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THE ONE-PHASE SUPERCOOLED STEFAN PROBLEM WITH TEMPERATURE BOUNDARY CONDITION

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Abstract .— We consider the supercooled one-phase Stefan problem with temperature boundary condition at the fixed face. We analyse the relation between the temperature boundary data and the possibility of continuing the solution for arbitrarily large time intervals. We also give a family of explicit solutions.

Key Words .— Supercooled one-phase Stefan problem, phase-change problem, parabolic free boundary problem, explicit solutions.

I. — Introduction .

We study a one-phase supercooled Stefan problem in one space dimension. The initial temperature of the material is equal to $h(x)$. We impose a temperature boundary condition on $x = 0$, where the temperature f is a function of time.

The classical Stefan problem ($f \geq 0$, $h \geq 0$) is well studied in the literature, as for example [3, 11, 18]. Here we will treat the case $f \leq 0$ and $h \leq 0$ that corresponds to a supercooled liquid. The existence and uniqueness for this problem is proved in [7], where a class of free-boundary problems for the heat equation in one space dimension was analyzed, releasing the sign restrictions on the data and the latent heat usually required in the Stefan problem. In the next sections we relate the possibility of continuing the solution for arbitrarily large time intervals to the sign of

$$R(t) = \frac{1}{2} + \int_0^1 x\, h(x)\, dx + \int_0^t f(\tau)\, d\tau \,.$$

Problems of this kind have been studied by other authors also in conection with the freezing of a supercooled liquid. A one-phase Stefan problem with initial temperature h(x) and a heat flux equals to zero on $x = 0$ was studied in [9]. In [5, 17] a one-phase Stefan problem with initial temperature equal to zero and the heat flux g(t) on $x = 0$ was considered. Other condition is considered in [6]. On the other hand, convexity and smoothness properties of the free boundary are analyzed in [8, 13] and [12, 14] respectively, and a review on this subject were given in [16, 20]. Moreover, in [2] an application to the liquid-phase epitaxy is presented and in [15, 21] a two-phase supercooled or superheated Stefan problem is analyzed. An explicit solution is given in [4].

In section II we give the preliminaries corresponding to the description of the problem and we consider the case of a temperature boundary condition with a determined sign (in our case, negative) and we give some results in order to characterize the three possible cases [7, 19].

In section III we study the case in which the temperature on $x = 0$ is a constant in time, say $f(t) = -B < 0$ and we give a family of explicit solutions.

II. — The one-phase Stefan problem.

Let us consider problem (P) (one-phase Stefan problem with temperature boundary condition on the fixed face x=0) which consist of finding (T,s,z) such that :

(i) $T > 0$.

(ii) $s \in C([0,T])$, $s \in C^1((0,T))$; $0 < s(t) < 1$ for $0 < t < T$.

(iii) z(x,t) is a function, bounded in $0 \le x \le s(t)$, $0 \le t \le T$ and continuous on the same region , except perhaps at the points (0,0) and (s(0),0) ; $z_x(x,t)$ is a continuous function in $0 \le x \le s(t)$, $0 < t < T$; z_{xx}, u_t are continuous functions in $0 < x < s(t)$, $0 < t < T$.

(iv) The following conditions are satisfied :

(2.1)	$z_{xx} - z_t = 0$	in $D_T = \left\{ (x,t) : 0 < x < s(t), 0 < t < T \right\}$,
(2.2)	$s(0) = 1$,	
(2.3)	$z(x,0) = h(x)$,	$0 < x < 1$,
(2.4)	$z(0,t) = f(t)$,	$0 < t < T$,
(2.5)	$z(s(t),t) = 0$,	$0 < t < T$,
(2.6)	$z_x(s(t),t) = -\dot{s}(t)$	$0 < t < T$,

where the function f is a non positive piecewise continuous function on every interval (0,t), $t>0$ and h(x) is a given non positive continuous function in [0,1].

