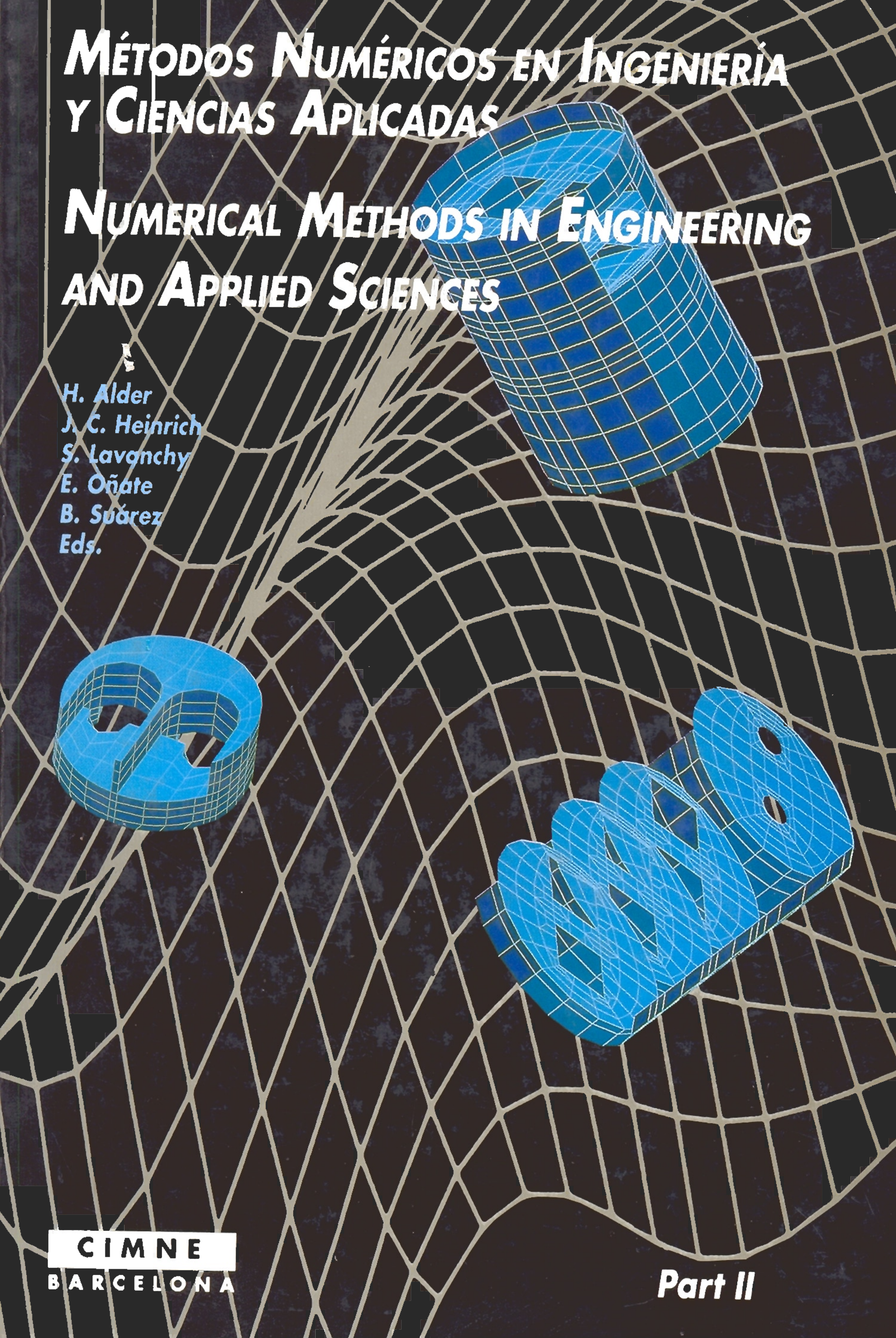
Métodos Numéricos en Ingeniería y Ciencias Aplicadas

Numerical Methods in Engineering and Applied Sciences



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CIMNE
BARCELONA

Part II

Estos dos volúmenes contienen los 136 trabajos presentados al Congreso Internacional sobre Métodos Numéricos en Ingeniería y Ciencias Aplicadas, celebrado en la Universidad de Concepción, Chile, del 16 al 20 de Noviembre de 1992. Los trabajos incluidos constituyen una referencia obligada del alcance y posibilidades actuales de los métodos numéricos en la resolución de problemas científicos y tecnológicos en el ámbito Iberoamericano e internacional.

These two volumes contain the 136 papers presented at the International Congress on Numerical Methods in Engineering and Applied Sciences held at the Universidad de la Concepción, Chile, from 16-20 November 1992. The papers included herein provide an overview of the current state of numerical methods for the solution of scientific and technological problems in both Iberoamerican and international levels.

