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Free boundary models for root growth of crops

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ABSTRACT

A model for root growth through a free boundary problems is studied. The effects resulting from differences in nutrient availability and transport of one only mobile or one only relatively immobile ion between the root surface and the rhizosphere produced by a absorption Michaelis-Menten for low and high concentrations are studied. The model equations are solved by two methods: the quasi-stationary method and the balance integral method. The numerical solutions are used to compute radial root growth. Curves of nutrient concentration at the root-soil interface vs. root radius as well as curves representing root radius vs. time as a function of some relevant parameters of system soil-plant are plotted. Theoretical results for radial root growth in the range of low and high concentrations are showed by the two methods, providing a more detailed information the balance integral method.

INTRODUCTION

One of the methods for studying the nutrient uptake by plant roots, which can be a satisfactory method of modelling the plant-root system is by use of the partial differential equation for convective and diffusive flow to a root, and simulation models

for nutrient uptake have frequently been used in the evaluation of the effect of soil and root characteristics on nutrient uptake (Nye and Marriott [1]; Claassen and Barber [2]; Cushman [3,4,5]). These models have only considered nutrient uptake and have supposed an exponential root growth, no taking into account effects of coupling between nutrient transport and root growth. The goal of the present communication is to compute the root radius $s(t)$ as a function of t and the interface concentration $C(s(t),t)$ as a function of $s(t)$ through the quasi-stationary and the balance integral methods with a constant nutrient concentration condition on rhizosphere border for relatively immobile ions and a null flux condition on rhizosphere border valid for mobile ions, in order to estimate the effects of different parameters on root growth for low and high concentrations (not saline conditions).

Because of the rather complicated scenario of the plant-root system, as a first step toward achieving some understanding of the physical and chemical processes involved, and idealized one-dimension diffusion-convection ion transport and radial root growth model was chosen for this study. As described for us (Reginato, Tarzia and Cantero [6,7]) is assumed a vertical cylindrical root emersed in a porous homogeneous and isotropic medium (the soil). Moisture, temperature, and light conditions are assumed maintained at a steady state. Nutrient uptake is assumed occurs at the root absorption zone, and the root hairs are not considered in the present model. The rate of uptake can be described by a Michaelis Menten equation, and the rate of influx at infinite concentration (J_m) and the Michaelis Menten constant (K_m) are independent of the flux velocity of soil solution at the root (v_o). The nutrient transport occurs via convection and diffusion in the radial direction only (the latter takes place in soil solution phase only). The diffusion coefficient (D) and the buffer power (b) are assumed independent of nutrient concentration which implies that there is a linear relation between C and C_1 , where C is the total diffusible ion concentration and C_1 is the ion concentration in soil solution. No allowance for a change in J_m , K_m , or E (efflux) with age is made. Also is assumed that the coefficient diffusion is independent of the flux, and the convective velocity of water at root surface is not affected by nutrient concentration. Production or depletion of nutrient by microbial or

other activity is considered null, and owing to the proposed model not takes into account the energy balance implicit in the root metabolism, we assumed that the nutrient taken up is totally available for root growth. Actually, only a fraction is available for root growth and the remaining nutrient is available for shoot growth. So, we arrive to only qualitative conclusions.

FREE BOUNDARY MODELS AND GOVERNING EQUATIONS

From now on, we shall denote C_1 by C for convenience in the notation. Taking into account the idea of the model used for the shrinking core problem for noncatalytic gas-solid reactions (Wen [8], Tarzia and Villa [9]), we propose to coupler the nutrient transport and the root growth through a free boundary problem (An extensive bibliography for moving and free boundary problems for the heat-diffusion equation is given in Tarzia [10]). The governing equations for convective and diffusive transport of nutrient to the root (Cushman [4]) as well as the governing equations for root growth at the root-soil interface are given in the following free boundary problem to one phase (the soil) (Stefan [11]; Lamé and Clayperon [12]; Carslaw and Jaeger [13]; Crank [14,15]; Tarzia [16]) (In order to simplify the model, without loss of generality, radial growth in cylindrical coordinates is considered) by:

$$\text{i)} \quad DC_{rr} + D(1+\epsilon)\frac{C_r}{r} = C_t, \quad s(t) < r < R, \quad 0 < t < T$$

$$\text{ii)} \quad C(r,0) = \phi(r), \quad s_0 < r < R$$

$$(1) \quad \text{iii)} \quad C(R,t) = C_\infty, \quad \text{or} \quad \text{iv)} \quad -DbC_r(R,t) + v_0C(r,t) = 0, \quad 0 < t < T$$

$$\text{v)} \quad DbC_r(s(t),t) + v_0C(s(t),t) = \frac{kC(s(t),t)}{1 + \frac{kC(s(t),t)}{J_m}} - E = aC(s(t),t)\dot{s}(t), \quad 0 < t < T$$

$$\text{vi)} \quad s(0) = s_0, \quad 0 < s_0 < R$$

where: $C_{rr} = \frac{\partial^2 C}{\partial r^2}$, $C_r = \frac{\partial C}{\partial r}$, $C_t = \frac{\partial C}{\partial t}$, r is the position coordinate, t the time, and T is the time for which the solution exists. Equation i) represents a simple application of the principle of conservation of mass (in soil) under steady moisture conditions with the nutrient flux consisting of two components (diffusive and convective) (Cushman, [4]); Equation ii) is the initial concentration profile; Equation iii) is the boundary condition on the rhizosphere radius taking into constant concentration for relatively immobile ions; Equation iv) is the boundary condition of total flux nul which take into account inter-root competition, which is a more realistic condition when the more mobile nutrients are considered (can input ions but no water); Equation v) representing the mass nutrient balance conditions at the soil-root interface, where $DbC_r(s(t),t) + v_o C(s(t),t)$ is the total flux arriving to root, $kC(s(t),t)/(1 + kC(s(t),t)/J_m)$ is the influx owing to absorption kinetics and E is a constant efflux; Equation vi) is the initial condition for the free boundary $s(t)$; $\dot{s}(t) = ds(t)/dt$ is the rate of radial root growth; a is a stoichiometric coefficient; R is the rhizosphere radius, and s_o is the initial root radius. The parameter ϵ is given by $\epsilon = v_o s_o / Db$ and $\phi(r)$ is the initial concentration profile.