Moreover, if the solution exists, then three cases can occur [7, 19] :

(A) The problem has a solution with arbitrarily large T.

(B) There exists a constant $T_B > 0$ such that $\lim_{t \to T_B^-} s(t) = 0$.

(C) There exists a constant $T_C > 0$ such that $\lim_{t \to T_C^-} s(t) > 0$ and $\lim_{t \to T_C^-} \dot{s}(t) = -\infty$.

As we shall see, any of these cases can actually occur with an appropiate choice of the functions $h = h(x)$ and $f = f(t)$ in (2.3) and (2.4) respectively.

If (T,s,z) solve (2.1)−(2.6) then it is well known that the following integral representations are satisfied :

(2.7) $$s(t) = 1 + \int_0^1 h(x)\,dx - \int_0^t z_x(0,\tau)\,d\tau - \int_0^{s(t)} z(x,t)\,dx ,$$

(2.8) $$\frac{(s(t)^2 - 1)}{2} = \int_0^1 x\,h(x)\,dx + \int_0^t f(\tau)\,d\tau - \int_0^{s(t)} x\,z(x,t)\,dx ,$$

(2.9) $$\frac{(s(t)^3 - 1)}{3} = 2\int_0^t d\tau \int_0^{s(\tau)} z(x,\tau)\,dx - \int_0^{s(t)} z(x,t)\,x^2\,dx + \int_0^1 h(x)\,x^2\,dx .$$

The first simple properties of the solutions of (2.1)−(2.6) are summarized in the following :

PROPOSITION 2.1 .— If (T,s,z) is a solution of Problem (P), then :

(i) $z \leq 0$ in D_T .

(ii) $s(t)$ is a decreasing function in $(0,T)$.

(iii) If $h'(x) \geq 0$ and $\dot{f}(t) \leq 0$, then $z_x(x,t) \geq 0$ in D_T.

(iv) If $h(x) = C - 1$, with $C = \text{Const.} < 0$ for $0 \leq x \leq 1$ and $f(t) > C - 1$ for $t > 0$, then no solution to Problem (P) can exist.

PROOF.— (i)−(iii) follow from the maximum principle.

(iv) For any solution (T, s, z) of the Problem (P) it would be $0 \leq s(t) \leq 1$, $0 \leq t \leq T$ and $z(x,t) > C-1$ in D_T because of (2.1)−(2.6) and the maximum principle. Thus, from (2.8) we have

$$\frac{(s^2(t) - 1)}{2} < \frac{C-1}{2} - (C-1)\,\frac{s^2(t)}{2} ,$$

i. e. $\frac{(s^2(t) - 1)\,C}{2} < 0$, which is a contradiction to $C < 0$ and $s(t) \leq 1$.

REMARK 1 .— A general result of non−existence was given in [10]. A sufficient condition of non-existence is given by $h \leq -1$ in a left neighbourhood of $x = 1$, independently of the behavior of the boundary temperature $f = f(t)$ on $x = 0$.

PROPOSITION 2.2 .— If (T,s,z) is a solution of Problem (P), and

(2.10) $$h(x) \geq m(f)\,(1-x),\ 0 \leq x \leq 1\,,$$

then

(2.11) $$z(x,t) \geq m(f)\,(1-x) \text{ in } D_T\,,$$

(2.12) $$\frac{s^2(t)}{2}\,[1+m(f)] - m(f)\,\frac{s^3(t)}{3} \leq \tfrac{1}{2} + \int_0^t f(\tau)\,d\tau + \int_0^1 x\,h(x)\,dx,\ 0 < t < T,$$

where $m(f) = \min_{0 \leq t \leq T} f(t)$.

PROOF .— (i) We define the function :

$$W(x,t) = m(f)\,(x-1), \text{ in } D = \left\{ (x,t) : 0 < x < 1,\ 0 < t < T \right\}.$$

By comparing W with z and using the maximum principle we obtain (2.11). By using (2.8) we obtain (2.12).

