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ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF THE DEAD-CORE PROBLEM

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1. INTRODUCTION

LET $p < 1$ and consider the equation

$$u_t - u_{xx} + \lambda^2 u_+^p = 0. \tag{1.1}$$

This equation originates from the study of a class of models for the reaction–diffusion processes of a gas inside a chemical reactor [2, 8].

From the mathematical point of view the major interest of equation (1.1) lies in the competition of the linear diffusion and the nonlinear "fast" ($p < 1$) reaction term.

To illustrate this phenomenon let us consider a model problem, i.e. equation (1.1) in the first quadrant $x > 0$, $t > 0$, with initial and boundary conditions

$$u(x, 0) = 0, \qquad x > 0; \tag{1.2}$$

$$u(0, t) = 1, \qquad t > 0. \tag{1.3}$$

It is easy to construct a stationary solution of (1.1), satisfying (1.3), which is the unique bounded C^1 solution [1]. This is given by (we take $\lambda > 0$):

$$u^*(x) = \left[1 - \frac{\lambda x}{L}\right]_+^{2/(1-p)}, \qquad 0 < x, \qquad L = \frac{\sqrt{2(1+p)}}{1-p}, \tag{1.4}$$

where $[s]_+ = \max\{0, s\}$.

The stationary solution has compact support, contrary to what happens in the linear absorption case ($p = 1$), where the stationary solution is $u^*(x) = e^{-\lambda x}$.

As a consequence of the comparison principle for equation (1.1), see for instance [3], the solution $u(x, t)$ of (1.1)–(1.3) satisfies

$$0 \le u(x, t) \le u^*(x), \qquad x > 0, t > 0. \tag{1.5}$$

This means that $u(\cdot, t)$ has compact support for any $t > 0$. In fact the supremum of the support of $u(\cdot, t)$ is a free boundary $s(t)$, moving with finite speed (except for $t = 0$) and its behavior for small t has been analyzed in [5], where a lower bound and an asymptotic ($t \to 0^+$) expansion of $s(t)$ are given. An explicit and sharp upper bound for small t can also be obtained by means of the comparison principle [6], namely, for $\lambda = 1$,

$$s(t) \leq 2\left[\frac{t\log(1/t)}{1-p}\right]^{1/2} + \frac{[2(1+p)]^{1/2}}{1-p}t^{1/2}\left[\frac{1-p}{\pi\log(1/t)}\right]^{(1-p)/4}, \qquad 0 < t. \tag{1.6}$$

This estimate is of course ineffective for large t when we know that $s(t) \to L/\lambda$.

The first motivation for this paper was to give an estimate of how fast the free boundary $s(t)$ tends to its limit L/λ as $t \to +\infty$. The estimate we get implies that this convergence is exponentially fast in time.

The proof is based on the construction of a subsolution for (1.1), which converges to the stationary solution. The same construction gives also supersolutions for (1.1). This makes it possible to obtain an exponential estimate for the decay of the solution of the Dirichlet problem for equation (1.1) in $0 < x < a$, with $a > 2L/\lambda$, boundary data $u(0, t) = u(a, t) = 1$, and initial datum $u(x, 0) = 1$, which is the one dimensional version of the so-called "dead-core" problem.

The next two sections are devoted to the construction of these sub- and supersolutions for the equation in the more general form

$$u_t - (\phi(u))_{xx} + f(u) = 0, \qquad x > 0,\ t > 0 \tag{1.7}$$

with conditions

$$\phi(u(0, t)) = 1, \qquad t > 0; \tag{1.8}$$

$$u(x, 0) = u_0(x), \qquad x > 0. \tag{1.9}$$

Concerning the functions ϕ and f we assume that they satisfy the following assumptions:

$$\phi \in C^0(\mathbb{R}) \cap C^2(\mathbb{R}\setminus\{0\}),\ \phi(0) \geq 0,\ \phi'(s) > 0 \text{ for } s > 0; \tag{Φ}$$

(a typical ϕ is $\phi(u) = u^m$, $m > 0$)

$$f \in C^0(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{0\}),\ f(0) = 0,\ f'(s) > 0 \text{ for } s > 0; \tag{F}$$

(a typical f is $f(u) = u^p$, $0 < p < 1$).