A schematic diagram of model is shown by the Figure 1

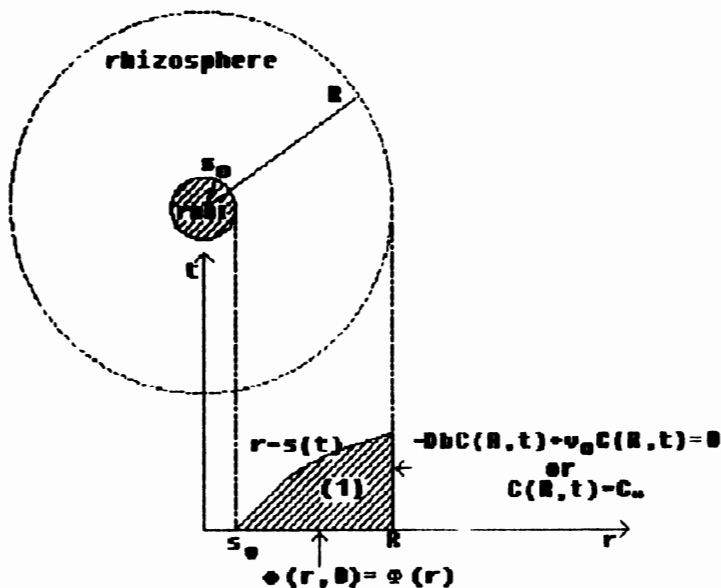


Figure 1.

The free boundary model

The boundary condition

$$DbC_r(R,t) + v_o C(R,t) = 0$$

is valid for mobile ions. The

condition $C(R,t) = C_\infty$ is the

boundary condition for

relatively immobile ions.

The two free boundary conditions (1-iv) can be written by:

$$(2) \quad C_r(s(t), t) = g(C(s(t), t)), \quad t > 0$$

$$(3) \quad \dot{s}(t) = f(C(s(t), t)), \quad t > 0$$

where functions g and f are given by:

$$(4) \quad g(C) = \frac{1}{Db} \left[\left(\frac{k}{1 + \frac{kC}{J_m}} - v_o \right) C - E \right]$$

$$(5) \quad f(C) = \frac{1}{a} \left[\frac{k}{1 + \frac{kC}{J_m}} - \frac{E}{C} \right]$$

which satisfy the following properties:

$$(6) \quad g(C) > 0 \Leftrightarrow C_m^- = \frac{-J_m + \delta^2(J_m - E) - d}{2k} < C < C_m^+ = \frac{-J_m + \delta^2(J_m - E) + d}{2k}$$

$$\text{where } d = \sqrt{[-J_m + \delta^2(J_m - E)]^2 - 4\delta^2 J_m E}$$

with the hypothesis: $\delta^2 = \frac{k}{v_o} > 1$, $\frac{E}{J_m} < \left(1 - \frac{1}{\delta^2}\right)^2$, and

$$(7) \quad f(C) > 0 \Leftrightarrow C > C_p = \frac{E}{k \left[1 - \frac{E}{J_m} \right]}$$

with the hypothesis $\frac{E}{J_m} < 1$

In order to solve the problem (1) are applied the quasistationary (Stefan [11]; Crank [14,15]; Reginato, Tarzia and Cantero [6,7]) and balance integral (Goodman [17,18]; Reginato and Tarzia [19]) methods.

QUASISTATIONARY METHOD

Firstly is applied the quasistationary method for **RELATIVELY IMMOBILE IONS FOR LOW CONCENTRATIONS** (K^+ , potassium, phosphorum) (the absorption kinetics is approximated by $kC/(1+kC/J_m) \sim kC$ which is a normal situation found at field for these type of ions).

Now, functions g and f can be approximate by:

$$(8) \quad g(C) = \frac{1}{Db} [(k - v_0)C - E]$$

$$(9) \quad f(C) = \frac{1}{a} \left[k - \frac{E}{C} \right]$$

and verifies the following properties:

$$(10) \quad g(C) > 0 \Leftrightarrow C > C_m = \frac{E}{(k - v_0)}, \quad (C_m > C_p)$$

$$(11) \quad f(C) > 0 \Leftrightarrow C > C_p = \frac{E}{k}$$

i.e, C_p is the threshold ion concentration above which root growth is possible.

The quasistationary method assumes that the soil solution concentration is that corresponding to the stationary case in the interval $(s(t), R)$. Thus is solved the equation $DC_{rr} + D(1+\epsilon)C_r/r = 0$, $s(t) < r < R$, $0 < t < T$ with the conditions (1-ii,iii,vi, and v with the approximation $J \sim kC$).

For this case, the solution of problem (1) is given by:

$$(12) \quad C(r,t) = \beta(t) - \frac{\alpha(t)}{r^\epsilon}, \quad s(t) < r < R, \quad t > 0$$

where:

$$\alpha(t) = \left[\frac{1}{Db} \right] \frac{[(k - v_0)C_\infty - E]}{\frac{\epsilon}{s(t)^{1+\epsilon}} + \frac{(k - v_0)}{Db} \left[\frac{1}{s(t)^\epsilon} - \frac{1}{R^\epsilon} \right]},$$

$$\beta(t) = C_{\infty} + \frac{\alpha(t)}{R^{\epsilon}},$$

$$\Phi(r) = C_{\infty} - \frac{[(k - v_0)C_{\infty} - E]}{\frac{v_0}{s_0^{\epsilon}} + (k - v_0)\left[\frac{1}{s_0^{\epsilon}} - \frac{1}{R^{\epsilon}}\right]} \left[\frac{1}{r^{\epsilon}} - \frac{1}{R^{\epsilon}}\right].$$

and $s(t)$ is the unique solution of the following Cauchy problem:

$$(13) \quad \dot{s}(t) = F(s(t)), \quad s(0) = s_0 \in (0, R), \quad t > 0$$

with:

$$F(s) = \frac{k}{\alpha} [1 - \alpha_3 H(s)]$$

$$H(s) = \frac{[1 + \alpha_2 G(s)]}{[1 + \alpha_1 G(s)]}, \quad G(s) = s \left[1 - \left(\frac{s}{R}\right)^{\epsilon}\right]$$

$$\alpha_1 = \frac{E}{v_0 s_0 C_{\infty}} > 0, \quad \alpha_2 = \frac{(k - v_0)}{v_0 s_0} > 0, \quad \alpha_3 = \frac{E}{k C_{\infty}} = \frac{C_p}{C_{\infty}} > 0$$

Therefore, we obtain, after some elementary manipulations, that the interface concentration is given by the following expression:

$$(14) \quad C(s(t), t) = \frac{C_{\infty}}{H(s(t))} (= C(s(t))), \quad t > 0$$

that is, the interface concentration does not depend explicitly on variable t .

The solution of Cauchy problem is computed numerically by the Runge-Kutta method for ordinary differential equations. Figures 2 and 3 represents some results for the interface concentration $C(s(t), t)$ vs. s and the interface position $s(t)$ vs. t respectively as a function of the absorption power k .

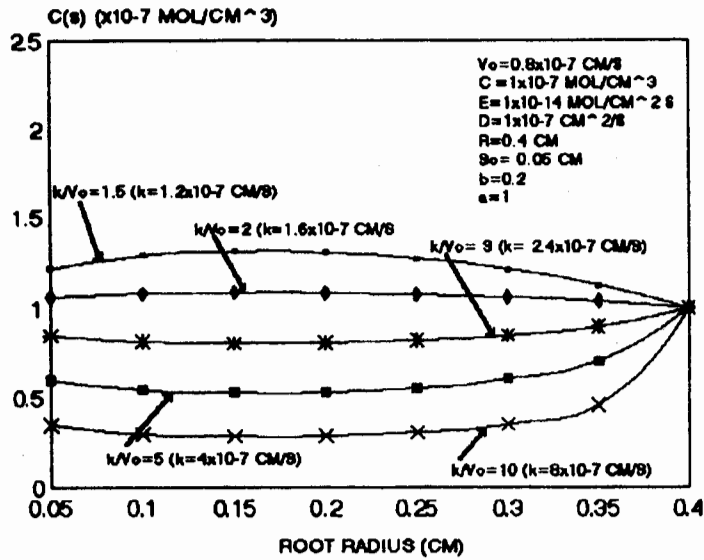


Figure 2.

