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NUMERICAL ANALYSIS OF A FAMILY OF OPTIMAL DISTRIBUTED CONTROL PROBLEMS GOVERNED BY AN ELLIPTIC VARIATIONAL INEQUALITY

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Abstract

The numerical analysis of a family of distributed mixed optimal control problems governed by elliptic variational inequalities (with parameter $\alpha > 0$) is obtained by considering the finite element method

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with parameter $h > 0$. A commutative diagram for two continuous optimal control problems and the corresponding two discrete optimal control problems is obtained when $h \to 0$, $\alpha \to \infty$, and $(h, \alpha) \to (0, \infty)$.

1. Introduction

Following [5], we consider a bounded domain $\Omega \subset \mathbb{R}^n$ whose regular boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ consists of the union of two disjoint portions Γ_1 and Γ_2 with $meas(\Gamma_1) > 0$, and we state, for each $\alpha > 0$, the following free boundary system:

$$u \geq 0; \quad u(-\Delta u - g) = 0; \quad -\Delta u - g \geq 0 \text{ in } \Omega; \tag{1.1}$$

$$-\frac{\partial u}{\partial n} = \alpha(u - b) \text{ on } \Gamma_1; \quad -\frac{\partial u}{\partial n} = q \text{ on } \Gamma_2; \tag{1.2}$$

where the function g in (1.1) can be considered as the internal energy in Ω, $\alpha > 0$ is the heat transfer coefficient on Γ_1, $b > 0$ is the constant environment temperature, and q is the heat flux on Γ_2. The variational formulation of the above problem is given as (system (S_α)):

Find $u = u_{\alpha g} \in K_+$ such that $\forall v \in K_+$,

$$a_\alpha(u_{\alpha g}, v - u_{\alpha g}) \geq (g, v - u_{\alpha g})_H - (q, v - u_{\alpha g})_Q + \alpha(b, v - u_{\alpha g})_R, \tag{1.3}$$

where

$$K_+ = \{v \in H^1(\Omega) : v \geq 0 \text{ in } \Omega\}, \quad H = L^2(\Omega),$$

$$Q = L^2(\Gamma_2) \text{ and } R = L^2(\Gamma_1),$$

$(u, v)_A$ is the usual inner functional product over the set A (with $A = H$, Q, R). The application a is defined as $a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$ and

$$a_\alpha(u, v) = a(u, v) + \alpha(u, v)_R. \tag{1.4}$$

We note that a_1, and therefore a_α, is a bilinear, continuous, symmetric and coercive form on V with λ_1 and $\lambda_\alpha = \lambda_1 \min\{1, \alpha\} > 0$ the corresponding coercive constants [16, 22].

In [5], the following family of continuous distributed optimal control problems associated with the system (S_α) was considered for each $\alpha > 0$:

Problem (P_α): Find the distributed optimal control $g_{op_\alpha} \in H$ such that

$$J_\alpha(g_{op_\alpha}) = \min_{g \in H} J_\alpha(g) = \frac{1}{2} \| u_{\alpha g} \|_H^2 + \frac{M}{2} \| g \|_H^2, \tag{1.5}$$

where the quadratic cost functional $J_\alpha : H \to \mathbb{R}_0^+$, $M > 0$ a given constant and $u_{\alpha g}$ is the corresponding solution of the elliptic variational inequality (1.3) associated to the control $g \in H$.

Several optimal control problems are governed by elliptic variational inequalities [1-3, 9, 19, 20, 27] and there exists an abundant literature about continuous and numerical analysis of optimal control problems governed by elliptic variational equalities or inequalities [7, 10-15, 18, 23, 24] and by parabolic variational equalities or inequalities [4].

The objective of this work is to make the numerical analysis of the continuous optimal control problem (P_α) which is governed by the elliptic variational inequality (1.3) by proving the convergence of a discrete solution to the solution of the continuous optimal control problem.

In Section 2, we establish the discrete expression for the continuous elliptic variational inequality (1.3), and we obtain that these discrete problems possess unique solutions for all positive h. Moreover, we define a family $(P_{h\alpha})$ of discrete optimal control problems (2.3) and, we obtain several properties for the state system (2.1) and for the discrete cost functional $J_{h\alpha}$ defined in (2.5).