2. REMARKS ON THE STATIONARY SOLUTIONS

In this section we deduce some properties of the solutions of the following family (S_λ) of stationary problems:

$$(\phi(u))_{xx} - \lambda^2 f(u) = 0, \qquad x > 0; \tag{2.1}$$

$$\phi(u(0)) = 1. \tag{2.2}$$

For $\lambda = 1$ this is the stationary problem for equation (1.7). Letting $g(v) = f(\phi^{-1}(v))$, problems (S_λ) transform into the problems (S'_λ):

$$v_{xx} - \lambda^2 g(v) = 0, \qquad x > 0; \tag{2.1'}$$

$$v(0) = 1; \tag{2.2'}$$

with $v(x) = \phi(u(x))$.

As it is well known, see for instance [4], (S'_λ) has a solution with compact support iff the function g satisfies the following hypothesis (H_1):

$$\int_{0^+}^{v} \frac{d\xi}{\sqrt{G(\xi)}} < +\infty,\ v > 0, \qquad \text{with } G(\xi) = \int_0^{\xi} g(s)\,ds. \tag{H$_1$}$$

Remark. If $\phi(u) = u^m$ and $f(u) = u^p$, hypothesis (H_1) simply means that $m > p$.

We assume that hypothesis (H_1) holds in the following.
We indicate by $u_\lambda^*(x)$ the solution of (S_λ) and by $L > 0$ the length of the support of $u_1^*(x)$, i.e.

$$L = \sup\{x\colon u_1^*(x) > 0\}, \qquad (L < +\infty, \text{ because of } (H_1)). \tag{2.3}$$

The functions $u_\lambda^*(x)$ are related by the following scaling property:

$$u_\lambda^*(x) = u_1^*(\lambda x). \tag{2.4}$$

Setting $U(x/L) = u_1^*(x)$ then the solution of (S_λ) is given by

$$u_\lambda^*(x) = U\left(\frac{\lambda x}{L}\right). \tag{2.5}$$

In the next section we will make use of the following estimate.

LEMMA 2.1. Let $F(u) = (\int_{0^+}^{u} f(s)\phi'(s)\,ds)^{1/2}$, and assume the following hypothesis holds:

$$\frac{d}{du}F(u) \geq c(\phi, f) > 0 \qquad \text{for } u \in (0, 1), \tag{H$_2$}$$

where $c(\phi, f)$ is a positive constant depending on the functions ϕ and f.
Then the function $H(y) = -U'(y)/f(U(y))$ is bounded from above by a constant $K(\phi, f)$ for $y \in (0, 1)$.

Remark. $H(y)$ bounded implies that the derivative u_{1x}^* is uniformly bounded in $[0, +\infty)$ by some constant depending on ϕ and f.

Remark. If $\phi(u) = u^m$ and $f(u) = u^p$, assumption (H_2) is equivalent to

$$m + p \leq 2, \tag{H$'_2$}$$

in particular if $m = 1$, then (H'_2) is implied by (H_1).

Proof. We have just to compute $H(y)$. From the definition of $U(y)$ we have $H(y) = -\theta'(x)L/f(\theta(x))$ where, for the sake of simplicity, $\theta(x) = u_1^*(x)$, and $x = Ly$.
θ' can be expressed in closed form in terms of f and ϕ. In fact, since θ solves $(\phi(u))_{xx} - f(u) = 0$, the function $\eta([L - x]_+) = \phi(\theta(x))$ solves $\eta_{xx} - f(\phi^{-1}(\eta)) = 0$, $\eta(0) = 0$, and then