Interface concentration $C(s)$ vs. root radius s as a function of absorption power k through the quasistationary method for relatively immobile ions for low concentrations.

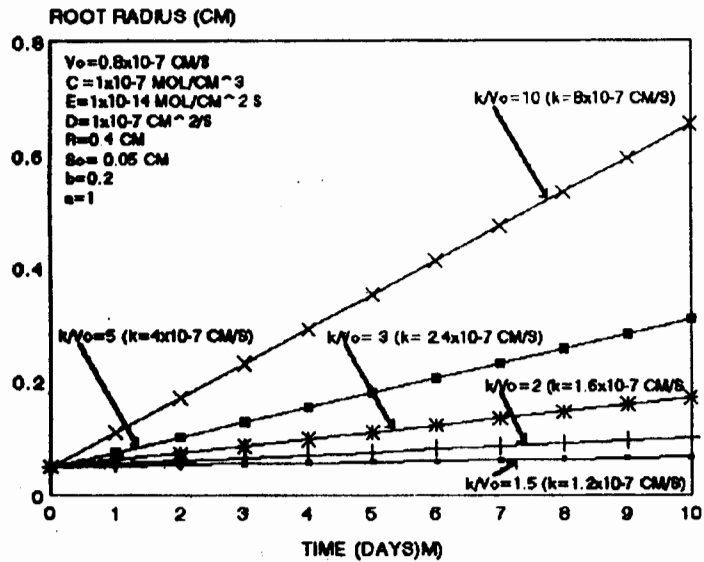


Figure 3.

Root radius $s(t)$ vs. time as a function of absorption power k through the quasistationary method for relatively immobile ions for low concentrations.

From the results of figures 2 and 3 we deduce that if the ratio k/v_0 is small (e.g.: 1.5, 2) accumulation of nutrient is produced in the interface root-soil, then there is counterdiffusion and the root growth is low. On the other hand, for large values of k/v_0 (e.g.: 10) the root growth is fast and the counterdiffusion is null. The limit value of k/v_0 which produces the counterdiffusion effect depends on the remaining parameters.

Let γ be the parameter defined by $\gamma = E/[(k - v_0) C_{\infty}]$ ($=\alpha_1/\alpha_2$), can be

demonstrated that:

- i) $\gamma = E/[(k - v_o) C_\infty] < 1$ implies that $C(s(t), t)$ has a minimum value because the absorption power k is large with respect to v_o and there is no counterdiffusion;
- ii) $\gamma = E/[(k - v_o) C_\infty] < 1$ implies that $C(s(t), t)$ is constant;
- iii) $\gamma = E/[(k - v_o) C_\infty] > 1$ implies that $C(s(t), t)$ has a maximum value because k is small and the root can not absorb all the arriving nutrient and there is a counterdiffusion effect. These results agree with Cushman's conclusions [5].

Moreover, it can be demonstrated that:

$$(15) \quad \frac{1}{\alpha_3} \geq \frac{1 + \alpha_2 R}{1 + \alpha_1 R} \Rightarrow \dot{s}(t) > 0$$

A more detailed information can be found in Reginato, Tarzia and Cantero [6].

For **RELATIVELY IMMOBILE IONS FOR HIGH CONCENTRATIONS** ($J \sim J_m$) functions g and f are given by:

$$(16) \quad g(C) = \frac{1}{D_b} [J_m - E - v_o C]$$

$$(17) \quad f(C) = \frac{J_m - E}{a} \left[\frac{1}{C} \right]$$

which satisfy the following properties:

$$(18) \quad g(C) > 0 \Leftrightarrow C < C_q = \frac{J_m - E}{v_o}$$

$$(19) \quad f(C) > 0, \quad \forall C > 0$$

and, solution of problem (1) is given by Eq. (12) with:

$$\alpha(t) = \frac{R^\epsilon \left[\frac{J_m - E}{v_o} - C_\infty \right]}{1 - \left[\left(\frac{R}{s(t)} \right)^\epsilon \left(1 - \frac{s_o}{s(t)} \right) \right]},$$

$$\beta(t) = C_{\infty} + \frac{\alpha(t)}{R^{\epsilon}},$$

$$\Phi(r) = C_{\infty} + \left[\frac{J_m - E}{v_o} - C_{\infty} \right] \left[1 - \left(\frac{R}{r} \right)^{\epsilon} \right].$$

and $s(t)$ is the unique solution of the following Cauchy problem:

$$(20) \quad \dot{s}(t) = F(s(t)), \quad s(0) = s_0 \in (0, R), \quad 0 < t < T$$

with:

$$F(s) = \left[\frac{J_m - E}{s} \right] \frac{1}{C(s)}$$

where:

$$C(s) = C_{\infty} + \left[\frac{J_m - E}{v_o} - C_{\infty} \right] \left[1 - \left(\frac{R}{s} \right)^{\epsilon} \right], \quad 0 < t \leq T$$

After some elementary manipulations, we obtain that the interface concentration is given by the following expression:

$$(21) \quad C(s(t), t) = C(s(t))$$

that is, the interface concentration does not depend explicitly on variable t .

The solution of Cauchy problem is computed numerically by the Runge-Kutta method for ordinary differential equations. Figures 4 and 5 represent theoretical results for the interface concentration $C(s)$ vs. the root radius s and the root radius $s(t)$ vs. the time t as a function of diffusion coefficient D , respectively. The values of parameters used are given for each.

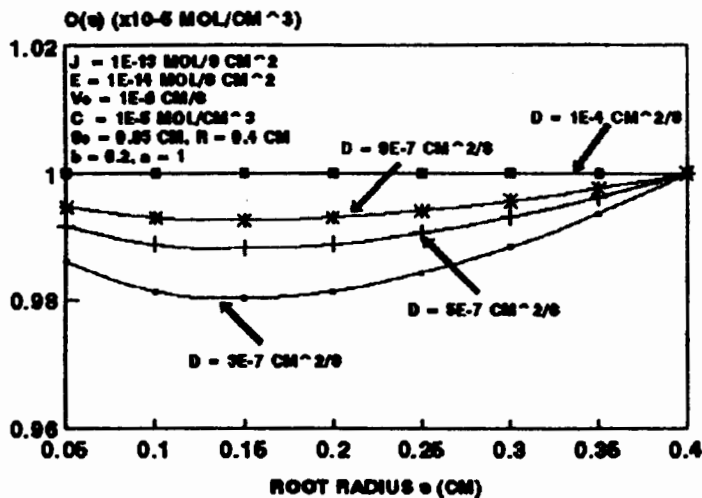


Figure 4.