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NUMERICAL METHODS IN ENGINEERING AND APPLIED SCIENCES
H. Alder, J.C. Heinrich, S. Lavanchy, E. Oñate and B. Suárez (Eds.)
CIMNE, Barcelona 1992

A FREE BOUNDARY MODEL FOR ROOT GROWTH OF CROPS OWING TO ABSORPTION OF ONE MORE MOBILE AND INMOBILE IONS

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SUMMARY

A model for root growth owing to absorption of mobile and inmobile ions through a free boundary problem is studied. The resultant effects from differences in nutrient availability and transport between the root surface and the rhizosphere produced by a active absorption type Michaelis-Menten are studied. The model equations are solved by the balance integral method. The numerical solutions are used to compute growth of root radius. Examples of concentrations for both ions at the root-soil interface curves as a function of root radius and root radius as a function of time are plotted.

1. INTRODUCTION

In recent articles [1,2] a method for compute the root growth owing to absorption of a unique nutrient through a free boundary problem [6] has been proposed. In these articles, the root soil interface $s(t)$ (the root radius) as a function of time and the interface ion concentration $C(s)$ as a function of the interface position $s(t)$ has been computed [1,2] by the quasi-stationary method [3]. Also, the interface concentration $C(s(t),t)$ as a explicit function of $s(t)$ and the time t, as a the root radius $s(t)$ vs. t have been computed [5] through the balance integral method [4]. The goal of the present communication is to compute the interface position $s(t)$ as a function of t and the interface concentration $C_i(s(t),t)$ as a function of $s(t)$ owing to absorption of mobile and inmobile ions through the balance integral method, in order to estimate the effect of different soil-plant parameters on root growth.

2. PHYSICAL MODEL AND GOVERNING EQUATIONS

As described in [1,2,4] it is assumed a vertical cylindrical root summersed in a porous homogeneous and isotropic medium (the soil). Moisture, temperature, and light conditions are assumed maintained at a steady state. Nutrient uptake is assumed occurs at the root surface of the absorption zone, and the root hairs are not considered in the present model. The rates of uptake of ions can be described by Michaelis Menten type equations, and the rates of influx at infinite concentrations (J_{m_i}) and the Michaelis Menten constants (K_{m_i}) are independent of the velocity of soil water at the root (v_o). The nutrient transport occurs via convection and diffusion in the radial direction only (the latter takes place in soil solution phase only). The diffusion coefficients (Di) and the buffer powers b_i are assumed independent of concentrations. No allowance for a change en J_{m_i}, k_i (absorption power $k_i = J_{m_i}/K_{m_i}$), or E_i (efflux) with age is made. Also is assumed that the diffusion coefficients (D_i) are independent of the flux, and the convective velocity of water at root surface is not affected by nutrient concentrations. Production or depletion of nutrient by microbial or other activity is considered null, and the net uptake of nutrients is totally available for root growth.

Although this model can be generalized, for simplicity we consider the root growth owing to absorption of a unique inmobile nutrient (i=1) and a unique mobile nutrient (i=2) for low concentrations ($J_{m_i} \sim k_i C_i$). We assume that R_1 and R_2 are the rhizosphere fixed radius for the inmobile and mobile ions respectively with $R_1 < R_2$. The following free boundary problem we propose below will have a mathematical meaning in the domain $s(t) < r < R_1$, $0 < t < T$ with $s(T) = R_1$. From a physical point of view we shall replace the information of C_2 or the flux on $r = R_1$ (which is a priori an unknown) for $0 < t < T$ by the information of the null flux on R_2. Owing to the approximate method used for the solution, we can define $C_2 = C_2(r,t)$ for $s(t) < r < R_2$, $0 < t < T$. Thus, the governing equations for nutrient fluxes to the root and root growth are given in the following free boundary problem [6] (in cylindrical coordinates) by:

$$
(1)\quad
\begin{array}{lll}
\text{i)} & D_1 C_{1_{rr}} + D_1 (1+\epsilon_1)\dfrac{C_{1_r}}{r} = C_{1_t}, & s(t)<r<R_1,\ 0<t<T, \\
\text{ii)} & C_1(r,0) = \Phi_1(r), & s_o \le r \le R_1, \\
\text{iii)} & C_1(R_1,t) = C_{1\infty}, & 0<t<T, \\
\text{iv)} & D_2 C_{2_{rr}} + D_2 (1+\epsilon_2)\dfrac{C_{2_r}}{r} = C_{2_t}, & s(t)<r<R_2,\ 0<t<T, \\
\text{v)} & C_2(r,0) = \Phi_2(r), & s_o \le r \le R_2, \\
\text{vi)} & -D_2 b_2 C_{2_r}(R_2,t) + v_o C_2(R_2,t) = 0, & 0<t<T, \\
\text{vii)} & D_1 b_1 C_{1_r}(s(t),t) + v_o C_1(s(t),t) = k_1 C_1(s(t),t) - E_1, & \\
\text{viii)} & D_2 b_2 C_{2_r}(s(t),t) + v_o C_2(s(t),t) = k_2 C_2(s(t),t) - E_2, & \\
\text{ix)} & k_1 C_1(s(t),t) + k_2 C_2(s(t),t) - E_1 - E_2 = \left[a_1 C_1(s(t),t) + a_2 C_2(s(t),t)\right] \dot{s}(t), & \\
\text{ix)} & s(0) = s_o, & 0<s_o<R_1<R_2
\end{array}
$$