We proceed to characterize cases (A), (B) and (C) depending on the value of R(t), where

(2.13) $$R(t) = \tfrac{1}{2} + \int_0^1 x\,h(x)\,dx + \int_0^t f(\tau)\,d\tau\,,$$

We remark that $\dot{R}(t) = f(t) \leq 0,\ 0 \leq t \leq T$.

PROPOSITION 2.3 .— We have that

(i) Case (B) $\Rightarrow R(T_B) = 0$.

(ii) Case (B) $\Rightarrow \int_0^{T_B} z_x(0,\tau)\,d\tau = 1 + \int_0^1 h(x)\,dx$.

(iii) Case (B) $\Rightarrow 2 \iint_{D_{T_B}} z(x,\tau)\,dx d\tau = -\tfrac{1}{3} - \int_0^1 x^2\,h(x)\,dx$.

PROOF .— Owing to Proposition 2.2, we can perform the limit for $t \to T_B$ in (2.7), (2.8) and (2.9) in order to obtain the above three relations.

PROPOSITION 2.4 .— Assume h(x) satisfies : There exists a positive constant H such that

(2.14) $$h(x) \geq -H\,(1-x),\ 0 \leq x \leq 1,$$

and let (T,s,z) be a solution of Problem (P) such that

(2.15) $$s_T = \inf_{t \in (0,T)} s(t) \geq 0 .$$

If there exists two constants $d \in (0, s_T)$, $z_0 \in (0, 1)$ such that $H\,d \leq z_0 \ln(2)$ and

(2.16) $$z(s(t) - d, t) \geq -z_0, \; 0 \leq t \leq T,$$

then

(2.17) $$\dot{s}(t) \geq \min[-H / z_0 , \ln(1 - z_0) / d] .$$

PROOF .— It is the same as the one for the Lemma (2.4) in [9].

COROLLARY 2.5 .— If case (C) occurs, the isotherm $z = -1$ exists and reaches the free-boundary at $t=T_C$.

PROPOSITION 2.6 .— Let (T,s,z) be a solution of the Problem (P). If $R(T_B) = 0$ and $-1 \leq m(f) \leq 0$ then we have the case (B).

PROOF .— We replace $R(T_B) = 0$ in (2.12) of Proposition (2.2) then we get

$$\frac{s^2(T_B)}{2}(1 + m(f)) - m(f)\frac{s^3(T_B)}{2} \leq 0 .$$

Since $1 + m(f) \geq 0$ and $m(f) \leq 0$ we conclude that $s(T_B) = 0$, i.e., case (B).

PROPOSITION 2.7 .— If (T,s,z) is a solution of Problem (P), with $f \in L^1(0,\infty)$, h verifies inequality (2.10) and we have case (A), then

(2.18) $$R(t) \geq 0, \qquad t \geq 0 .$$

PROOF .— Using the equality (2.8) and the hypothese we obtain the following inequality

(2.19) $$-\int_0^{s(t)} x\, z(x,t)\, dx \leq \|f\|_1 + m(f)\int_0^1 x\,(x-1)\, dx = \|f\|_1 - \tfrac{1}{6}\, m(f) \equiv C ,$$

where $C \geq 0$. From the above inequality we conclude

$$\int_0^{s(t)} x^2 \Big(-z(x,t)\Big)\, dx \leq \int_0^{s(t)} x \Big(-z(x,t)\Big)\, dx \leq C .$$

The following estimation is obtained by replacing the above inequality in (2.9)

$$(2.20) \qquad 2\int_0^T d\tau \int_0^{s(\tau)} z(x,\tau)\, dx\, d\tau \geq -C - \tfrac{1}{3}\,,\ t \geq 0\,.$$

Now suppose that there exists a T_o such that $R(T_o) < 0$, then from (2.8), it follows that

$$(2.21) \qquad \int_0^{s(t)} \big(-z(x,t)\big)\, dx \geq \int_0^{s(t)} -\big(x\, z(x,t)\big)\, dx = \frac{s^2(t)}{2} - R(t) \geq -R(T_o) > 0,\ t \geq T_o.$$

If we integrate with respect to time the last inequality, it follows

$$(2.22) \qquad \iint_{D_T} z(x,\tau)\, dx\, d\tau \leq R(T_o)\, t\,,\ t \geq T_o$$

contradicting (2.20).