$\eta(s) = \psi^{-1}(s)$ where

$$\psi(s) = \int_{0^+}^{s} \frac{d\xi}{\sqrt{2G(\xi)}},$$

G as in (H_1). Then

$$\theta'(x) = \frac{\eta'([L-x]_+)}{\phi'(\theta(x))} = \frac{-\sqrt{2G(\eta)}}{\phi'(\theta)}$$

and finally

$$H(y) = L\frac{-\sqrt{2G(\phi^{-1}(\theta(x)))}}{\phi'(\theta(x))f(\theta(x))} = \sqrt{2}L\left(\int_{0^+}^{\phi^{-1}(\theta(x))} f(\phi^{-1}(s))\,ds\right)^{1/2} /(\phi'(\theta(x))f(\theta(x)))$$

$$= \sqrt{2}L\left(\int_{0^+}^{\theta(x)} f(s)\phi'(s)\,ds\right)^{1/2} /(\phi'(\theta(x))f(\theta(x))) = \frac{L}{\sqrt{2}}\frac{1}{F'(\theta(x))}, \qquad \theta(x) \in (0, 1). \quad \blacksquare$$

3. CONSTRUCTION OF SUB- AND SUPERSOLUTIONS

Let $r(t)$ be a positive function defined for any $t \geq 0$, and let $h(x, t)$ be defined by

$$\begin{aligned} h(x,t) &= U\left(\frac{x}{r(t)}\right), \qquad t > 0, \qquad 0 < x < r(t); \\ h(x,t) &= 0, \qquad t > 0, \qquad x > r(t). \end{aligned} \tag{3.1}$$

Then for any fixed t, the function $h(\cdot, t)$ is a solution of problem (S_λ) with $\lambda = L/r(t)$.

We want to prove that, with an appropriate choice of the function $r(t)$, $h(x, t)$ turns out to be either a subsolution or a supersolution for equation (1.7).

THEOREM 3.1. Let $r(t)$ be given by

$$r^2(t) = L^2 - (L^2 - r^2(0))e^{-2t/K}, \tag{3.2}$$

where $K = K(\phi, f)$ is the constant in lemma 2.1.

Then $h(x, t)$ is either a subsolution or a supersolution if $r(0) < L$ or $r(0) > L$ respectively.

Proof. Let us compute

$$\mathcal{L}(h) = h_t - (\phi(h))_{xx} + f(h).$$

Because of the definition of h, $(\phi(h(x, t)))_{xx} = L^2/(r^2(t))f(h)$, so that

$$\begin{aligned} \mathcal{L} &= h_t(x,t) + \left(1 - \frac{L^2}{r^2(t)}\right) f(h(x,t)) \\ &= \frac{f(h(x,t))}{r^2(t)}\left\{\frac{-U'(x/r(t))}{f(U(x/r(t)))} x\dot{r}(t) + r^2(t) - L^2\right\} \end{aligned} \tag{3.3}$$

for any $x \in (0, r(t))$, and

$$\mathcal{L}(h) = 0, \qquad t > 0, \qquad x > r(t). \tag{3.4}$$

Now if $\dot{r}(t) > 0$ and (H_2) holds, we have

$$\mathcal{L}(h) \leq r^2(t) f(h(x,t)) \{K(\phi, f) r(t) \dot{r}(t) + r^2(t) - L^2\}, \qquad t > 0,\ 0 < x < r(t). \tag{3.5}$$

Taking $r(t)$ as in (3.2) we have

$$\mathcal{L}(h(x,t)) \leq 0, \qquad t > 0,\ x > 0 \tag{3.6}$$

and the condition $\dot{r}(t) > 0$ is satisfied if $r(0) < L$.

This means that $h(x, t)$ is a subsolution of (1.7), with boundary datum $h(0, t) = 1$, $t > 0$.

On the contrary, if $r(0) > L$, then $\dot{r}(t) < 0$ and inequalities (3.5) and (3.6) are both reversed, i.e. $h(x, t)$ is a supersolution.

Remark. If $\phi(u) = u^m$ and $f(u) = u^p$, one can be more precise about the value of the constant $K(\phi, f)$, namely [7]

$$\tfrac{1}{2} K(\phi, f) = (m - p)\beta$$

where

$$\beta = \frac{(1+\gamma)^{(1+\gamma)}}{\gamma^\gamma}, \qquad \gamma = \frac{2-(m+p)}{m-p}, \qquad \text{if } \gamma > 0; \qquad \beta = 1 \text{ if } \gamma = 0.$$

(remember that (H_2) implies $2 \geq m + p$).