In Section 3, on adequate functional spaces, we obtain a result of global strong convergence when the parameter $h \to 0$ (for each $\alpha > 0$) and when

$\alpha \to \infty$ (for each $h > 0$) for the discrete state systems and for the discrete optimal problem corresponding to (P_α). We end this work proving the double convergence of the discrete optimal solutions of $(P_{h\alpha})$ when $(h, \alpha) \to (0, \infty)$ obtaining a complete commutative diagram among two discrete and two continuous optimal control problems given in Figure 1. We generalize recent results obtained for optimal control problems governed by elliptic variational equalities given in [25, 26].

2. Properties of the Discretization of the Problem (P_α)

Following the considerations given in [21], we approximate the sets V and K_+ by:

$$V_h = \{v_h \in C^0(\overline{\Omega}) : v_h/T \in P_1(T), \forall T \in \tau_h\},$$

$$K_{+h} = \{v_h \in V_h : v_h \geq 0 \text{ in } \Omega\}.$$

The discrete formulation $(S_{h\alpha})$ of the continuous system (S_α) is, for each $\alpha > 0$, defined as: Find $u_{h\alpha g} \in K_{+h}$ such that for all $v_h \in K_{+h}$,

$$a_\alpha(u_{h\alpha g}, v_h - u_{h\alpha g}) \geq (g, v_h - u_{h\alpha g})_H - (q, v_h - u_{h\alpha g})_Q + \alpha(b, v_h - u_{h\alpha g})_R. \tag{2.1}$$

Theorem 2.1. *Let $g \in H$ and $q \in Q$. Then there exists a unique solution of the elliptic variational inequality* (2.1).

Proof. It follows from the application of Lax-Milgram Theorem [16]. □

Lemma 2.1. (a) *Let g_n and $g \in H$, and $u_{h\alpha g_n}$ and $u_{h\alpha g} \in K_{+h}$ be the associated solutions of the system $(S_{h\alpha})$ for each $\alpha > 0$. If $g_n \rightharpoonup g$ in H weak, then we have that*

(i) $\exists C > 0$ (*independent of h, α and of n*) *such that*

$$\| u_{h\alpha g_n} \|_V \leq C; \tag{2.2}$$

(ii) $\forall h > 0,$

$$\lim_{n\to\infty} \| u_{h\alpha g_n} - u_{h\alpha g} \|_V = 0. \tag{2.3}$$

(b) *We have that*

$$\| u_{h\alpha g_2} - u_{h\alpha g_1} \|_V \leq \frac{1}{\lambda_\alpha} \| g_2 - g_1 \|_H,$$

where $u_{h\alpha g_i}$ *is the associated solution of the system* $(S_{h\alpha})$ *for* g_i, $i = 1, 2.$

Proof. We follow a similar methodology as in [10, 21]. □

Lemma 2.2. *Let* $u_{\alpha g} \in K_+ \cap H^r(\Omega)$ $(1 < r \leq 2)$ *and* $u_{h\alpha g} \in K_{+h}$ *be the solutions of the elliptic variational inequalities* (1.3) *and* (2.1), *respectively*, *for the control* $g \in H$. *Then there exists a positive constant C such that*

$$\| u_{h\alpha g} - u_{\alpha g} \|_V \leq C(\alpha) h^{(r-1)/2}. \tag{2.4}$$

Proof. If we consider $v = u_{h\alpha g} \in K_{+h} \subset K_+$ in the elliptic variational inequality (1.3) and $v_h = \Pi_h(u_{\alpha g}) \in K_{+h}$ in (2.1) (where Π_h is the interpolation operator [6, 21]), and calling $w = \Pi_h(u_{\alpha g}) - u_{\alpha g}$, we have that

$$a_\alpha(u_{h\alpha g} - u_{\alpha g}, u_{h\alpha g} - u_{\alpha g}) \leq a_\alpha(u_{h\alpha g}, w) - (g, w)_H + (q, w)_Q - \alpha(b, w)_R.$$