PROPOSITION 2.8 .– If (T,s,z) is a solution of the Problem (P) and the functions h and f satisfy the following hypotheses :

(i) $h = h(x) \leq 0$ is an increasing function in $[0,1]$;

(ii) $f(t) \leq 0$ is a decreasing function of t, $t \geq 0$;

then Case (C) implies $R(T_C) \leq 0$.

PROOF .– From Proposition (2.1)(iii) we have $z_x(x,t) \geq 0$ in D_T. From Corollary 2.5 the isotherm $z = -1$ must reach the free boundary at $T = T_C$, then the domain is divided in two regions, and $z(x,t) \leq -1$ to the left of the isotherm $z = -1$. If we replace this estimation in (2.8) we get

$$(2.23) \qquad \frac{s(T_C)^2 - 1}{2} \geq \int_0^1 x\, h(x)\, dx + \int_0^{T_C} f(\tau)\, d\tau + \int_0^{s(T_C)} x\ dx\ ,$$

i.e. $\frac{s^2(T_C)}{2} \geq R(T_C) + \frac{s^2(T_C)}{2}$. Then $R(T_C) \leq 0$.

COROLLARY 2.9 .–If (T,s,z) is a solution of the problem P and the functions h and f satisfy the following hypotheses:

(i) $h = h(x) \leq -1$ in $(0,b)$, $h(b)=-1$ and $-1<h(x)<0$ in $(b,1)$, with $b\in(0,1)$.

(ii) $-1<f(t)<0$ in $(0,T_o)$, $f(T_o)=-1$ in (T_o,T_c) ; $f(t)<-1$ in (T_o,T_c).

then Case (C) implies $R(T_C) \leq 0$.

PROOF .– Using the same methods from Proposition 2.8, one obtains a region where $z \leq -1$ to left of one isotherm $z=-1$. In this region the inequality (2.23) is hold.

COROLLARY 2.10 .– If $R(t) > 0$ for every $t > 0$, then we have case (A).

REMARK 2 .— In the same way we can obtain the same results for the solid phase overheated.

III.— The case $f(t) = $ Constant.

In this section we consider the case in which the temperature $z(0,t)$ is a constant in any time, say $f(t) = -B < 0$ ($B>0$) and the initial temperature $h(x) = 0$.

As a trivial consequence of Proposition 2.7 no global solution exist in this case, so either (B) or (C) must occur. Moreover, one can easily prove that the solution, for a given $B > 0$, exists for any $t \leq T$, with $T \geq \frac{1}{2B}$.

PROPOSITION 3.1 .— Let (T,s,z) be a solution of the problem (P). If $0 \leq B \leq 1$ then we have case (B).

PROOF .— Using the maximum principle it follows that $z(x,t) \geq -1$. From Proposition 2.4 case (C) is excluded. Then only case (B) is possible.

PROPOSITION 3.2 .— Let (T,s,z) be a solution of the problem (P), then

(3.1) $$z(x,t) \geq -B \ \mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right), \text{ in } D_T .$$

PROOF .— This follows from the maximum principle applied to $z(x,t) - w(x,t)$, where w is the solution of the heat equation in the first quadrant $x > 0$ and $t > 0$, with the following boundary conditions : $w(x,0) = 0$, $x > 0$ and $w(0,t) = -B$, $t > 0$. By the other hand, w is given by

(3.2) $$w(x,t) = -B \ \mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right), \quad x \geq 0, \quad t \geq 0.$$

As a consequence of Lemma 3.2, we have the following estimate on B for the case (C).