Interface concentration $C(s)$ vs. the root radius s as a function of diffusion coefficient D through the quasistationary method for relatively immobile ions for high concentrations.

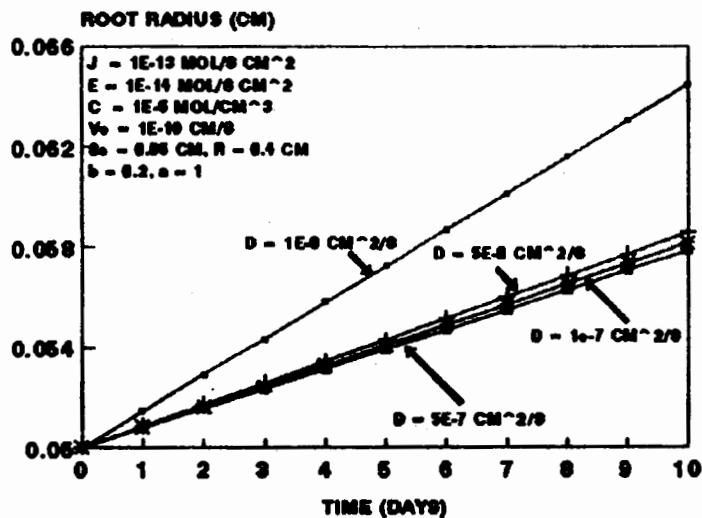


Figure 5.

Root radius $s(t)$ vs. time as a function of diffusion coefficient D through the quasistationary method for relatively immobile ions for high concentrations.

A more detailed information can be found in Reginato, Tarzia and Cantero [7].

From the comparison between the results of present model for root growth for high concentrations and the corresponding model for low concentrations, we obtain that the absorption for low concentrations is more efficient than the mechanism for high concentrations owing to the range of values for $C(s)$ and $s(t)$ for each. This result is valid in the concentration range for the highly specific mechanisms I (Epstein [20,21]), although total growth can be higher when both mechanisms I and II (not specific) operate in parallel (Welch and Epstein [22,23]).

For **MOBILE IONS FOR LOW CONCENTRATIONS** (NO_3^- , nitrate, i.e., for

example, ions which can be the result of mineralization process of organic matter) the solution of problem (1) is given by Eq. (12) with:

$$\alpha(t) = \left[\frac{E}{Db} \right] \frac{1}{\frac{(k-v_0)}{Db} \left[\frac{1}{R^\epsilon} \alpha_0 - \frac{1}{s(t)^\epsilon} \right] - \frac{\epsilon}{s(t)^{\epsilon+1}}} = \alpha_2 \frac{s(t)^{\epsilon+1}}{\beta_1 I\left(\frac{s(t)}{R}\right) - 1},$$

$$\beta(t) = \frac{\alpha(t)}{R^\epsilon} \left[1 + \frac{s_0}{R} \right] = \alpha_0 \frac{\alpha(t)}{R^\epsilon} = \alpha_0 \frac{\alpha_2}{R^\epsilon} \frac{s(t)^{\epsilon+1}}{\beta_1 I\left(\frac{s(t)}{R}\right) - 1},$$

$$\Phi(r) = E \frac{\left[\frac{1}{R^\epsilon} \left(1 + \frac{s_0}{R} \right) - \frac{1}{r^\epsilon} \right]}{(k-v_0) \left[\frac{1}{R^\epsilon} \left(1 + \frac{s_0}{R} \right) \right] - \frac{k}{s_0^\epsilon}} = \alpha(0) \left[\frac{\alpha_0}{R} - \frac{1}{r^\epsilon} \right] = \frac{\beta_2}{\beta_1 I\left(\frac{s_0}{R}\right) - 1} \left[\frac{s_0}{r} \right]^{\epsilon+1} I\left(\frac{r}{R}\right)$$

and $s(t)$ is the unique solution of the following Cauchy problem:

$$(22) \quad \dot{s}(t) = \frac{v_0}{s} L\left(\frac{s(t)}{R}\right), \quad s(0) = s_0 \in (0, R), \quad t > 0$$

where:

$$\alpha_2 = \frac{E}{v_0 s_0}, \quad \beta_1 = \frac{R(k-v_0)}{v_0 s_0} > 0, \quad \beta_2 = \frac{RE}{v_0 s_0} > 0, \quad \alpha_0 = 1 + \frac{s_0}{R},$$

$$I(x) = x(\alpha_0 x^\epsilon - 1), \quad L(x) = 1 + \frac{s_0}{R} \frac{1}{I(x)} \quad \text{with } x \in (0, 1)$$

Moreover, the interface concentration is given by the following expression:

$$(23) \quad C(s(t), t) = \frac{\beta_2 I\left(\frac{s(t)}{R}\right)}{\beta_1 I\left(\frac{s(t)}{R}\right) - 1}, \quad t > 0$$

that is, the interface concentration does not depend explicitly on variable t .

We can remark that the solution, given above, for the problem (1) there exists and has a physical meaning if the conditions

$$(24) \quad x_3(w) < \xi < 1, \quad \xi = \frac{s_0}{R} \in (0, 1),$$

$$(25) \quad \frac{k}{v_0} > \psi(\rho(\xi)) = \frac{\rho(\frac{s_0}{R})}{\rho(\frac{s_0}{R}) - 1} = \frac{(1+\xi)\xi^\epsilon}{(1+\xi)\xi^\epsilon - 1}, \quad \epsilon = \theta\xi, \quad \theta = \frac{Rv_0}{Db} > 0$$

among parameters $\frac{k}{v_0}$, $\frac{s_0}{R}$, θ are satisfied, and $x_3 = x_3(\theta) > 0$ is the unique solution of the equation:

$$(26) \quad \rho(x) = 1, \quad x \in (0, 1)$$

with:

$$\rho(x) = (1+x) x^{\theta x} \quad \text{with } x \in (0, 1),$$

$$\psi(x) = \frac{x}{x-1} \quad \text{with } x \in (1, 2).$$

The solution of the Cauchy problem (22) is computed numerically by the Runge-Kutta method for ordinary differential equations. Figures 6 and 7 represent some theoretical results for the interface concentration $C(s(t), t)$ vs. s and the root radius $s(t)$ vs. t as a function of absorption power k for low concentrations and parameters which satisfy the condition (25).