By using the coerciveness of a_α, [6] and by some mathematical computation, we obtain that

$$\| u_{h\alpha g} - u_{\alpha g} \|_V^2 \leq \frac{C}{\lambda_\alpha} \| \Pi_h(u_{\alpha g}) - u_{\alpha g} \|_V \leq \frac{C}{\lambda_\alpha} h^{r-1} \| u_{\alpha g} \|_r. \qquad \square$$

Now, we consider the continuous optimal control problem which was established in (1.5) and we establish the following discrete distributed optimal control problem $(P_{h\alpha})$:

Find $g_{op_{h\alpha}} \in H$ such that

$$J_{h\alpha}(g_{op_{h\alpha}}) = \min_{g \in H} J_{h\alpha}(g) = \min_{g \in H} \frac{1}{2} \| u_{h\alpha g} \|_H^2 + \frac{M}{2} \| g \|_H^2, \tag{2.5}$$

where $u_{h\alpha g}$ is the unique solution of the elliptic variational inequality (2.1) for a given control $g \in H$ and a given parameter $\alpha > 0$. We remark that the discrete (in the space) distributed optimal control problem $(P_{h\alpha})$ is still an infinite dimensional optimal control problem since the control space H is not discretized.

Theorem 2.2. *For the control* $g \in H$, *the parameters* $\alpha > 0$ *and* $h > 0$, *we have*:

(a) $\lim_{\| g \|_H \to \infty} J_{h\alpha}(g) = \infty.$

(b) $J_{h\alpha}(g) \geq \frac{M}{2} \| g \|_H^2 - C \| g \|_H$ *for some constant C independent of* $h > 0$.

(c) *The functional* $J_{h\alpha}$ *is a lower weakly semi-continuous application in H*.

(d) *For each* $h > 0$ *and* $\alpha > 0$, *there exists a solution of the discrete distributed optimal control problem* (2.5).

Proof. From the definition of $J_{h\alpha}(g)$, we obtain (a) and (b).

(c) Let $g_n \rightharpoonup g$ in H weak. Then by using the equality $\| g_n \|_H^2 = \| g_n - g \|_H^2 - \| g \|_H^2 + 2(g_n, g)_H$, we obtain that

$$\| g \|_H \leq \liminf_{n \to \infty} \| g_n \|_H.$$

Therefore, we have

$$\liminf_{n \to \infty} J_{h\alpha}(g_n) \geq \frac{1}{2} \| u_{h\alpha g} \|_H^2 + \frac{M}{2} \| g \|_H^2 = J_{h\alpha}(g).$$

(d) It follows from [17]. □

Lemma 2.3. *If the continuous state system has the regularity* $u_{\alpha g} \in H^r(\Omega)$ $(1 < r \leq 2)$ *for* $g \in H$ *and* $\alpha > 0,$ *then we have the following estimation* $\forall g \in H$:

$$| J_{h\alpha}(g) - J_{\alpha}(g) | \leq C(\alpha) h^{\frac{r-1}{2}}, \tag{2.6}$$

where C is a positive constant independent of $h > 0.$

Proof. It follows from the definition of $J_{h\alpha}$ and by (2.4). □

Remark. In general, the solution of the discrete optimal control problem (2.5) is not unique. Following [21], we can also define an open problem, for each $\alpha > 0,$ in order to have the uniqueness (see (39)-(42) and Remarks 8 and 9 in [21]).

3. Results of Convergence

3.1. Convergence when $h \to 0$

Theorem 3.1. *Let* $u_{\alpha g} \in K_+ \cap H^r(\Omega)$ $(1 < r \leq 2)$ *and* $u_{h\alpha g} \in K_{+h}$ *be the solutions of the elliptic variational inequalities* (1.3) *and* (2.1), *respectively, for the control* $g \in H$. *Then* $u_{h\alpha g} \to u_{\alpha g}$ *in V when* $h \to 0^+$.