PROPOSITION 3.3 .— Let (T,s,z) be a solution of the problem (P). Then, $B \leq \frac{\pi}{2}$ is a necessary condition in order to have case (B).

PROOF .— From Lemma 3.2 we can deduce that

(3.3) $$z_x(0,t) \geq w_x(0,t) = \frac{B}{\sqrt{\pi t}} .$$

We replace the above estimation in (2.7) and it follows

$$(3.4) \qquad 1 \geq \frac{B}{\sqrt{\pi}} \int_0^{1/2B} \frac{1}{\sqrt{\tau}}\, d\tau = \sqrt{\frac{2B}{\pi}} .$$

PROPOSITION 3.4 .— The following inequalities hold :

$$(3.5) \qquad s(t) \geq \sqrt{1-2\,B\,t}\,, \; 0 \leq t \leq \frac{1}{2B} .$$

$$(3.6) \qquad s(t) \leq \frac{1 - 2\,B\sqrt{\frac{t}{\pi}}}{1 - B}\,, \; t \geq 0, \;\; 0 < B < 1.$$

$$(3.7) \qquad s(t) \leq \frac{\sqrt{1-2\,B\,t}}{\sqrt{1 - B}}\,, \; 0 \leq t \leq \frac{1}{2B}\,, \; 0 < B < 1.$$

PROOF .— (3.5) and (3.7) follow by replacing $z(x,t) \leq 0$ and $z(x,t) \geq -B$ respectively in (2.8). Moreover, (3.6) follows by replacing $z(x,t) \geq -B$ and the estimation (3.3) in (2.7).

From now on, we shall consider the particular case

$$(3.8) \qquad f(t) = -B < 0,\; t > 0 \;\; (B > 0)$$

corresponding to conditions (2.1)−(2.6) with an initial temperature $h = h(x)$. This problem will be denoted by (P_B).

We shall give an explicit solution to problem (P_B), when the condition

$$(3.9) \qquad \dot{s}(0) < 0$$

is considered.

PROPOSITION 3.5 .— The real number $T > 0$ and the functions $x = s(t)$ and $z=z(x,t)$, defined by

$$(3.10) \qquad z(x,t) = \Phi\Big(\frac{x}{s(t)}\Big)\,, \quad 0 < x < s(t),\; 0 < t < T,$$

is a solution of (P_B) if and only if the function $\Phi = \Phi(\xi)$ is given by

$$(3.11) \qquad \Phi(\xi) = -B + \frac{B}{F(\eta)}\, F(\eta\xi)$$

where $\eta > 0$ is a solution of the equation

(3.12) $$G(x) = \frac{B}{2}, x > 0$$

with

(3.13) $$F(x) = \int_0^x e^{r^2}\, dr, \qquad G(x) = x \exp(-x^2)\, F(x).$$

Moreover, in this case, we have

(3.14i) $$s(t) = \sqrt{1 - 4\eta^2 t} = 2\eta \sqrt{T - t},\ 0 \le t \le T,$$

(3.14ii) $$T = \frac{1}{4\eta^2} > 0, \quad \dot{s}(0) = -2\eta^2 < 0,$$

(3.14iii) $$h(x) = \Phi(x),\ 0 \le x \le 1.$$

PROOF .— The function $\Phi = \Phi(\xi)$ must satisfy the following ordinary differential problem :

(3.15) $$\Phi''(\xi) + \dot{s}(0)\, \xi\, \Phi'(\xi) = 0,\ \Phi(0) = -B,\ \Phi(1) = 0.$$

From (3.15) we obtain the thesis.

REMARK 3 .— (i) Let $f_1 = f_1(x)$ be the function defined by

(3.16) $$f_1(x) = \exp(-x^2)\, F(x) > 0,\ x > 0 \quad (\text{Dawson's integral [1]}).$$

Then, we have the following properties.