where: $C_{i_r} = \partial C/\partial r$, $C_{i_{rr}} = \partial^2 C/\partial r^2$, $C_{i_t} = \partial C/\partial t$, r is the position coordinate, t the time, T is the time for which there exists solution, i) is the Cushman equation for the inmobile ion 1, which is a simple application of the principle of conservation of mass (in soil) under steady moisture conditions with the nutrient flux consisting of two components (diffusive and convective) [7]; ii) and iii) are the initial condition and the boundary condition on the rhizosphere radius (constant concentration) for the ion 1, respectively. iv) is the Cushman equation for the mobile ion 2; v) and vi) are the initial condition and the boundary condition (null flux) for the ion 2; vii) and viii) are the interface conditions representing the mass nutrient balance for ions 1 and 2, respectively; which expresses the equality of the rate of net mass absorption of the ion considered in the active kinetics (right hand side) and the incoming total mass and diffusive flux (left hand side), ix) states the same net balance in terms of the free boundary velocity, since a $C_i(s(t),t)\,\dot{s}(t)$ is again the rate of the mass absorption of the ion i; x) is the initial condition for the free boundary s(t) (interface root-soil or root radius). $\dot{s}(t) = ds(t)/dt$ is the interface velocity, a_i is the stoichiometric coefficient for the i-ion, and s_o is the initial radius. The parameters ϵ_i are given by ϵ_i

$= v_o s_o / D_i b_i > 0$. The $\Phi_i(r)$ are the initial concentrations profiles (given by the equations (23) and (24) below).

The three free boundary conditions (1-vii-viii-ix) can be written by:

(2) $C_{1_r}(s(t),t) = g_1(C_1(s(t),t))$, $t > 0$

(3) $C_{2_r}(s(t),t) = g_2(C_2(s(t),t))$, $t > 0$

(4) $\dot{s}(t) = f(C(s(t),t))$, $t > 0$

where functions g and f are given by:

(5) $g_1(C_1) = \frac{1}{D_1 b_1}\left[(k_1 - v_o)\, C_1 - E_1\right]$, defined for $C_1 > 0$,

(6) $g_2(C_2) = \frac{1}{D_2 b_2}\left[(k_2 - v_o)\, C_2 - E_2\right]$, defined for $C_2 > 0$,

(7) $f(C_1,C_2) = \frac{1}{\left[a_1C_1 + a_2C_2\right]}\left[k_1C_1 + k_2C_2 - E_1 - E_2\right]$, defined for $C_1 > 0$, $C_2 > 0$,

which satisfy the following properties:

(8) $g_1(C_1) > 0 \Leftrightarrow C_1 > C_{m_1} = \frac{E_1}{k_1 - v_o}$, with the hipothesis $k_1 > v_o$,

(9) $g_2(C_2) > 0 \Leftrightarrow C_2 > C_{m_2} = \frac{E_2}{k_2 - v_o}$, with the hipothesis $k_2 > v_o$,

(10) $f(C_1,C_2) > 0 \Leftrightarrow k_1C_1 + k_2C_2 > E_1 + E_2$

To solve (1) (that is, to compute $C_1 = C_1(r,t)$ and $C_2 = C_2(r,t)$ (in particular, $C_1 = C_1(s(t),t)$, $C_2 = C_2(s(t),t)$), and the free boundary interface $r = s(t)$ (a priori unknown) we apply the mass balance integral method [4] to the present case for root growth. The solution is found integring the partial differential equations (1-i) and (1-iv) in the variable r on the domains (s(t), R_1) and (s(t), R_2) respectively. Thus:

(11) $$\int_{s(t)}^{R_1} C_{1_t}(r,t)\, dr = D_1 \int_{s(t)}^{R_1} C_{1_{rr}}(r,t) dr + D_1(1+\epsilon_1) \int_{s(t)}^{R_1} \frac{C_{1_r}(r,t)}{r} dr$$

(12) $$\int_{s(t)}^{R_2} C_{2_t}(r,t)\, dr = D_2 \int_{s(t)}^{R_2} C_{2_{rr}}(r,t) dr + D_2(1+\epsilon_2) \int_{s(t)}^{R_2} \frac{C_{2_r}(r,t)}{r}\, dr$$

and we propose:

(13) $C_1(r,t) = \Phi_1(r)\,[1 + \beta(t)(R_1 - r)^2]$ with Φ_1 and β to be determinated

(14) $C_2(r,t) = \Phi_2(r)\,[1 + \gamma(t)(R_2 - r)^2]$ with Φ_2 and γ to be determinated

which depends on the parameters of the system and satisfy the initial and boundary conditions 1-ii) and 1-iii) for the inmobile ion and the conditions 1-

v) and 1-vi) for the mobile ion, that is:

(15) $C_1(r,0) = \Phi_1(r)\ ,\ \forall\ r \in [s_o, R_1] \quad \Leftrightarrow \quad \beta(0) = 0,$

(16) $C_1(R_1,t) = C_{1_\infty}\ ,\ 0 < t < T \quad \Leftrightarrow \quad \Phi_1(R_1) = C_{1_\infty},$

(17) $C_2(r,0) = \Phi_2(r)\ ,\ \forall\ r \in [s_o, R_2] \quad \Leftrightarrow \quad \gamma(0) = 0,$

