_1 u_{1x}(s(t),t) - k_2 u_{2x}(s(t),t) = \rho l\, \dot{s}(t) & t > 0. \\ u_1(s(t),t) = u_2(s(t),t) = u_m & t > 0, \\ u_1(x,0) = u_i & 0 < x < \infty \\ u_2(0,t) = u_0 & t > 0. \\ s(0) = 0 \end{cases} \tag{46}$$

■

Remark 3.4. *It is an open problem to prove that F_2 is an increasing function, which is a sufficient condition to could ensure the uniqueness of the solution to equation (16). By using Maple we show below some graphs for different values of $0 < \alpha < 1$, from which it can be seen that F_2 is an increasing function on $\mathbb{R}^+$.*



(a) F_2 is an increasing function for $\alpha = 1/16, 1/8, 1/4, 3/8$ and $1/2$.



(b) F_2 is an increasing function for: $\alpha = 1/2, 5/8, 7/8, 3/4$ and $15/16$.

4 Conclusions

By using the Wright and Mainardi functions and the fractional error function $1 - W(-x, -\alpha/2, 1)$, a generalized Neumann solution for the two-phase fractional Lamé-Clapeyron-Stefan problem is obtained for each $0 < \alpha < 1$. Moreover, the classical Neumann solution is recovered through the limit when $\alpha \nearrow 1$.

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