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Asymptotic Behaviour of the Solutions of a Class of Diffusion-Reaction Equations

The purpose of this paper is to give an explicit estimate for the asymptotic behaviour of the solution of the problem:

$$\mathcal{L}(u) = u_t - (u^m)_{xx} + u^p = 0, \; x > 0, \; t > 0; \tag{1}$$

$$u(0,t) = 1, \; t > 0; \tag{2}$$

$$u(x,0) = u_0(x) \geq 0, \; x > 0. \tag{3}$$

If $0 < p < m$, it is well known [1], [3] that equation (1) has a stationary solution corresponding to datum (2), which has compact support in $[0, +\infty)$, namely

$$u^*(x) = \left[1 - \frac{x}{L}\right]_+^{\frac{2}{m-p}}, \tag{4}$$

where the constant L is given by

$$L = L(m,p) = \frac{\sqrt{2m(m+p)}}{m-p} \tag{5}$$

The main result we prove here is the existence of super- and sub-solutions for equation (1), which satisfy condition (2) at any time and which exponentially approach the stationary solution $u^*(x)$. The exact statement of this result is the following:

Theorem. If $m + p \leq 2$, then the functions

$$\overline{u}(x,t) = \left[1 - \frac{x}{\overline{s}(t)}\right]_+^{\frac{2}{m-p}}, \tag{6}$$

$$\underline{u}(x,t) = \left[1 - \frac{x}{\underline{s}(t)}\right]_+^{\frac{2}{m-p}}, \tag{7}$$

are respectively a super-solution and a sub-solution of (1) when the functions $\overline{s}(t)$, $\underline{s}(t)$ are defined by

$\overline{s}(t) = r(t)$ if $L_0 > L$ and $\underline{s}(t) = r(t)$ if $0 \leq L_0 < L$, where

$$r(t) = \sqrt{L^2 + (L_0^2 - L^2)\exp[-\beta(m-p)t]}, \tag{8}$$

where the constant β is given by

$$\beta = \frac{(1+\gamma)^{(1+\gamma)}}{\gamma^\gamma}, \; \gamma = \frac{2-(m+p)}{m-p}, \text{ if } \gamma > 0; \; \beta = 1 \text{ if } \gamma = 0. \tag{9}$$

Proof. We define

$$h(x,t)=\left[1-\frac{x}{\rho(t)}\right]_+^{\frac{2}{m-p}}, \tag{10}$$

and we compute $\mathcal{L}(h)$,

$$\mathcal{L}(h)=\left[1-\frac{x}{\rho(t)}\right]_+^{\frac{2p}{m-p}}\frac{1}{\rho^2(t)}\left\{\frac{2}{m-p}x\left[1-\frac{x}{\rho(t)}\right]_+^{\frac{2-(m+p)}{m-p}}\rho'(t)+\rho^2(t)-L^2\right\}. \tag{11}$$

Now we substitute the x-dependent part of (11) by its maximum in the interval $[0,\rho(t)]$, which is given by $\frac{\rho(t)}{\beta}$, with β defined by (9). We thus obtain either a lower bound for $\mathcal{L}(h)$ if ρ' is negative or an upper bound if ρ' is positive. Let us suppose, for instance, that ρ' is negative: then

$$\mathcal{L}(h)\geq\left[1-\frac{x}{\rho(t)}\right]_+^{\frac{2m}{m-p}}\frac{1}{\rho^2(t)}\left\{\frac{2}{m-p}\frac{\rho(t)}{\beta}\rho'(t)+\rho^2(t)-L^2\right\}. \tag{12}$$

Finally we equate to zero the expression in brackets and solve the O. D. E. thus obtained with initial condition

$$\rho(0)=L_0>L. \tag{13}$$

The solution of this O. D. E. is given by (8), and it satisfies the condition $r'<0$. Then we can conclude that the function $h(x,t)$ defined by (6) is a super-solution for the operator $\mathcal{L}$. In the same way, if we take the initial condition L_0 less than L we get a sub-solution. □

We can now apply the comparison principle for the operator $\mathcal{L}$ [2] and we obtain the following result:

Corollary. Suppose $m+p\leq 2$ and that there exists an A such that the initial datum satisfies

$$0\leq u_0(x)\leq\left[1-\frac{x}{A}\right]_+^{\frac{2}{m-p}}. \tag{14}$$

Then the solution to (1) – (3) exponentially converges to the stationary solution $u^*(x)$.

The rate of convergence is faster than $\beta\cdot(m-p)$ since the difference between the upper and lower solution which bounds the actual solution behaves like Const. $\cdot\exp[-\beta(m-p)t]$.

Remark 1. The result of the corollary can be used, together with another estimate for the vanishing of the solution inside the dead core [5], to prove that the solution of the problem $u_t-u_{xx}+u^p=0$, $0<x<A$, $t>0$, $(A>2L)$, with boundary condition $u(0,t)=u(A,t)=1$ and initial condition $u(x,0)=1$ in $0<x<A$, converges exponentially to the steady state solution [4].

Remark 2. To our knowledge, the condition $m+p\leq 2$ has no special interpretation outside of this proof. This leaves open the question of characterizing the asymptotic behaviour of the solution of (1) – (3) when $m+p>2$.

Remark 3. Our results generalize to the case $u_t - (\phi(u))_{xx} + f(u) = 0$, with appropriate conditions on the functions ϕ and f [4].

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