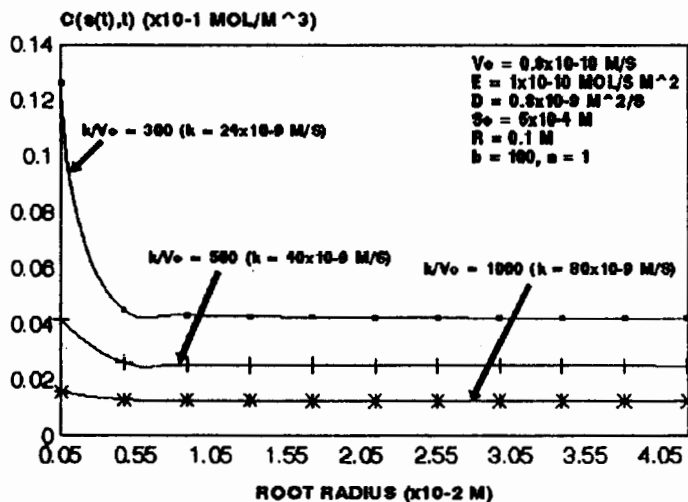


Figure 6.

The interface concentration $C(s)$ vs. root radius s as a function of absorption power k through the quasistationary method for mobile ions for low concentrations.

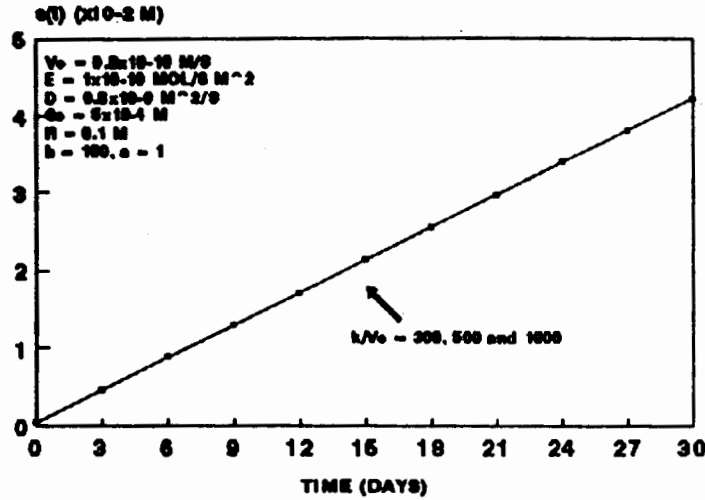


Figure 7.

Root radius s vs. time as a function of absorption power k through the quasistationary method for mobile ions for low concentrations.

A more detailed information can be found in Reginato, Tarzia and Dzioba [24].

A similar solutions for **MOBILE IONS FOR HIGH CONCENTRATIONS** are given by Eq (12) with:

$$\alpha(t) = \left[\frac{J_m - E}{Db} \right] \frac{1}{\frac{v_0}{Db} \left[\frac{1}{R^\epsilon} \left(1 + \frac{s_0}{R} \right) - \frac{1}{s(t)^\epsilon} \right] + \frac{\epsilon}{s(t)^{\epsilon+1}}} = \frac{J_m - E}{v_0} \frac{R^\epsilon}{\alpha_0 + \left(\frac{R}{s(t)} \right)^\epsilon \left(\frac{s_0}{s(t)} - 1 \right)}$$

$$\beta(t) = \frac{\delta(t)}{R^\epsilon} \left[1 + \frac{s_0}{R} \right] = \alpha_0 \frac{\delta(t)}{R^\epsilon},$$

$$\Phi(r) = \frac{(J_m - E)}{v_0} \left[1 - \left(\frac{R}{r} \right)^\epsilon \frac{1}{\alpha_0} \right].$$

and $s(t)$ is the unique solution of the following Cauchy problem:

$$(27) \quad \dot{s}(t) = \frac{v_0}{a} L\left(\frac{s(t)}{R}\right), \quad s(0) = s_0 \in (0, R), \quad t > 0$$

where I and L are defined before. Function $\Phi = \Phi(r)$ satisfies the following conditions

$$\Phi(0^+) = -\infty, \quad \Phi(R_1) = \frac{(J_m - E)}{v_0} \frac{\xi}{1 + \xi} > 0, \quad \Phi' > 0 \text{ in } (0, R).$$

Moreover, the coefficient $\xi = \frac{s_0}{R}$ must verifies the condition:

$$\Phi(s_0) = \frac{(J_m - E)}{v_0} \left[1 - \frac{1}{\rho(\xi)} \right] > 0$$

which is equivalent to the condition $x_3(\theta) < \xi = \frac{s_0}{R} < 1$, where $x_3 = x_3(\theta)$ and function ρ are defined before. Therefore, we obtain, that the interface concentration is given by the following expression:

$$(28) \quad C(s(t), t) = \frac{(J_m - E)}{v_0} \frac{1}{L\left(\frac{s(t)}{R}\right)} = \frac{(J_m - E)}{v_0} \frac{1}{\dot{s}(t)}, \quad t > 0$$

Moreover, we see that $\dot{s}(t) > 0$, $\forall t > 0$ because $\xi = \frac{s_0}{R} > x_0 = \frac{1}{(1+\xi)^{\theta\xi}}$ is obviously verified.

The solution of Cauchy problem (27) is computed numerically by the Runge-Kutta method for ordinary differential equations. Figures 8 and 9 represent some theoretical results for the interface concentration $C(s)$ vs. s and the root radius $s(t)$ vs. t as a function of rhizosphere radius R for high concentrations.

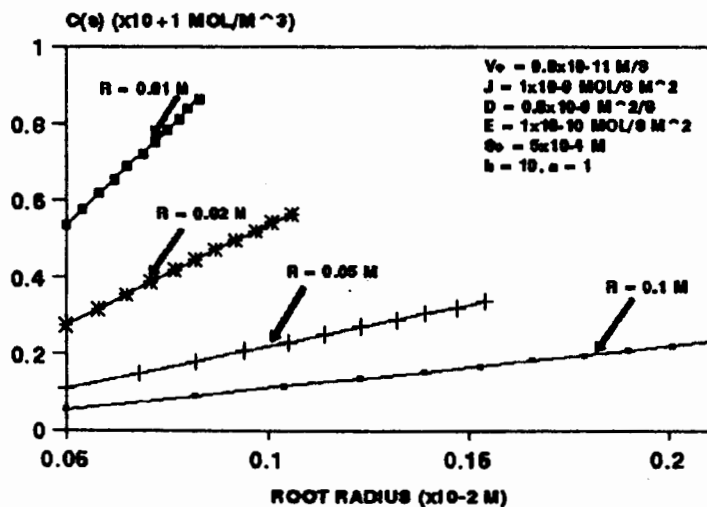


Figure 8.

The interface concentration $C(s)$ vs. root radius s as a function of rhizosphere radius R through the quasistationary method for mobile ions for high

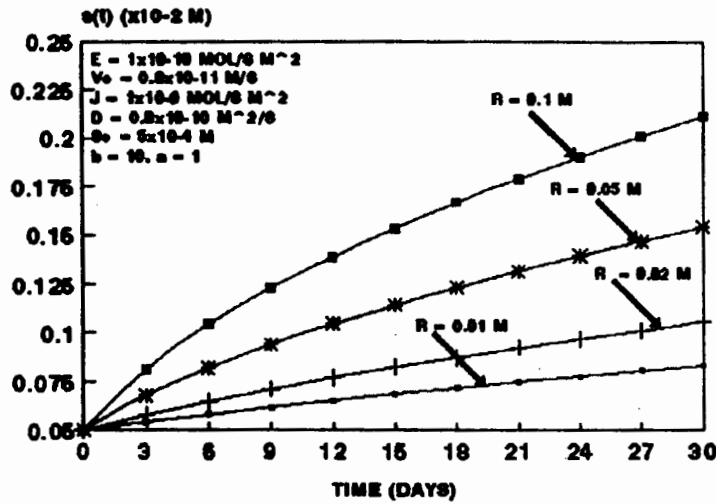


Figure 9.