(18) $-D_2 b_2 C_{2_r}(R_2,t) + v_o C_2(R_2,t) = 0,\ 0 < t < T \Leftrightarrow -D_2 b_2 \Phi_2'(R_2) + v_o \Phi_2(R_2) = 0.$

We denote $\alpha_1 = \alpha_1(t)$ and $\alpha_2 = \alpha_2(t)$ by:

(19) $\alpha_1(t) = C_1(s(t),t) = \Phi_1(s(t))\left[1 + \beta(t)(R_1 - s(t))^2\right]$

(20) $\alpha_2(t) = C_2(s(t),t) = \Phi_2(s(t))\left[1 + \gamma(t)(R_2 - s(t))^2\right]$

which depends on the parameters of the system through $s(t)$, $\Phi_1(s(t))$, $\Phi_2(s(t))$, $\beta(t)$ and $\gamma(t)$. Replacing (19), (20) in Eqs. (11), (12) after some elementary manipulations, the problem (1) reduces to:

$$\int_{s(t)}^{R_1} C_{1_t}(r,t)dr = D_1\left[C_{1_r}(R_1,t) - g_1(\alpha_1(t))\right] + D_1(1+\epsilon_1)\left[\frac{C_{1\infty}}{R_1} - \frac{\alpha_1(t)}{s(t)} + \int_{s(t)}^{R_1} \frac{C_1(r,t)}{r^2}dr\right],$$

(21)

$$\int_{s(t)}^{R_2} C_{2_t}(r,t)dr = D_2\left[C_{2_r}(R_2,t) - g_2(\alpha_2(t))\right] + D_2(1+\epsilon_2)\left[\frac{\Phi_2(R_2)}{R_2} - \frac{\alpha_2(t)}{s(t)} + \int_{s(t)}^{R_2} \frac{C_2(r,t)}{r^2}dr\right]$$

$$\dot{s}(t) = f(\alpha_1(t),\alpha_2(t)), \qquad s(0) = s_o, \qquad t>0.$$

Replacing (13) and (14) in Eq. (21), after some elementary manipulations we obtain the following system of three coupled ordinary differential equations (see **Appendix.**) (valid for the cases $\epsilon \neq 1, 2, 3$):

$$\frac{d\beta(t)}{dt} = \frac{F_1+F_2+D_1(1+\epsilon_1)\,(F_3+F_4+F_5+F_6+F_7)}{(F_8+F_9+F_{10})}, \qquad \beta(0) = 0$$

(22) $$\frac{d\gamma(t)}{dt} = \frac{F_{11}+F_{12}+F_{13}+D_2\,(1+\epsilon_2)\,(F_{14}+F_{15}+F_{16}+F_{17})}{(F_{18}+F_{19}+F_{20})}, \qquad \gamma(0) = 0$$

$$\frac{ds(t)}{dt} = \frac{1}{\left[a_1\alpha_1(t)+a_2\alpha_2(t)\right]}\left[k_1\alpha_1(t)+k_2\alpha_2(t)-E_1-E_2\right], \qquad s(0) = 0$$

where:

(23) $$\Phi_1(r) = C_{1_\infty} + A\left[1-\left[\frac{R_1}{r}\right]^{\epsilon_1}\right], \qquad \text{with: } A = \frac{E_1 - (k_1-v_o)C_{1\infty}}{k_1\left[1-\left[\frac{R_1}{s_o}\right]^{\epsilon_1}\right] - v_o}.$$

(24) $$\Phi_2(r) = FB - \frac{F}{r^{\epsilon_2}}, \qquad \text{with: } B = \frac{1}{R_2^{\epsilon_2}}\left[1+\frac{s_o}{R_2}\right],\quad F = \frac{E_2}{(k_2-v_o)B - \frac{k_2}{s_o^{\epsilon_2}}}.$$

The initial profiles concentration $\Phi_1(r)$, $\Phi_2(r)$ given by the Eqs. (23) and (24) above has been computed by the quasi-stationary method [1,2] for low concentrations and are determinated by the system, similarly to the Cushman's prediction [7]. The functions F_i are given by:

$$F_1 = \frac{D_1 A \epsilon_1}{R_1} + D_1(1+\epsilon_1)\left[\frac{C_{1\infty}}{R_1} - \frac{\Phi_1(s(t))\left[1+\beta(t)\,(R_1-s(t))^2\right]}{s(t)}\right],$$

$$F_2 = -\frac{1}{b_1}\left[(k_1-v_o)\,\Phi_1(s(t))\left[1+\beta(t)(R_1-s(t))^2\right]-E_1\right],$$

$$F_3 = (C_{1\infty}+A)\beta(t)\left[R_1-s(t)\right] - 2\,(C_{1\infty}+A)\,\beta(t)\,R_1 \ln\left[\frac{R_1}{s(t)}\right],$$

$$F_4 = (C_{1\infty}+A)\left[1+\beta(t)R_1^2\right]\left[\frac{1}{s(t)} - \frac{1}{R_1}\right],$$