As a major consequence of theorem 3.1 we have the following.

COROLLARY 3.2. Let $u(x, t)$ be the solution of (1.7)–(1.9), and suppose that $\lambda_2 \leq 1$, and $\lambda_1 \geq 1$ exist such that

$$u^*_{\lambda_1}(x) \leq u_0(x) \leq u^*_{\lambda_2}(x) \tag{3.7}$$

then there exist two constants C_1, C_2, depending on ϕ, f and on λ_1, λ_2, such that

$$|u(x,t) - u^*_1(x)| \sim C_1 e^{-C_2 t} \qquad \text{as } t \to +\infty.$$

The free boundary $s(t)$ of $u(x, t)$ satisfies

$$|s(t) - L| \sim C_1 e^{-C_2 t} \qquad \text{as } t \to +\infty.$$

Proof. Apply the comparison principle using the subsolution such that $h_1(x, 0) = u^*_{\lambda_1}(x)$, and the supersolution such that $h_2(x, 0) = u^*_{\lambda_2}(x)$, then

$$|u(x,t) - u^*_1(x)| \leq |h_1(x,t) - h_2(x,t)| = \left| u^*_1\left(\frac{Lx}{r_1(t)}\right) - u^*_1\left(\frac{Lx}{r_2(t)}\right) \right|$$

$$\leq \sup |u^*_{1x}| Lx \left| \frac{1}{r_1(t)} - \frac{1}{r_2(t)} \right|.$$

But from hypothesis (H_2), u^*_{1x} is bounded by some constant depending on ϕ and f, times the function $f(u^*_1)$. The estimate then follows from the expression of the r's.. The statement about the free boundary is a consequence of the form of the sub- and supersolutions.

4. APPLICATION TO THE DEAD-CORE PROBLEM

The sub- and supersolutions h's can be used to describe the asymptotic behavior of the solution of the parabolic problem.

$$u_t - (\phi(u))_{xx} + f(u) = 0, \qquad 0 < x < a,\ t > 0, \tag{4.1}$$

with boundary and initial data

$$\phi(u(0,t)) = \phi(u(a,t)) = 1, \qquad t > 0; \tag{4.2}$$

$$u(x,0) = u_0(x), \qquad 0 < x < a, \tag{4.3}$$

where

$$a > 2L, \qquad (L \text{ is defined in (2.3)}) \tag{4.4}$$

$$0 \leq \phi(u_0(x)) \leq 1. \tag{4.5}$$

Because of (4.4), equation (4.1) has a stationary solution $u^*(x)$ corresponding to boundary conditions (4.2), which has a nonvoid dead core, i.e. which vanishes in the interval $D = [L, a - L]$, given by

$$u^*(x) = U\left(\frac{x}{L}\right), \qquad x < L, \qquad u^*(x) = U\left(\frac{a-x}{L}\right), \qquad x > a - L, \qquad \text{and}$$

$$u^*(x) \equiv 0, \qquad L < x < a - L.$$

Here we assume that the function f satisfies also the condition (H_3):

$$\Phi(u) = \int_{0^+}^{u} \frac{d\xi}{f(\xi)} < +\infty, \qquad u > 0. \tag{H$_3$}$$

Notice that if $\phi(u) = u^m$ and $f(u) = u^p$, then (H_1) and (H_2) imply (H_3).

Under assumption (H_3) equation (4.1) has a family of nontrivial spatially homogeneous solutions which vanish in finite time, given by

$$\Psi(t;\bar{t}) = \Phi^{-1}(\bar{t} - t), \qquad 0 < t < \bar{t}, \qquad \Psi(t;\bar{t}) = 0, \qquad t > \bar{t}. \tag{4.6}$$

We denote simply by $\Psi(t)$ the solution corresponding to $\bar{t} = t_1 = \int_{0^+}^{1} d\xi/f(\xi)$, i.e. the spatially homogeneous solution with initial datum $u_0(x) = 1$.