Root radius $s(t)$ vx. time as a function of rhizosphere radius R through the quasistationary method for mobile ions for high concentrations.

A more detailed information can be found in Reginato, Tarzia and Dzioba [24].

BALANCE INTEGRAL METHOD

To solve (1) (that is, to compute $C = C(r,t)$ (in particular, $C = C(s(t),t)$) and the free boundary interface $r = s(t)$ a priori unknown) we apply the mass balance integral method (Goodman [17,18]) to the present case for root growth. The solution is found integrating the partial differential equation (1-i) in r on the domain $(s(t), R)$. Thus:

$$(29) \quad \int_{s(t)}^R C_t(r,t) dr = D \int_{s(t)}^R C_{rr}(r,t) dr + D(1 + \epsilon) \int_{s(t)}^R \frac{C_r(r,t)}{r} dr$$

and we propose:

$$(30) \quad C(r,t) = \Phi(r) [1 + \beta(t)(R - r)]$$

which depend on the parameters of the system and satisfy the initial and boundary conditions 1-ii) and 1-iii), that is:

$$(31) \quad C(r,0) = \Phi(r) \quad \Leftrightarrow \quad \beta(0) = 0,$$

$$(32) \quad C(R,t) = C_\infty \quad \Leftrightarrow \quad \Phi(R) = C_\infty.$$

We denote $\alpha = \alpha(t)$ by $\alpha(t) = C(s(t),t)$ which depends on the parameters of the system through $s(t)$, $\Phi(s(t))$ and $\beta(t)$.

Replacing (31) and (32) in Eq. (29), after some elementary manipulations, the problem (1) reduces to:

$$(33) \quad \int_{s(t)}^R C_t(r,t) dr = D [C_r(R,t) - g(\alpha(t))] + D(1+\epsilon) \left[\frac{C_\infty}{R} - \frac{\alpha(t)}{s(t)} + \int_{s(t)}^R \frac{C(r,t)}{r^2} dr \right], \quad t > 0$$

$$\dot{s}(t) = f(\alpha(t)), \quad t > 0, \quad s(0) = s_0.$$

Using the approximation $k C/[1 + (k/J_m) C] \approx k C$ (valid for low concentration), and replacing (30) in Eq. (33), after some elementary manipulations we obtain the following system of two coupled ordinary differential equations (valid for the cases $\epsilon \neq 1, 2$):

$$(34) \quad \begin{aligned} \frac{d\beta(t)}{dt} &= F_1 (F_2 + F_3 + F_4 + F_5 + F_6 + F_7), & \beta(0) &= 0 \\ \frac{ds(t)}{dt} &= \frac{1}{a} \left[k - \frac{E}{\Phi(s(t)) [1 + \beta(t) (R - s(t))]} \right], & s(0) &= s_0 \end{aligned}$$

where:

$$(35) \quad \Phi(r) = C_\infty + A \left[1 - \left(\frac{R}{r} \right)^\epsilon \right],$$

and

$$F_1 = \frac{D}{R \left\{ (C_\infty + A)[R - s(t)] + \frac{AR^\epsilon s^{(1-\epsilon)}(t) - AR}{(1-\epsilon)} \right\} \left\{ \frac{C_\infty + A}{2} (R^2 - s^2(t)) + \frac{AR^\epsilon s^{(2-\epsilon)}(t) - AR^2}{(2-\epsilon)} \right\}}$$

$$F_2 = \frac{A\epsilon}{R} - C_\infty \beta(t) - \frac{1}{D b} [\Phi(s(t)) [1 + \beta(t) (R - s(t))] (k - v_o) - E],$$

$$F_3 = (1 + \epsilon) \frac{C_\infty}{R} - (1 + \epsilon) \frac{\Phi(s(t)) [1 + \beta(t) (R - s(t))]}{s(t)},$$

$$F_4 = (1 + \epsilon) (C_\infty + A) (1 + \beta(t) R) \left[\frac{1}{s(t)} - \frac{1}{R} \right],$$

$$F_5 = - (1 + \epsilon) \beta(t) (C_\infty + A) \ln \left(\frac{R}{s(t)} \right),$$

$$F_6 = \frac{(1 + \epsilon)}{\epsilon} A \beta(t) R^\epsilon \left[\frac{1}{s^\epsilon(t)} - \frac{1}{R^\epsilon} \right],$$

$$F_7 = - A R^\epsilon [1 + \beta(t) R] \left[\frac{1}{s^{\epsilon+1}(t)} - \frac{1}{R^{\epsilon+1}} \right],$$

$$A = \frac{E - (k - v_o) C_\infty}{k \left[1 - \left(\frac{R}{s_o} \right)^\epsilon \right] - v_o}.$$

Remark 1: The initial profile concentration $\Phi(r)$ given by the Eq. (35) above has been computed by the quasi-stationary method [6] and it is determined by the system, similarly to the Cushman's prediction [4].

Remark 2: For the particular cases $\epsilon = 1$ and 2 , we can obtain a similar system to (34) of two ordinary differential equations.

Remark 3: For the general rank of concentration C we can obtain a similar system to (34) of two ordinary differential equations.

The solution of system (34) is computed numerically by the Runge-Kutta method for a system of ordinary differential equations. The figures 10 and 11 represent some results for the interface concentration $C(s(t), t)$ vs. s and the interface position $s(t)$ vs. t respectively as a function of absorption power k for relatively immobile ions for low concentrations.

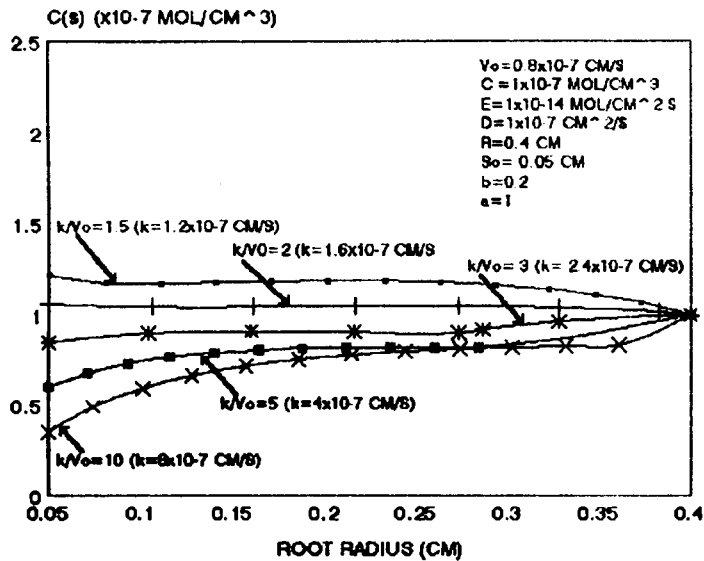


Figure 10.