$$F_5 = -\frac{A\left[R_1^{\epsilon_1} + \beta(t)\,R_1^{\epsilon_1+2}\right]}{(\epsilon_1+1)}\left[\frac{1}{s^{(\epsilon_1+1)}(t)} - \frac{1}{R_1^{(\epsilon_1+1)}}\right],$$

$$F_6 = \frac{2}{\epsilon_1}\,A\,\beta(t)\,R_1^{\epsilon_1+1}\left[\frac{1}{s^{\epsilon_1}(t)} - \frac{1}{R_1^{\epsilon_1}}\right],$$

$$F_7 = \frac{A\,\beta(t)\,R_1^{\epsilon_1}}{(1-\epsilon_1)}\left[\frac{1}{s^{(\epsilon_1-1)}(t)} - \frac{1}{R_1^{(\epsilon_1-1)}}\right],$$

$$F_8 = (C_{1\infty}+A)R_1^2[R_1-s(t)] - \frac{A\,R_1^{\epsilon_1+2}}{(1-\epsilon_1)}\left[R_1^{(1-\epsilon_1)}-s^{(1-\epsilon_1)}(t)\right],$$

$$F_9 = -(C_{1\infty}+A)R_1\left[R_1^2-s^2(t)\right] + \frac{2A\,R_1^{\epsilon_1+1}}{(2-\epsilon_1)}\left[R_1^{(2-\epsilon_1)}-s^{(2-\epsilon_1)}(t)\right],$$

$$F_{10} = \frac{(C_{1\infty}+A)}{3}\left[R_1^3-s^3(t)\right] - \frac{A R_1^{\epsilon_1}}{(3-\epsilon_1)}\left[R_1^{(3-\epsilon_1)}-s^{(3-\epsilon_1)}(t)\right],$$

$$F_{11} = \left[\frac{v_o}{b_2} + \frac{D_2(1+\epsilon_2)}{R_2}\right]\frac{F\,s_o}{R_2^{\epsilon_2+1}},$$

$$F_{12} = -\frac{1}{b_2}\left[(k_2-v_o)\,\Phi_2(s(t))\left[1+\gamma(t)(R_2-s(t))^2\right]-E_2\right],$$

$$F_{13} = -D_2(1+\epsilon_2)\,\frac{\Phi_2(s(t))\left[1+\gamma(t)\,(R_2-s(t))^2\right]}{s(t)},$$

$$F_{14} = FB\left[1+\gamma(t)\,R_2^2\right]\left[\frac{1}{s(t)} - \frac{1}{R_2}\right] - 2FB\gamma(t)R_2 \ln\left[\frac{R_2}{s(t)}\right] + FB\gamma(t)\left[R_2-s(t)\right],$$

$$F_{15} = -\frac{F\left[1+\gamma(t)\,R_2^2\right]}{(\epsilon_2+1)}\left[\frac{1}{s^{(\epsilon_2+1)}(t)} - \frac{1}{R_2^{(\epsilon_2+1)}}\right],$$

$$F_{16} = \frac{2}{\epsilon_2} F\, \gamma(t)\, R_2 \left[\frac{1}{s^{\epsilon_2}(t)} - \frac{1}{R_2^{\epsilon_2}}\right],$$

$$F_{17} = \frac{F\, \gamma(t)}{(1-\epsilon_2)} \left[\frac{1}{s^{(\epsilon_2-1)}(t)} - \frac{1}{R_2^{(\epsilon_2-1)}}\right],$$

$$F_{18} = FB\, R_2^2\, [R_2 - s(t)] - \frac{FR_2^2}{(1-\epsilon_2)} \left[R_2^{(1-\epsilon_2)} - s^{(1-\epsilon_2)}(t)\right],$$

$$F_{19} = -FB\, R_2 \left[R_2^2 - s^2(t)\right] + \frac{2FR_2}{(2-\epsilon_2)} \left[R_2^{(2-\epsilon_2)} - s^{(2-\epsilon_2)}(t)\right],$$

$$F_{20} = \frac{FB}{3} \left[R_2^3 - s^3(t)\right] - \frac{F}{(3-\epsilon_2)} \left[R_2^{(3-\epsilon_2)} - s^{(3-\epsilon_2)}(t)\right].$$

We can remark that for the particular cases $\epsilon = 1$, 2 and 3, can be obtained a similar system to (22) of three ordinary differential equations.

The solution of system (22) is computed numerically by the Runge-Kutta method for a system of ordinary differential equations. The figures 1 to 6 represents theoretical results for the interface concentration $C(s(t),t)$ vs. s and the interface position $s(t)$ vs. t as a function of some characteristic parameters of system soil-plant such as the absorption powers k_1 y k_2 respectively, both related with the choice seed.