(3.17i) $$f_1(0) = 0, \qquad f_1(+\infty) = 0,$$

(3.17ii) $$f_2(x) = f_1'(x) = 1 - 2x f_1(x) = \begin{cases} > 0 & \text{if } 0 < x < x_1, \\ = 0 & \text{if } x = x_1, \\ < 0 & \text{if } x > x_1, \end{cases}$$

where

(3.17iii) $$x_1 \doteq 0.924, \qquad f_1(x_1) \doteq 0.541.$$

Moreover, we have

$$\text{(3.17iv)} \qquad f_1''(x) = \begin{vmatrix} < 0 & \text{if} & 0 < x < x_2\,, \\ 0 & \text{if} & x = x_2\,, \\ > 0 & \text{if} & x > x_2\,, \end{vmatrix}$$

where

$$\text{(3.17v)} \qquad x_2 \doteqdot 1.502\,, \qquad f_1(x_2) \doteqdot 0.428\,.$$

(ii) The function $G = G(x) = x\, f_1(x)$, defined in (3.13) verifies the following properties :

$$\text{(3.18i)} \qquad G(x) = \tfrac{1}{2}\,(\,1 - f_2(x)\,), \quad G(0) = 0,\ G(x_1) = G(+\infty) = \tfrac{1}{2}\,,$$

$$\text{(3.18ii)} \qquad G'(x) = f_1(x) + x\, f_2(x) = -\,\tfrac{1}{2}\, f_2'(x) = -\,\tfrac{1}{2}\, f_1''(x),\ G'(0) = 0,$$

where

$$\text{(3.18iii)} \qquad G_M = \operatorname*{Max}_{x > 0} G(x) = G(x_2) \doteqdot 0.645\,.$$

<u>COROLLARY 3.6</u> .—i) The equation (3.12) has a solution $\eta > 0$ if and only if the datum B satisfies the inequalities

$$\text{(3.19)} \qquad 0 < B \le 2\, G_M\,, \qquad (\text{ with } 2\, G_M \doteqdot 1.29\,)\,.$$

(ii) If $0 < B \le 1$ then the number $\eta > 0$ is unique.

(iii)If $1 < B < 2\, G_M$ there exist two numbers $\eta_1 = \eta_1(B)$ and $\eta_2 = \eta_2(B)$ which satisfy the following inequalities

$$\text{(3.20)} \qquad 0 < x_1 < \eta_1 < x_2 < \eta_2\,,$$

and the limits

$$\text{(3.21i)} \qquad \lim_{B \to 1^+} \eta_1(B) = x_1, \qquad \lim_{B \to 1^+} \eta_2(B) = +\infty,$$

$$\text{(3.21ii)} \qquad \lim_{B \to 2G_M} \eta_1(B) = \lim_{B \to 2G_M} \eta_2(B) = x_2\,.$$

(iv) If $B = 2\, G_M$ then the number $\eta = \eta(B)$ is unique and given by

$$\text{(3.22)} \qquad \eta = \eta(2G_M) = x_2\,.$$

<u>PROOF</u> .—(i), (ii) and (iv) follow from the properties of the function G and (3.18i,ii,iii).
(iii) Since G is an increasing function from 0 to G in $[0,x_2]$ and decreasing function from G_M to $\frac{1}{2}$ in $[x_2,\infty]$, then for the case $1<B<2G_M$ there exist two numbers $\eta_1=\eta_1(B)$ and $\eta_2=\eta_2(B)$ which satisfy the inequality (3.20) and the limits (3.21 i,ii).