Proof. Similarly to the part (a) of Lemma 2.1, we can show that there exists a constant $C > 0$ such that $\| u_{h\alpha g} \|_V \leq C,$ $\forall h > 0.$ Therefore, we conclude that there exists $\eta_\alpha \in V$ so that $u_{h\alpha g} \rightharpoonup \eta_\alpha$ in V (in H strong) as $h \to 0^+$ and $\eta \in K_+$. On the other hand, given $v \in K_+,$ let $v_h = \Pi(v)$ $\in K_{+h}$ for each h such that $v_h \to v$ in V when h goes to zero. Now, by considering $v_h \in K_{+h}$ in the discrete elliptic variational inequality (2.1), we get:

$$a_\alpha(u_{h\alpha g}, v_h - u_{h\alpha g}) \geq (g, v_h - u_{h\alpha g})_H - (q, v_h - u_{h\alpha g})_Q$$
$$+ \alpha(b, v_h - u_{h\alpha g})_R \tag{3.1}$$

and when we pass to the limit as $h \to 0^+$ in (3.1) by using that the bilinear form a is lower weak semi-continuous in V, we obtain:

$$a_\alpha(\eta_\alpha, v - \eta_\alpha) \geq (g, v - \eta_\alpha)_H - (q, v - \eta_\alpha)_Q$$
$$+ \alpha(b, v - \eta_\alpha)_R, \quad \forall v \in K_+$$

and from the uniqueness of the solution of the discrete elliptic variational inequality (1.3), we obtain that $\eta = u_{\alpha g}$.

Now, we will prove the strong convergence. As a consequence of Lemma 2.2, by passing to the limit when $h \to 0^+$ in the inequality (2.4), it results:

$$\lim_{h \to 0^+} \| u_{h\alpha g} - u_{\alpha g} \|_V = 0. \qquad \square$$

Theorem 3.2. *Let* $u_{\alpha g_{op}} \in K_+$ *be the continuous state system associated to the optimal control* $g_{op_\alpha} \in H$ *which is the solution of the continuous distributed optimal control problem* (1.5). *If, for each* $h > 0$, *we choose a discrete optimal control* $g_{op_{h\alpha}} \in H$ *which is a solution of the discrete distributed optimal control problem* (2.5) *and its corresponding discrete state system* $u_{h\alpha g_{op_{h\alpha}}} \in K_{+h}$, *we obtain that*

$$u_{h\alpha g_{op_{h\alpha}}} \to u_{\alpha g_{op_\alpha}} \text{ in } V \text{ strong when } h \to 0^+ \tag{3.2}$$

and

$$g_{op_{h\alpha}} \to g_{op_\alpha} \text{ in } H \text{ strong when } h \to 0^+. \tag{3.3}$$

Proof. Now, we consider a fixed value of the heat transfer coefficient $\alpha > 0$. Let $h > 0$ and $g_{op_{h\alpha}}$ be a solution of (2.5) and $u_{h\alpha g_{op_{h\alpha}}}$ its associated discrete optimal state system which is the solution of the problem defined in (2.1) for each $h > 0$. From (2.3), we have that for all $g \in H$,

$$J_{h\alpha}(g_{op_{h\alpha}}) \leq \frac{1}{2} \| u_{h\alpha g} \|_H^2 + \frac{M}{2} \| g \|_H^2.$$

Then, if we consider $g = 0$ and $u_{h\alpha 0}$ its corresponding associated state system, then it results that

$$J_{h\alpha}(g_{op_{h\alpha}}) \leq \frac{1}{2} \| u_{h\alpha 0} \|_H^2 .$$

Since $\| u_{h\alpha 0} \|_H \leq C, \forall h,$ we can obtain:

$$\| u_{h\alpha g_{op_{h\alpha}}} \|_H \leq C \text{ and } \| g_{op_{h\alpha}} \|_H \leq \frac{1}{\sqrt{M}} C, \quad \forall h.$$

If we consider $v_h = b \in K_{+h}$ in the inequality (2.1) for $g_{op_{h\alpha}}$, then we obtain, because of the coerciveness of the application a_α:

$$\| u_{h\alpha g_{op_{h\alpha}}} \|_V \leq C,$$

where the constant C is independent of the parameter h and $\alpha > 0$. Now we can say that there exist $\eta_\alpha \in V$ and $f_\alpha \in H$ such that $u_{h\alpha g_{op_{h\alpha}}} \rightharpoonup \eta_\alpha$ in V weak (in H strong), and $g_{op_{h\alpha}} \rightharpoonup f_\alpha$ in H weak when $h \to 0^+$. Moreover, $\eta_\alpha \in K_+$. Then, as in Theorem 3.1, we can obtain that $\eta_\alpha = u_{\alpha f_\alpha}$.