Interface concentration $C(s(t),t)$ vs. root radius $s(t)$ as a function of absorption power k through the balance integral method for relatively immobile ions for low concentrations.

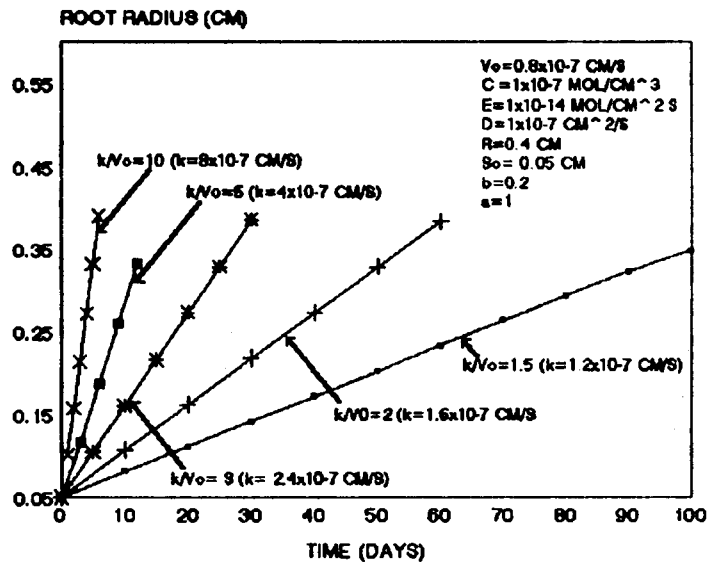


Figure 11.

Root radius $s(t)$ vs. time as a function of absorption power k through the balance integral method for relatively immobile ions for low concentrations.

A more detailed information can be found in Reginato and Tarzia [19]

For **MOBILE IONS FOR LOW CONCENTRATIONS** the balance integral method give us a similar solution with:

$$(36) \quad C(r,t) = \Phi(r) [1 + \eta(t)(R-r)^2]$$

which depend on the parameters of the system and satisfy the initial and boundary

conditions 1-ii) and 1-iii), that is:

$$(37) \quad C(r,0) = \Phi(r) \quad \Leftrightarrow \quad \eta(0) = 0,$$

$$(38) \quad -DbC_r(R,t) + v_o C(R,t) = 0 \quad \Leftrightarrow \quad -Db\Phi'(R) + v_o \Phi(R) = 0.$$

Figures 12 and 13 represent some results for the interface concentration $C(s(t),t)$ vs. $s(t)$ and the interface position $s(t)$ vs. t respectively as a function of absorption power k mobile ions for low concentrations.

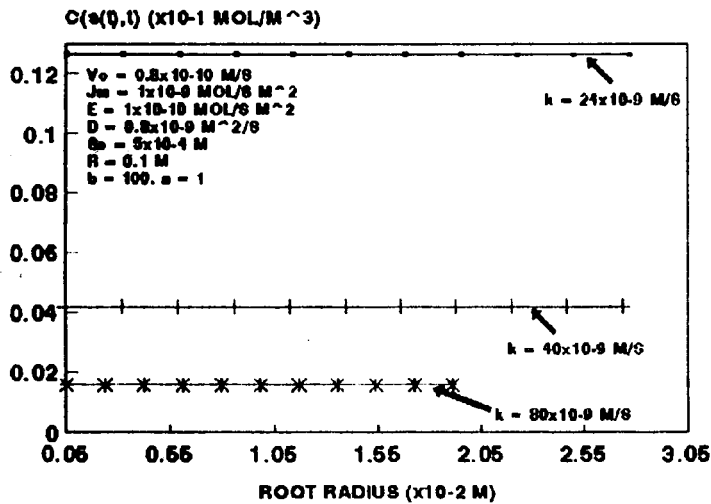


Figure 12.

Interface concentration $C(s(t),t)$ vs. root radius $s(t)$ as a function of absorption power k through the balance integral method for mobile ions for low concentrations

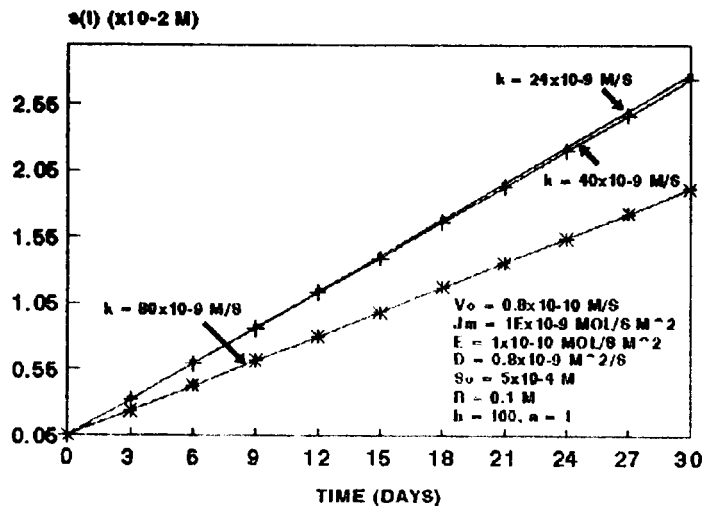


Figure 13.

Root radius $s(t)$ vs. time as a function of absorption power k through the balance integral method for mobile ions for low concentrations.

A more detailed information can be found in Reginato, Tarzia and Dzioba [24]

CONCLUSIONS

From the analysis of total results of cases presented above we conclude that for *ions relatively immobile for low concentrations* $s=s(t)$ increases when parameter k (Figure 3) or C_∞ increases; $s=s(t)$ decreases when parameter E increases; $s=s(t)$ increases when parameter (k/v_0) increases and, k and v_0 are large. $s=s(t)$ does not vary in function of the parameters v_0 , b and D because we did not have variations in the corresponding diagrams in a wide range of order of magnitude (1 to 10^5 for each). $\dot{s}=\dot{s}(t)$ decreases when parameter γ increases, because from (13) we have for $\dot{s}(t)$ the following representation in function of the parameter γ :

$$\dot{s} = \frac{k}{a} \left[1 - \frac{(k - v_0)}{k} \frac{G(s(t) + \frac{1}{\alpha_2})}{G(s(t) + \frac{1}{\gamma \alpha_2})} \right], \quad t > 0.$$

This conclusion agrees with the first three conclusions.

Some of the above theoretical results have been observed from an experimental point of view (Barley, [25], Nye and Tinker, [26]).

