CONSTRAINTS TO FUTURES-STYLE OPTION PREMIUMS

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Abstract: Many options on futures are subject to futures-style premium posting: the premium is not paid up front but marked to market, and the last settlement premium paid upon exercise. The previous literature has derived pricing models for such options. Only after that derivation, and using the resulting model or its assumptions, has it deducted pricing constraints like a put-call parity or the positivity of time value. We show the full generality of many rational option-pricing constraints.

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1 INTRODUCTION

This paper deals with *futures-style options on futures*. For parsimony, we use the abbreviated terminology initiated by [2]: *pure futures options* (PFO). All options on futures traded on EUREX, Euronext, Sydney Futures Exchange, and South African Exchange are PFO.

The literature on PFO options is limited and the seminal paper is [3]. Assuming a Black market [1] where the futures price follows a geometric Brownian motion and the interest rate is constant, the paper [3] derives a pricing formula and some properties of option premiums.

Let τ be an index indicating the closing time of the exercise date, and F_τ, the corresponding settlement price. A call on futures with exercise price K is an option whose exercise generates

- a payoff $F_\tau - K$ that is cash settled at time τ, and
- a long futures contract initiated at F_τ.

$F_\tau - K$ is referred to as the *exercise cash flow*, and the first mark-to-market cash flow (MTM CF) of the futures contract takes place the day after exercise. If the settlement price accurately represents the futures price at τ, the futures contract originated has zero net present value at τ. In the case of a put, the long receives an exercise cash flow of $K - F_\tau$ and a short futures contract initiated at F_τ.

If an option contract is settled futures-style, the premium is not paid cash; instead, any increase (decrease) of the premium generates a positive (negative) MTM CF to the long, and a negative (positive) MTM CF to the short. Thus, a futures-style option works exactly as a futures contract while the option is alive, i.e., while neither exercise time τ nor expiration T has taken place.

For concreteness, we will consider a call option. Assuming the buyer exercises the call, the MTM CF are:

Day	Cash flow
1 (Transaction day)	$C_1 - C_0$
2	$C_2 - C_1$
...	...
$\tau - 1$	$C_{\tau-1} - C_{\tau-2}$
τ (Exercise day)	$C_\tau - C_{\tau-1}$

where C_0 is the premium negotiated during trading hours, C_1 is the settlement premium of the day of trade, C_2 the settlement premium of the next day, and so on. On the day the option is exercised, the long receives an exercise cash flow of $F_\tau - K$, and pays the call's settlement price of that day, C_τ. Summing up, the total cash flows for the buyer can be arranged into three components:

B1: $C_\tau - C_0$, the accumulated MTM CF,

B2: the payment of $-C_\tau$, the settlement price of the day of exercise, and

B3: $F_\tau - K$, the exercise cash flow.

The sum of B2 and B3, $F_\tau - K - C_\tau$, will be referred to as *net exercise cash flow*. B1 plus B2 add up to $-C_0$, where C_0 is the premium originally negotiated.

For an option expiring at or out of the money, the exchange sets $C_T = 0$. When the option expires in the money, the exchange sets $C_T = F_T - K$ based on a no-arbitrage argument. The following formula presents, in one expression, the settlement premiums at expiration:

$$C_T = (F_T - K)^+ . \tag{1}$$

The explanations of this section are valid for puts by only substituting P for C and $F_T - K$ for $K - F_T$. Thus, the settlement premium at expiration of a put is

$$P_T = (K - F_T)^+ . \tag{2}$$

2 Some restrictions on futures-option premiums

We use the following assumptions:

A1: There are no arbitrage opportunities,

A2: The futures and options markets are frictionless.

Let G_t and H_t be futures prices, European futures-style option premiums, or weighted sums of the prices of the instruments included in portfolios composed of them, where each weight is the amount of the corresponding underlying.