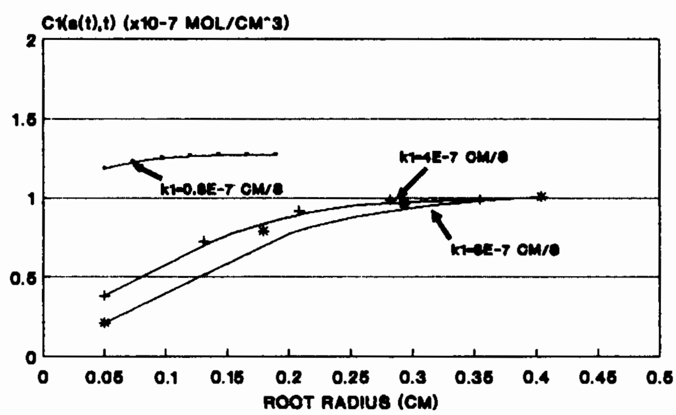

Fig.1: Interface concentration for inmobile ion 1 vs. root radius as a function of absorption power k1

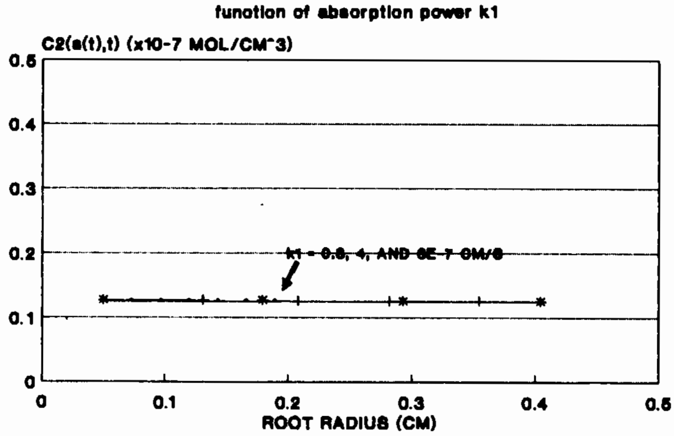

Fig.2: Interface concentration of mobile ion 2 vs. root radius as a function o absorption power k1

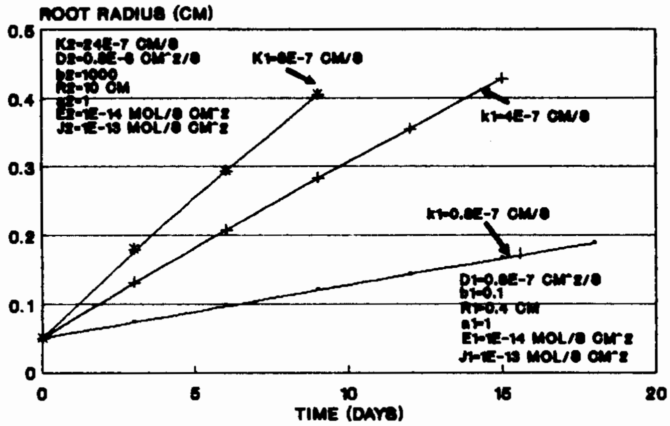

Fig.3: Root radius vs. time as a function of absorption power k1

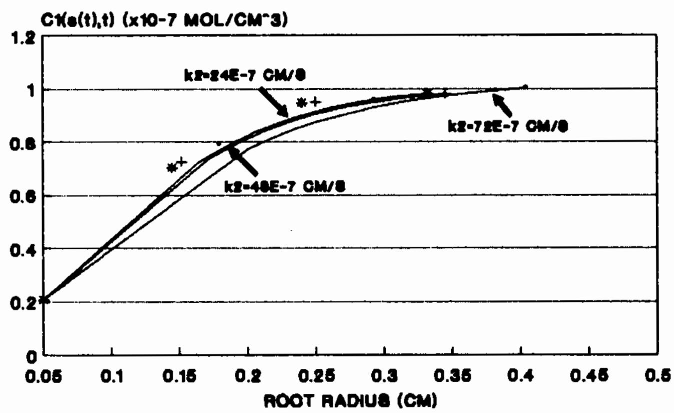

Fig.4: Interface concentration for inmobile ion 1 C1(s(t),t) as a function of absorption power k2

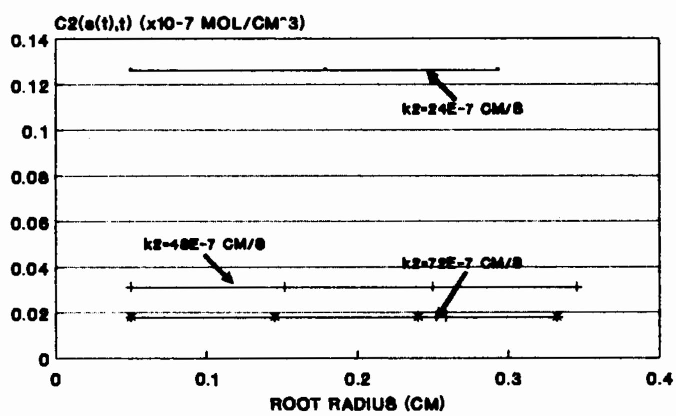

Fig.5: Interface concentration for mobile ion 2 C2(s(t),t) as a function of absorption power k2