By using that the functional cost J_α is semi-continuous in H weak (see [5]) and Theorem 3.1, it results that $f = u_{\alpha g_{op_\alpha}}$ and $\eta_\alpha = u_{g_{op_\alpha}}$.

Now, we consider $v = u_{h\alpha g_{op_{h\alpha}}} \in K_{+h} \subset K_+$ in the system (S_α) with control g_{op_α}, and $v_h = \Pi_h(u_{\alpha g_{op_\alpha}})$ in the discrete system $(S_{h\alpha})$ for the control $g_{op_{h\alpha}}$ and define $w_h = u_{h\alpha g_{op_{h\alpha}}} - u_{\alpha g_{op_\alpha}}$. After some mathematical work, we obtain that

$$\begin{aligned} a_\alpha(w_h, w_h) \leq & -a_\alpha(u_{h\alpha g_{op_{h\alpha}}}, \Pi_h(u_{\alpha g_{op_\alpha}}) - u_{\alpha g_{op_\alpha}}) \\ & + (q, \Pi_h(u_{\alpha g_{op_\alpha}}) - u_{\alpha g_{op_\alpha}})_Q - \alpha(b, \Pi_h(u_{\alpha g_{op_\alpha}}) - u_{\alpha g_{op_\alpha}})_R \\ & + (g_{op_{h\alpha}}, \Pi_h(u_{\alpha g_{op_\alpha}}) - u_{h\alpha g_{op_\alpha}})_H - (g_{op_\alpha}, w_h)_H . \end{aligned}$$

From the coerciveness of the application a_α, and $u_{h\alpha g_{op_{h\alpha}}} \to u_{\alpha g_{op_\alpha}}$ in H and $\Pi_h(u_{\alpha g_{op_\alpha}}) \to u_{\alpha g_{op_\alpha}}$ in H, we obtain that $\| w_h \|_V \to 0$ if $h \to 0$. Thus, (3.2) holds. It is easy to see that (3.3) holds too. □

3.2. Convergence when $\alpha \to \infty$

Now, under the same hypothesis in Section 1, we consider the following free boundary system [5]:

$$u \geq 0; \quad u(-\Delta u - g) = 0; \quad -\Delta u - g \geq 0 \text{ in } \Omega; \tag{3.4}$$

$$u = b \text{ on } \Gamma_1; \quad -\frac{\partial u}{\partial n} = q \text{ on } \Gamma_2. \tag{3.5}$$

The variational formulation of the above problem is given as (S). Find $u_g \in K$ such that

$$a(u, v - u_g) \geq (g, v - u_g)_H - (q, v - u_g)_Q, \quad \forall v \in K, \tag{3.6}$$

where

$$K = \{v \in V : v \geq 0 \text{ in } \Omega, v/\Gamma_1 = b\}.$$

In [5], the following continuous distributed optimal control problem (P) associated with the elliptic variational inequality (3.6) was considered: Find the continuous distributed optimal control $g_{op} \in H$ such that

$$J(g_{op}) = \min_{g \in H} J(g) = \min_{g \in H} \frac{1}{2} \| u_g \|_H^2 + \frac{M}{2} \| g \|_H^2 \tag{3.7}$$

as in (2.5) with $M > 0$ a given constant and u_g is the corresponding solution of the elliptic variational inequality (3.6) associated to the control $g \in H$. Therefore, as in Section 2, we define the discrete variational inequality formulation (S_h) of the system (S) as follows: Find $u_{hg} \in K_h$ such that

$$a(u_{hg}, v_h - u_{hg}) \geq (g, v_h - u_{hg})_H - (q, v_h - u_{hg})_Q, \quad \forall v_h \in K_h, \tag{3.8}$$