Moreover, for *ions relatively immobile for high concentrations* we conclude that no limit conditions for root growth in opposition to the case for low concentrations exists (for low concentrations there exist a minimum concentration above which the growth take place, i.e., the growth stop when $C(s) = E/k$; instead, for high concentrations the interface velocity $\dot{s}(t)$ is positive for all value of $C(s)$). One important difference is that there exists counterdiffusion for E and C_∞ for high concentrations. Instead, for low concentrations counterdiffusion is possible for all parameters. Moreover, we assume that both mechanisms (low and high concentrations) could be found in a radical system.

From the analysis of total results given by the quasi-stationary method, we conclude that for **mobile ions for low concentrations** that predicted $s(t)$ increases rapidly with increases in v_0 and R . Values for k (Figure 7), b , D , E , and s_0 were without effect because the mathematical approximation of method is poor. For **mobile ions for high concentrations** we conclude that predicted $s(t)$ increases in v_0 and R . Values for b , D , E and J_m were without effect or had little effect. The predicted $s(t)$ decreases with respect to s_0 .

From the analysis of result obtained by the **BALANCE INTEGRAL** method for **ions relatively immobile for low concentrations** we conclude that the free boundary $s = s(t)$ increases when absorption power k increases (Figure 11) (i.e., the dimensionless parameter k/v_0 increases with $v_0 = \text{const.}$) or when the soil solution concentration C_∞ increases. Moreover $s = s(t)$ decreases when efflux E increases. Likewise, $s = s(t)$ decreases when the buffer power b or the diffusion coefficient D increases, although this effect is meaningless for high values (e.g. 10) of the dimensionless parameter k/v_0 . Moreover, $s = s(t)$ increases when the flux velocity at the root surface v_0 increases (i.e., the parameter k/v_0 decreases with $k = \text{Const.}$).

The behavior of interface concentration $C(s(t), t)$ as a function of k , v_0 , C_∞ , E , D , and b is quite similar to results obtained by the quasi-stationary method applied to the same model.

On the other hand, for **mobile ions for low concentrations** the results balance integral indicate that predicted $s(t)$ increases rapidly with v_0 and R . The increase in v_0 represent an increase in total nutrient available for the plant, since if the remaining parameters are held constants, increasing v_0 increases the convective flux to root. Increasing R increases the rhizosphere volume and so a greater ion quantity is available. Also, we obtain that the predicted $s(t)$ increases moderately with J_m . Values of b and D were without effect or had little effect, whilst increasing k , E , and s_0 the predicted $s(t)$ decreases. In case of k decrease occurs because the gradient at root-soil interfase (given by the Eq. (4)) as a function of k and $C(s)$ decreases (the expression $g(C(s(t), t) = (1/Db) [(k - v_0)C(s(t), t) - E]$ takes into account the combined effects of increase in k and the decrease of $C(s(t), t)$ as a function of k). The decrease with s_0

occurs because, although the root surface increases as s_0^2 , the rhizosphere volume available decreases as $L(R^2 - s_0^2)$ where L is the length root and taking into account that $R \gg s_0$ then $(R^2 - s_0^2) > s_0^2$. (No allowance for a change in L with time is made because we have only considered radial growth).

From the analysis of the results obtained by the quasi-stationary method, the interface concentration become nearly constant after a week elapsed (Figure 6), whilst from the results obtained by the balance integral method the interface concentration is constant throughout the period simulated (Figure 12) in agreement with the De Willigen's results (De Willigen [27]) on N uptake.

By comparing the results obtained by the quasi-stationary method and the balance integral method we can conclude that: firstly, the mathematics of the quasi-stationary method allow analytical expressions which must be satisfied by the system parameters, such as the inequality (25) and, secondly the general qualitative behavior of the results is similar for both methods, although the balance integral method gives us a more detailed theoretical information with respect to dependence on system parameters, for example, for mobile ions for low concentrations, the variation of $s(t)$ vs. t with respect to the parameter k is negligible for the quasi-stationary method (Figure 7) whilst the balance integral method gives us a appreciable change (Figure 13).

In conclusion, this paper represents a qualitative approach for the effects on root growth due to absorption of only one mobile nutrient or only one relatively immobile ion. Moreover, these models are useful as a basis for developing more complex models for transport of nutrients and their effect on root growth, as for example, these same models taking into account the root length (the effects of some parameters could be change) and simultaneous absorption of one or more immobile and mobile ions. Finally, these conclusions can be perfectible taking into account the dependence of v_0 with the nutrient concentration C_∞ and the water content θ . Thus, this method can to provide a very useful qualitative criterion for the crops technology through the adequate choice seed (k), a precise soil management and irrigation (v_0) as soon as fertilization (C_∞) between some parameters of system soil-plant (A more detailed information of effects of parameters on uptake nutrient and root growth can be found in Barber [28]).

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NOMENCLATURE

r:	Position (radius) coordinate	[cm]
t:	Time	[Sec]
J_m:	Rate of influx at infinite concentration	$\left[\frac{\text{Mol}}{\text{Sec-Cm}^2} \right]$
K_m:	Michaelis Menten constant	$\left[\frac{\text{Mol}}{\text{Cm}^3} \right]$
k:	Absorption power of root	$\left[\frac{\text{Cm}}{\text{Sec}} \right]$
v_o:	Flux velocity of soil solution at root surface	$\left[\frac{\text{Cm}}{\text{Sec}} \right]$
D:	Effective diffusion coefficient	$\left[\frac{\text{Cm}^2}{\text{Sec}} \right]$
C:	Total diffusible ion concentration	$\left[\frac{\text{Mol}}{\text{Cm}^3} \right]$
C_i:	Ion concentration in soil solution	$\left[\frac{\text{Mol}}{\text{Cm}^3} \right]$
C(s(t),t):	Ion concentration in soil solution at root-soil interface	$\left[\frac{\text{Mol}}{\text{Cm}^3} \right]$
Φ(r):	Initial soil solution concentration profile	$\left[\frac{\text{Mol}}{\text{Cm}^3} \right]$

s_0 :	Initial radius	[Cm]
E:	Constant efflux	$\left[\frac{\text{Mol}}{\text{Sec-Cm}^2} \right]$
$s(t)$:	Instantaneous root radius	[Cm]
$\dot{s}(t)$:	Rate of radial root growth	$\left[\frac{\text{Cm}}{\text{Sec}} \right]$
L:	Root length	[cm]
R:	Rhizosphere radius	[Cm]
b:	Buffer power	Dimensionless
a:	Stoichiometric coefficient	Dimensionless
ϵ	Parameter	Dimensionless
α_1	Parameter	$\left[\frac{1}{\text{Cm}} \right]$
α_2	Parameter	$\left[\frac{\text{Mol}}{\text{Cm}^4} \right]$
$\alpha(t)$	Coefficient	$\left[\frac{\text{Mol}}{\text{Cm}^2} \right]$
$\beta(t)$	Coefficient	$\left[\frac{\text{Mol}}{\text{Cm}^3} \right]$
$\eta(t)$	Coefficient	$\left[\frac{1}{\text{Cm}^2} \right]$