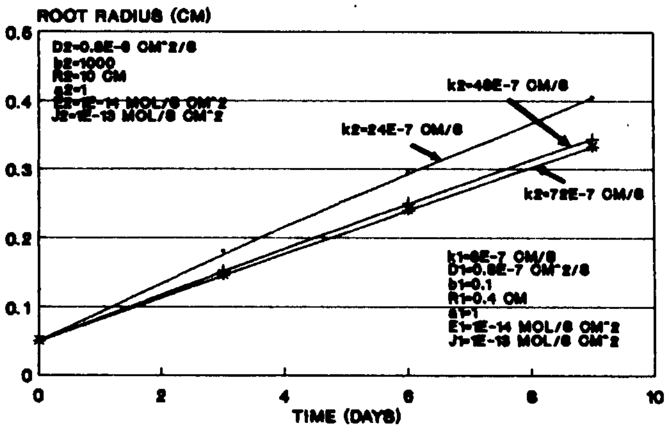

Fig.6: Root radius vs. time as a function of absorption power k2

3. CONCLUSIONS

From the analysis of s(t) and the root absorption $(k_iC_i(s(t),t)-E_i)$ for the results shown in the figures above, we conclude that:

— Root growth increases when k_1 increases and root absorption for ion 1 increases (e.g. Fig. 1 and 3);

— Root growth decreases when k_2 increases and root absorption of ion 2 decreases (e.g. Fig. 5 and 6).

A more exhaustive analysis of numerical results for the remaining parameters, we conclude that:

— Root growth decreases when $C_{1\infty}$ increases, although root absorption of ion 1 increases;

— Root growth do not vary when D_1 or D_2 increases, although root absorption for ion 1 increases and root absorption for ion 2 do not vary;

— Root growth decreases when b_1 or b_2 increases although root absorption for ion 1 do not vary and root absorption for ion 2 decreases;

— Root growth decreases when E_1 or E_2 increases although root absorption for ion 1 decreases and root absorption for ion 2 do not vary;

— Root growth increases when J_1 or J_2 increases and root absorption for ions 1 and 2 increases;

— Root growth increases when R_1 or R_2 increases, although the root absorption for ion 1 decreases and the root absorption for ion 2 increases;

— Root growth increases when v_0 increases and root absorption for ions 1 and 2 increases;

We remark that the qualitative behavior shown for these results above can vary if other different values of parameters for computing the root growth are used.

We can remark that this is a work of basic nature, and represent a qualitative approach for root growth owing to the effect of only one unique inmobile nutrient and a unique mobile nutrient. Moreover, this model can be, firstly, generalized whithout difficulty for to take into account more mobile and inmobile ions, and secondly, approximate to the more realistic situation of taking into account a variable rhizosphere (R_1,R_2 variable) for the root growth.

Moreover, these conclusions are useful to outline more complex models for nutrient transport and root growth. Thus, this method can to provide a very useful qualitative criterion for the crops technology.

4. REFERENCES

1. REGINATO, J.C.; TARZIA, D.A.; CANTERO, A. - On the free boundary problem for the Michaelis-Menten absorption model for root growth, *Soil Science*, 150 (4), 722-729, (1990).
2. REGINATO, J.C.; TARZIA, D.A.; CANTERO, A. - On the free boundary problem for the Michaelis-Menten absorption model for root growth. High concentrations, *Soil Science*, 152 (2), 63-71, (1991)
3. CRANK, J. - *"Free and moving Boundary Problems"*. Clarendon Press, Oxford, 1984.
4. GOODMAN, T.R. - The heat balance integral and its aplications to problems involving a change of phase, *Transf. of the ASME*, 80, 335-342 (1958)
5. REGINATO, J.C.; TARZIA, D.A. - The balance integral method applied to root growth of crops", *International Journal of Engineering Science* (1992) (To appear)
6. TARZIA, D.A. - *"A bibliography on moving-free boundary problems for the heat-diffusion equation. The Stefan problem", Progetto Nazionale M.P.I. "Equazione di Evoluzione e applicazioni fisica-matematiche", with 2528 references,* Firenze, 1988.
7. CUSHMANN, J.H. - Nutrient transport inside and outside the root rhizosphere theory, *Soil Science Society of America J.*,46 (4),704-709, (1982)

5. APPENDIX

Replacing (13) in the first equation of (21), after some manipulations, we obtain:

$$\int_{s(t)}^{R_1} C_{1_t}(r,t)dr = \int_{s(t)}^{R_1} \Phi_1(r)\dot{\beta}(t)[R_1 - r]^2 dr =$$

$$= R_1^2\dot{\beta}(t)\int_{s(t)}^{R_1} \Phi_1(r)dr - 2R_1\,\dot{\beta}(t)\int_{s(t)}^{R_1} r\Phi_1(r)dr + \dot{\beta}(t)\int_{s(t)}^{R_1} \Phi_1(r)r^2dr$$

and, taking into account (23), we obtain:

$$\int_{s(t)}^{R_1} \Phi_1(r)dr = (C_{1_\infty}+A)\int_{s(t)}^{R_1} dr - AR_1^{\epsilon_1}\int_{s(t)}^{R_1} r^{-\epsilon_1}dr =$$

$$=(C_{1_\infty}+A)[R_1 - s(t)] - \frac{AR_1^{\epsilon_1}}{(1-\epsilon_1)}\left[R_1^{(1-\epsilon_1)} - s^{(1-\epsilon_1)}(t)\right], \quad (\text{with } \epsilon \neq 1)$$