Lemma 1 *If, at $t < T$, we know that $G_T \geq H_T$ under any circumstance of time T, then $G_t \geq H_t$.*

Proof. We will follow a recursive no-arbitrage argument from expiration to the pricing time, going through all intermediate settlement times. In order to make the recursive argument, we will show that if, at time s, we know that

$$G_{s+1} \geq H_{s+1}, \tag{3}$$

under any circumstance of time $s+1$, then

$$G_s \geq H_s \tag{4}$$

By negation of (4), if

$$G_s < H_s, \tag{5}$$

then the following strategy is an arbitrage: buy portfolio G and sell portfolio H at time s. The MTM CF at $s+1$ is

$$(G_{s+1} - G_s) - (H_{s+1} - H_s) = \underbrace{H_s - G_s}_{>0 \text{ by (5)}} + \underbrace{G_{s+1} - H_{s+1}}_{\geq 0 \text{ by (3)}} > 0.$$

This is an arbitrage because there is zero investment at time s and a positive cash flow at time $s+1$. Therefore, (5) cannot be true, so (4) holds.

So far, we have shown that

$$G_{s+1} \geq H_{s+1} \implies G_s \geq H_s .$$

To complete the argument, it is enough to note that the lemma assumes that, at expiration time T, $G_T \geq H_T$ under any circumstance. $\square$

Lemma 2 *If, at $t < T$, we know that $G_T = H_T$ then $G_t = H_t$.*

Proof. Equality follows from the original and the inverted weak inequalities between G_T and H_T and between G_t and H_t of Lemma 1. $\square$

2.1 PUT-CALL PARITY AND TIME-VALUE EQUIVALENCE

Theorem 1 *Put-call parity for European pure futures options is*

$$c_t - p_t = F_t - K. \tag{6}$$

Proof. We start with the mathematical identity

$$(F_T - K)^+ - (K - F_T)^+ \equiv F_T - K,$$

and substitute using (1) and (2):

$$c_T - p_T = F_T - K. \tag{7}$$

By using Lemma 2, we derive (6) from (7). □

Theorem 2 *European pure futures puts and calls with the same expiration T and strike K have equal time values:*

$$c_t - (F_t - K)^+ = p_t - (K - F_t)^+ . \tag{8}$$

Proof. This result is obtained by subtracting the mathematical identity

$$(F_t - K)^+ - (K - F_t)^+ \equiv F_t - K$$

from the put-call parity (6) of the option premiums, and rearranging terms. □

2.2 BOUNDARIES OF RATIONAL OPTION PRICING AND EXERCISE POLICY

Lemma 3 *The premium of a European PFO cannot be negative:*

$$c_t \geq 0 \quad and \quad p_t \geq 0.$$

Proof. We know that, at expiration, $c_T = (F_T - K)^+ \geqslant 0$ and $p_T = (K - F_T)^+ \geqslant 0$. Therefore, by Lemma 1, $c_t \geqslant 0$ and $p_t \geqslant 0$. □

Theorem 3 *The time value of a European PFO cannot be negative:*

$$c_t \geqslant (F_t - K)^+ \quad and \quad p_t \geqslant (K - F_t)^+ .$$

Proof. (i) For an at- or out-of-the-money option, the non-negativity of the time value follows from the non-negativity of the premium stated in Lemma 3.A. (ii) To prove the theorem for an in-the-money *call*, consider an out-of-the-money put with the same strike and expiration. As shown by (i), this put has a non-negative time value. Therefore, making use of Theorem 2, the time value of an in-the-money *call* is also non-negative. A symmetric argument establishes the non-negativity of the time value of an in-the-money *put*. □

Now, we consider American PFO. C_t and P_t will stand for the premium of an American call and put, respectively. Of course, $C_\tau \geq c_\tau$ and $P_\tau \geq p_\tau$.

Theorem 4 *It is optimal to hold American PFO until expiration.*

Proof. Let τ be the settlement time of a potential early-exercise date. Because the American call premium is greater than or equal to that of the European version ($C_\tau \geq c_\tau$), Theorem 3 implies that $C_\tau \geq F_\tau - K$. Therefore, the net exercise cash flow of a call is not positive prior to expiration, $F_\tau - K - C_\tau \leqslant 0$. Since the net exercise cash flow cannot be positive and the generated futures contract has zero net present value, early exercise never dominates the holding strategy, i.e., the latter is an optimal strategy. The same argument can be made for a put using its net exercise cash flow $K - F_\tau - P_\tau \leqslant 0$. □

Corollary 1 *The early-exercise right has no value; therefore, $C_t = c_t$ and $P_t = p_t$, and all the results derived for European are valid for American PFO.*

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