<u>THEOREM 3.7</u> .— If we choose a parameter $\eta \geq 0$ we obtain that the following family of functions :

(3.23) $$z(x,t) = -\,2\,G(\eta)\,[\,1 - \frac{F(\eta\,\frac{x}{s(t)})}{F(\eta)}\,]\,, \qquad 0 < x < s(t),\ 0 < t < T,$$

(3.24) $$s(t) = \sqrt{1 - 4\,\eta^2 t}\,, \qquad 0 < t < T,$$

(3.25) $$T = \frac{1}{4\eta^2} > 0, \qquad B = 2\,G(\eta),$$

(3.26) $$h(x) = -\,2\,G(\eta)\,[\,1 - \frac{F(\eta\,x)}{F(\eta)}\,], \qquad 0 < x < 1,$$

is solution of the supercooled one-phase Stefan problem (P_B). Moreover, we have the case (B) and

(3.27) $$\{\,T\,\}_{\eta\,>\,0} = \mathbb{R}^+, \qquad \{\,B\,\}_{\eta\,>\,0} = (0,\,2\,G_M)\,,$$

where G_M is given by (3.18iii).

<u>PROOF</u> .— It follows from Propotition 3.5, Remark 2 and Corollary 3.6.

<u>REMARK 4</u> .— The solution (3.23)—(3.26), for each $\eta > 0$ verifies the following equalities

(3.28) $$\int_0^1 h(x)\,dx = -\,[\,1 - \exp(-\eta^2)\,],$$

(3.29) $$\int_0^1 x\,h(x)\,dx = -\,\frac{1}{2} + 8\,\eta^2\,G(\eta) = -\,\frac{1}{2} + B\,T,$$

(3.30) $$\int_0^t z_x(0,\tau)\,d\tau = \frac{B}{F(\eta)}\,\frac{t}{\sqrt{T} + \sqrt{T-t}}, \qquad 0 < t < T,$$

(3.31) $$\int_0^T z_x(0,\tau)\,d\tau = \exp(-\eta^2)\,.$$

It is important to remark what is the physical meaning of the coefficient B.

REMARK 5 .— We consider the supercooled one-phase Stefan problem in physical variables :

$$\text{(3.32)}\qquad \begin{aligned} &\rho\, c\, \theta_\tau - k\, \theta_{yy} = 0\,, && 0 < y < r(\tau), \quad 0 < \tau < \tau_o \\ &r(0) = b > 0\,, && \\ &\theta(y,0) = \phi(y)\,, && 0 < y < b, \\ &\theta(0,\tau) = -\,\theta_o < 0\,, && 0 < \tau < \tau_o\,, \\ &\theta(\, r(\tau),0\,) = 0\,, && 0 < \tau < \tau_o\,, \\ &k\,\theta_y(\, r(\tau),\tau) = -\,\rho\, l\, \dot{r}(\tau)\,, && 0 < \tau < \tau_o\,, \end{aligned}$$

where

(3.33)

θ : temperature, τ : time,

y : spatial variable k : thermal conductivity,

ρ : mass density c : specific heat,

l : latent heat of fusion.

We can obtain the dimensionless problem (P_B) by choosing the following variables :

$$\text{(3.34)}\qquad \begin{aligned} &x = \frac{y}{b}, && t = \frac{k}{\rho\, c\, b^2}\,\tau, && T = \frac{k}{\rho\, c\, b^2}\,\tau_o\,, \\ &z(x,t) = \frac{c}{l}\,\theta(y,\tau), && s(t) = \frac{r(\tau)}{b}\,, && \\ &h(x) = \frac{c}{l}\,\phi(y), && B = \frac{c\,\theta_o}{l}\,, && \end{aligned}$$

that is, B is the Stefan number.

REMARK 6 .— If, in problem (P_B) , we perform the classical transformation :

$$\text{(3.35)}\qquad u(x,t) = \int_x^{s(t)} \left\{ \int_\beta^{s(t)} [\, 1 + z(\alpha,t)\,]\, d\alpha \right\} d\beta\,,$$

then we obtain the following oxygen consumption problem

$$\text{(3.36)}\qquad \begin{aligned} &u_{xx} - u_t = 1\,, \qquad 0 < x < s(t), \qquad 0 < t < T\,, \\ &s(0) = 1, \\ &u(x,0) = H(x), \quad 0 < x < 1\,; \quad u(0,t) = G_o(t), \quad 0 < t < T, \\ &u(s(t),t) = u_x(s(t),t) = 0\,, \quad 0 < t < T, \end{aligned}$$