$$\int_{s(t)}^{R_1} r\Phi_1(r)dr = (C_{1_\infty}+A)\int_{s(t)}^{R_1} rdr - AR_1^{\epsilon_1}\int_{s(t)}^{R_1} r^{(1-\epsilon_1)}dr =$$

$$=\frac{(C_{1_\infty}+A)}{2}\left[R_1^2 - s^2(t)\right] - \frac{AR_1^{\epsilon_1}}{(2-\epsilon_1)}\left[R_1^{(2-\epsilon_1)} - s^{(2-\epsilon_1)}(t)\right], \quad (\text{with } \epsilon \neq 2)$$

$$\int_{s(t)}^{R_1} r^2\Phi_1(r)dr = (C_{1\infty}+A)\int_{s(t)}^{R_1} r^2 dr - AR_1^{\epsilon_1}\int_{s(t)}^{R_1} r^{(2-\epsilon_1)}dr =$$

$$=\frac{(C_{1\infty}+A)}{3}\left[R_1^3 - s^3(t)\right] - \frac{AR_1^{\epsilon_1}}{(3-\epsilon_1)}\left[R_1^{(3-\epsilon_1)} - s^{(3-\epsilon_1)}(t)\right], \text{(with } \epsilon\neq 3)$$

then:

$$\int_{s(t)}^{R_1} C_{1_t}(r,t)dr = R_1^2\dot{\beta}(t)(C_{1\infty}+A)[R_1 - s(t)] - \frac{AR_1^{\epsilon_1+2}\dot{\beta}(t)}{(1-\epsilon_1)}\left[R_1^{(1-\epsilon_1)} - s^{(1-\epsilon_1)}(t)\right] -$$

$$-2R_1\dot{\beta}(t)\frac{(C_{1\infty}+A)}{2}\left[R_1^2 - s^2(t)\right] +$$

$$+\frac{A2R_1^{\epsilon_1+1}\dot{\beta}(t)}{(2-\epsilon_1)}\left[R_1^{(2-\epsilon_1)} - s^{(2-\epsilon_1)}(t)\right] +$$

$$+\dot{\beta}(t)\frac{(C_{1\infty}+A)}{3}\left[R_1^3 - s^3(t)\right] - \frac{A\dot{\beta}(t)R^{\epsilon_1}}{(3-\epsilon_1)}\left[R_1^{(3-\epsilon_1)} - s^{(3-\epsilon_1)}(t)\right]. \quad \text{(D1)}$$

Similarly, we obtain:

$$\int_{s(t)}^{R_1} \frac{C_1(r,t)}{r^2}dr = \int_{s(t)}^{R_1} \frac{1}{r^2}\left\{C_{1\infty} + A\left[1 - \left(\frac{R_1}{r}\right)^{\epsilon_1}\right]\right\}\left[1+\beta(t)(R_1 - r)^2\right]dr =$$

$$= (C_{1\infty}+A)\left[1+\beta(t)R_1^2\right]\int_{s(t)}^{R_1}\frac{dr}{r^2} - \left[2(C_{1\infty}+A)\beta(t)R_1\right]\int_{s(t)}^{R_1}\frac{dr}{r} +$$

$$+ (C_{1\infty}+A)\beta(t)\int_{s(t)}^{R_1} dr - A\left[1+\beta(t)R_1^{\epsilon_1+2}\right]\int_{s(t)}^{R_1}\frac{dr}{r^{\epsilon_1+2}} +$$

$$+\left[2A\beta(t)R_1^{\epsilon_1+1}\right]\int_{s(t)}^{R_1}\frac{dr}{r^{\epsilon_1+1}} - AR_1^{\epsilon_1}\beta(t)\int_{s(t)}^{R_1}\frac{dr}{r^{\epsilon_1}}$$

$$\int_{s(t)}^{R_1}\frac{C_1(r,t)}{r^2}dr = (C_{1\infty}+A)\left[1+\beta(t)R_1^2\right]\left[\frac{1}{s(t)} - \frac{1}{R}\right] - 2(C_{1\infty}+A)\beta(t)R_1\ln\frac{R_1}{s(t)} -$$

$$-(C_{1\infty}+A)\beta(t)\left[R_1 - s(t)\right] - \frac{A\left[1+\beta(t)R^{\epsilon_1+2}\right]}{(1+\epsilon_1)}\left[\frac{1}{s^{(\epsilon_1+1)}(t)} - \frac{1}{R_1^{(\epsilon_1+1)}}\right] +$$

$$+\frac{2A\beta(t)R_1^{\epsilon_1+1}}{\epsilon_1}\left[\frac{1}{s^{\epsilon}(t)} - \frac{1}{R_1^{\epsilon}}\right] + \frac{AR_1^{\epsilon_1}\beta(t)}{(1-\epsilon_1)}\left[\frac{1}{s^{(\epsilon_1-1)}} - \frac{1}{R_1^{(\epsilon_1-1)}(t)}\right]. \quad \text{(D2)}$$

Finally, replacing (D1) and (D2) in the first equation of (21), after elementary manipulations, we obtain the first equation of system (22). Similarly, for $C_2(r,t)$ we obtain the second equation of system (22).