where

$$K_h = \{v_h \in V_h : v_h \geq 0 \text{ in } \Omega,\ v_h/\Gamma_1 = b\}.$$

The corresponding discrete distributed optimal control problem (P_h) of the continuous distributed optimal control problem (P) is defined as: Find the discrete distributed optimal control $g_{op_h} \in H$ such that

$$J_h(g_{op_h}) = \min_{g\in H} J_h(g) = \min_{g\in H} \frac{1}{2}\| u_{hg} \|_H^2 + \frac{M}{2}\| g \|_H^2, \tag{3.9}$$

where u_{hg} is the solution of the elliptic variational inequality (3.8).

Theorem 3.3. (i) *Let* $g \in H$ *and* $q \in Q$. *Then there exists a unique solution of elliptic variational inequality* (3.8).

(ii) *There exists a solution of the discrete optimal control problem* (3.9).

Proof. (i) It follows from the application of Lax-Milgram Theorem [16, 17].

(ii) It follows from [21]. □

Theorem 3.4. *Let* $g \in H$, $q \in Q$ *and* $h > 0$. *Then we have*

$$\lim_{\alpha\to\infty}\| u_{h\alpha g} - u_{hg} \|_V = 0.$$

Proof. Without loss of generality, we consider $\alpha > 1$ and we define $w = u_{h\alpha g} - u_{hg} \in V$. By definition of a_α, we have:

$$a_\alpha(w, w) - a_1(w, w) = (\alpha - 1)\| w \|_R^2.$$

After mathematical work, we obtain that

$$a_1(w, w) \leq a_1(w, w) + (\alpha - 1)\| w \|_R^2 \leq (g, w)_H - (q, w)_Q - a(u_{hg}, w) \tag{3.10}$$

and by coerciveness of a_1, it results that

$$\| u_{h\alpha g} - u_{hg} \|_R^2 \leq \frac{C}{\alpha - 1}$$

and $u_{h\alpha g} \to u_{hg}$ in Γ_1, when $\alpha \to \infty$.

Moreover, as a consequence of (3.10), we obtain that $\| u_{h\alpha g} \|_V \leq C$ (C constant independent of α and h). Then there exists $\eta \in V$ such that

$$u_{h\alpha g} \rightharpoonup \eta \text{ in } V \text{ (in } H \text{ strong).}$$

Then the strong convergence in V is obtained similarly to the one in Theorem 3.1. □

Theorem 3.5. *If, for each* $h > 0$, *we choose* $g_{op_{h\alpha}} \in H$ *a solution of the optimal control problem* $(P_{h\alpha})$ *and consider its respective discrete state system* $u_{h\alpha g_{op_{h\alpha}}} \in K_{+h}$ *the solution of* (2.1), *then we obtain that*

$$u_{h\alpha g_{op_{h\alpha}}} \to u_{hf_h} \text{ in } V \text{ when } \alpha \to \infty \tag{3.11}$$

and

$$g_{op_{h\alpha}} \to f_h \text{ in } H \text{ when } \alpha \to \infty, \tag{3.12}$$

where $f_h \in H$ *is a solution of the discrete optimal control problem* (P_h) *and* u_{hf_h} *is its corresponding discrete state system solution of the variational inequality* (3.8).

Proof. As in Theorem 3.2, we have

$$\| u_{h\alpha g_{op_{h\alpha}}} \|_H \leq C \text{ and } \| g_{op_{h\alpha}} \|_H \leq \frac{1}{\sqrt{M}} C, \quad \forall h.$$

Now, considering $v_h = b$ in (2.1) (and we take $\alpha > 1$ without loss of generality) for the control $g_{op_{h\alpha}}$ and $w_h = b - u_{h\alpha g_{op_{h\alpha}}}$, we obtain:

$$a_\alpha(u_{h\alpha g_{op_{h\alpha}}}, w_h) \geq (g_{op_{h\alpha}}, w_h)_H - (q, w_h)_Q + \alpha(b, w_h)_R,$$

that is to say:

$$a_1(-w_h, w_h) + a_1(b, w_h) \geq (g_{op_{h\alpha}}, w_h)_H - (q, w_h)_Q + (\alpha - 1)\| w_h \|_R. \tag{3.13}$$