where

$$\text{(3.37i)} \qquad H(x) = \frac{1-B}{2}\, x^2 + \frac{\exp(-\eta^2)}{\eta}\, [\, \frac{F(\eta)}{2} + W(\eta x)\,]\,,$$

$$\text{(3.37ii)} \qquad G_0(t) = \frac{f_1(\eta)}{2\,\eta}\, s^2(t) = B\,(T - t) > 0,$$

with

$$\text{(3.38)} \qquad W(x) = (x^2 - \tfrac{1}{2})\, F(x) - \tfrac{x}{2}\exp(x^2), \qquad x > 0.$$

Moreover, the function $u = u(x,t)$ is given by :

$$\text{(3.39)} \qquad u(x,t) = \frac{1-B}{2}\, x^2 + \frac{\exp(-\eta^2)}{\eta}\, [\, \frac{F(\eta)}{2} + W(\frac{\eta\, x}{s(t)})\,]\, s^2(t)\,.$$

On the other hand, we have

$$\text{(3.40)} \qquad \begin{aligned} u_x(x,t) &= (1 - B)\, x - \exp(-\eta^2)\exp(\frac{x^2\,\eta^2}{s^2(t)})\, f_2(\frac{\eta\, x}{s(t)})\, s(t)\,, \\ u_x(0,t) &= -\exp(-\eta^2)\, s(t) < 0\,. \end{aligned}$$

<u>REMARK 7</u> .— (i) The function $W = W(x)$, defined by (3.38) verifies the following properties :

$$\text{(3.41)} \qquad \begin{aligned} &W(0) = 0, \qquad W(+\infty) = +\infty, \qquad W(x_1) = \operatorname*{Min}_{x>0} W(x) = -\frac{\exp(x_1^{\,2})}{2}\, f_1(x_1), \\ &W(x) = -\frac{\exp(x^2)}{2}\, G'(x), \qquad W'(0) = -1, \\ &W'(x) = -\exp(x^2)\, f_2(x), \qquad W''(x) = 2\, F(x)\,. \end{aligned}$$

(ii) The function $H = H(x)$, defined in (3.37), verifies the following properties ($H''(x) = 1 + h(x)$) :

$$\text{(3.42)} \qquad \begin{aligned} &H(0) = \frac{f_1(\eta)}{2\,\eta}\,, \qquad H'(0) = -\exp(-\eta^2), \qquad H''(0) = 1 - B, \\ &H(1) = 0, \qquad H'(1) = 0, \qquad H''(1) = 1, \\ &H(x) > 0, \qquad H'(x) < 0, \qquad \forall\, x \in [0,1)\,. \end{aligned}$$

On the other hand, we have :

(3.43) $$H''(x) > 0, \qquad \forall\, x \in (0,1], \qquad \forall\, B \in (0,1]\,,$$

and $H'' < 0$ in $[0,\delta]$, $H''(\delta) = 0$ and $H'' > 0$ in $(\delta,1]$, where $\delta \in (0,1)$ is the unique solution of the equation

(3.44) $$h(x) = -1\,, \qquad x \in (0,1),$$

for the case $1 < B \leq 2\,G_M$.

(iii) The function $u = u(x,t)$, defined in (3.39), verifies the following properties :

(3.45) $$\begin{aligned} &u_x(x,t) < 0, \quad 0 < x < s(t), 0 < t < T\,, \\ &0 < u(x,t) < B\,(T - t), \quad 0 < x < s(t)\,, \qquad 0 < t < T\,, \\ &-B < u_t(x,t) = z(x,t) < 0, \quad 0 < x < s(t), \quad 0 < t < T\,. \end{aligned}$$

REMARK 8 .– If $f \geq 0$, $h \geq 0$ and we impose the condition $\dot{s}(0) > 0$ then we obtain the classical Lamé-Clapeyron solution instead of the one obtained in Proposition 3.5.

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