If $f = u^p$, then

$$t_1 = \frac{1}{1-p}, \qquad \text{and} \qquad \Psi(t) = [1 - (1-p)t]_+^{1/(1-p)}.$$

Now we can proceed as in [8], to construct a supersolution of (4.1), greater than the solution of (4.1)–(4.3), which vanishes in finite time in some subinterval of D (in fact it is enough to take the one which vanishes at $a/2$ only).

Using the notation of Section 2, we define $\tilde{u}(x) = u^*_{a/2L}(x)$ for $0 < x < a/2$, and by reflection in $a/2 < x < a$.

This is the "single point dead-core" solution, i.e. $\tilde{u}(x) = 0$ for $x = a/2$ only. Moreover it solves equation (2.1) with $\lambda = a/2L < 1$.

Finally, let $\alpha = 1 - (a/2L)^2$, and define $z(x, t) = \tilde{u}(x) + \Psi(\alpha, t)$. We claim that z is a supersolution, greater than $u(x, t)$.

z is trivially bigger than u for $t = 0$ and on $x = 0$, $x = a$, so it remains only to prove that $\mathcal{L}(z) \geq 0$. Let us compute

$$\mathcal{L}(z) = z_t - (\phi(z))_{xx} + f(z) = \alpha\Psi'(\alpha t) - (\phi(\tilde{u}(x))_{xx} + f(z)$$
$$= -\alpha f(\Psi(\alpha t)) - \left(\frac{a}{2L}\right)^2 f(\tilde{u}(x)) + f(z).$$

Recall that f is monotone increasing, so we have $\max\{f(s), f(t)\} = f(\max\{s, t\}) \leq f(s + t)$ (s and t positive), and then

$$\mathcal{L}(z) \geq -\left(\alpha + \left(\frac{a}{2L}\right)^2\right) f(z) + f(z) = 0.$$

At this point we know that after a finite time $\tilde{t} \leq t_1/\alpha$, the solution $u(x, t)$ is below the stationary solution $\tilde{u}$ corresponding to $\lambda = a/2L$.

To conclude we can apply corollary 3.1 to get an estimate of the convergence of u to u^*. This convergence is exponentially fast in time as in the case of linear diffusion and linear absorption ($f(u) = u$). In our case there is also a time-dependent dead-core whose boundary converges exponentially to the boundary of the dead-core of $u^*(x)$.

REFERENCES

1. BANDLE C. & STAKGOLD I., The formation of the dead-core in parabolic reaction–diffusion equations, *Trans. Am. math. Soc.* **286**, 275–293 (1984).
2. BANDLE C., SPERB R. P. & STAKGOLD I., Diffusion and reaction with monotone kinetics, *Nonlinear Analysis* **8**, 321–333 (1984).
3. BERTSCH M., A class of degenerate diffusion equations with a singular nonlinear term, *Nonlinear Analysis* **7**, 117–127 (1983).
4. DIAZ J. I., Nonlinear partial differential equations and free boundaries, I: Elliptic problems, *Research Notes in Mathematics* No. 106, Pitman, London (1985).
5. GRUNDY R. E. & PELETIER L. A., Short time behavior of a singular solution of the heat equation with absorption, *Proc. R. Soc. Edinb.* **107A**, 271–288 (1987).
6. RICCI R., Note on the short time behavior of the free boundary of the solution of a diffusion–reaction problem, Preprint (1987).
7. RICCI R. & TARZIA D. A., Asymptotic behavior of the solutions of a class of diffusion–reaction equations, in *Free Boundary Problems: Theory & Applications, 11–20 June, 1987*, Irsee/Bavaria, Germany (to appear).
8. STAKGOLD I., Reaction–diffusion problems in chemical engineering, in Nonlinear Diffusion Problems (Edited by *A. Fasano & M. Primicerio*), *Lecture Notes in mathematics* **1224**, 119–152, Springer, Berlin (1986).