By the coerciveness of the application a_1, it results that

$$\| u_{h\alpha g_{op_{h\alpha}}} \|_V \leq C, \quad \forall \alpha > 0. \tag{3.14}$$

Moreover,

$$\| u_{h\alpha g_{op_{h\alpha}}} \|_R \leq \frac{C}{\alpha - 1}, \quad \forall \alpha > 0. \tag{3.15}$$

Then there exist $f_h \in H$ and $\eta_h \in V$ (we can see that $\eta_h \in K_h$) such that

$$g_{op_{h\alpha}} \rightharpoonup f_h \text{ in } H \tag{3.16}$$

and

$$u_{h\alpha g_{op_{h\alpha}}} \rightharpoonup \eta_h \text{ in } V \text{ (in } H \text{ strong).} \tag{3.17}$$

Letting $v_h \in K_h \subset K_{+h}$ and given $w_h = v_h - u_{h\alpha g_{op_{h\alpha}}}$, we have:

$$a_\alpha(u_{h\alpha g_{op_{h\alpha}}}, w_h) \geq (g_{op_{h\alpha}}, w_h)_H - (q, w_h)_Q + \alpha(b, w_h)_R,$$

$$a(u_{h\alpha g_{op_{h\alpha}}}, w_h) \geq (g_{op_{h\alpha}}, w_h)_H - (q, w_h)_Q + \alpha(b - u_{h\alpha g_{op_{h\alpha}}}, w_h)_R$$

and because of (3.16), (3.17) and similar arguments given in Theorem 3.2, and the fact that the application a is semi-continuous in V weak, we obtain that η_h is a solution of (3.8) for the control f_h. Then (by item (i) in Theorem 3.3), $\eta_h = u_{hf_h}$.

If we consider $v_h = u_{hf_h} \in K_h \subset K_{+h}$ in (2.1) for the control $g_{op_{h\alpha}} \in H$ and $w_h = u_{hf_h} - u_{h\alpha g_{op_{h\alpha}}}$, then:

$$a_\alpha(u_{h\alpha g_{op_{h\alpha}}}, w_h) \geq (g_{op_{h\alpha}}, w_h)_H - (q, w_h)_Q + \alpha(b, w_h)_R,$$

$$\begin{aligned} a_1(w_h, w_h) \leq\ & a_1(u_{hf_h}, w_h) - (g_{op_{h\alpha}}, w_h)_H + (q, w_h)_Q \\ & - \alpha(b, w_h)_R + (\alpha - 1)(u_{h\alpha g_{op_{h\alpha}}}, w_h)_R. \end{aligned}$$

Again, as a consequence of the coerciveness of the application a_1 and by (3.16), (3.17), it results (3.11).

Now we see that f_h is a solution of (2.5): because of Theorem 2.2(c), and by the definition of optimum:

$$J_h(f_h) \leq \lim_{\alpha\to\infty} J_{h\alpha}(g_{op_{h\alpha}}) \leq \lim_{\alpha\to\infty} J_{h\alpha}(g), \quad \forall g \in H,$$

and by Theorem 3.4, we conclude that

$$J_h(f_h) \leq J_h(g), \quad \forall g \in H.$$

Finally, we see that

$$J_h(f_h) \leq \lim_{\alpha\to\infty} J_{h\alpha}(g_{op_{h\alpha}}) \leq J_h(g), \quad \forall g \in H,$$

then, if we consider $g = f_h$:

$$\lim_{\alpha\to\infty} J_{h\alpha}(g_{op_{h\alpha}}) = J_h(f_h)$$

and, because of (3.11),

$$\lim_{\alpha\to\infty} \| g_{op_{h\alpha}} \|_H = \| f \|_H, \tag{3.18}$$

then, by using (3.16) and (3.17), we obtain (3.12). □

Now, following the idea given in [26], we have this final theorem:

3.3. Double convergence when $(h, \alpha) \to (0^+, \infty)$

Theorem 3.6. *If, for each* $h > 0$, *we choose* $g_{op_{h\alpha}} \in H$ *a solution of the optimal control problem* $(P_{h\alpha})$ *and we consider its respective discrete state system* $u_{h\alpha g_{op_{h\alpha}}} \in K_{+h}$, *which is the unique solution of* (2.1), *then we obtain that*

$$u_{h\alpha g_{op_{h\alpha}}} \to u_{g_{op}} \text{ in } V \text{ when } (h, \alpha) \to (0^+, \infty) \tag{3.19}$$

and

$$g_{op_{h\alpha}} \to g_{op} \text{ in } H \text{ when } (h, \alpha) \to (0^+, \infty), \tag{3.20}$$

where $g_{op} \in H$ *is the solution of the optimal control problem* (P) *and* $u_{g_{op}}$ *is its corresponding state system solution of the variational inequality* (3.6).

Proof. As in Theorem 3.2, there exists $u^* \in V$ with $u^*/\Gamma_1 = b$ and $g^* \in H$ such that

$$u_{h\alpha g_{oph\alpha}} \rightharpoonup u^* \text{ (strong in } H) \tag{3.21}$$

and

$$g_{oph\alpha} \rightharpoonup g^* \tag{3.22}$$

when $(h, \alpha) \to (0^+, \infty)$ in both cases. Let $v \in K$ be such that $v/\Gamma_1 = b$. We consider $v_h = \Pi_h(v) \in K_{+h}$ in the state system (2.1) and we define $w_h = v_h - u_{h\alpha g_{oph\alpha}}$. Then we obtain

$$a(u_{h\alpha g_{oph\alpha}}, w_h) \geq (g_{oph\alpha}, w_h)_H - (q, w_h)_Q.$$

Because the application a is semi-continuous weak in V and $w_h \to v - u^*$ in H when $(h, \alpha) \to (0, \infty)$, it results that u^* is a solution of (3.6). But this problem has a unique solution, then we conclude that $u^* = u_{g^*}$. Moreover, we have that

$$\begin{aligned} a_\alpha(u_{h\alpha g_{oph\alpha}} - u_{g^*}, u_{h\alpha g_{oph\alpha}} - u_{g^*}) &\leq (g_{oph\alpha}, u_{h\alpha g_{oph\alpha}} - \Pi(u_{g^*}))_H \\ &+ (q, u_{h\alpha g_{oph\alpha}} - \Pi(u_{g^*}))_Q + \alpha(b, u_{h\alpha g_{oph\alpha}} - \Pi(u_{g^*}))_R \\ &- a_\alpha(u_{g^*}, u_{h\alpha g_{oph\alpha}} - \Pi(u_{g^*})) + a_\alpha(u_{h\alpha g_{oph\alpha}}, \Pi(u_{g^*}) - u_{g^*}) \\ &- a_\alpha(u_{g^*}, \Pi(u_{g^*}) - u_{g^*}). \end{aligned}$$

Because of the coerciveness of the application a_α in V and by (3.21) and (3.22), we obtain (3.19) when $(h, \alpha) \to (0^+, \infty)$.

As the functional $J_{h\alpha}$ is lower weakly semi-continuous in H (Theorem 2.2) and (3.22), we obtain that $g_{oph\alpha} \rightharpoonup g_{op}$.

We also have that $\lim_{(h,\alpha)\to(0,\infty)} J_{h\alpha}(g_{op_{h\alpha}}) = J(g_{op})$, and then $\lim_{(h,\alpha)\to(0,\infty)} \| g_{op_{h\alpha}} \|_H = \| g_{op} \|_H$, and by (3.22) and (3.20), the thesis holds. □

4. Conclusion

In conclusion, by using the previous results given in [5] and [21], we obtain the following commutative diagram among the two continuous optimal control problems (P) and (P_α), and two discrete optimal control problems (P_h) and $(P_{h\alpha})$ when $h \to 0$, $\alpha \to \infty$ and $(h, \alpha) \to (0^+, \infty)$, which can be summarized by the following figure (Figure 1):

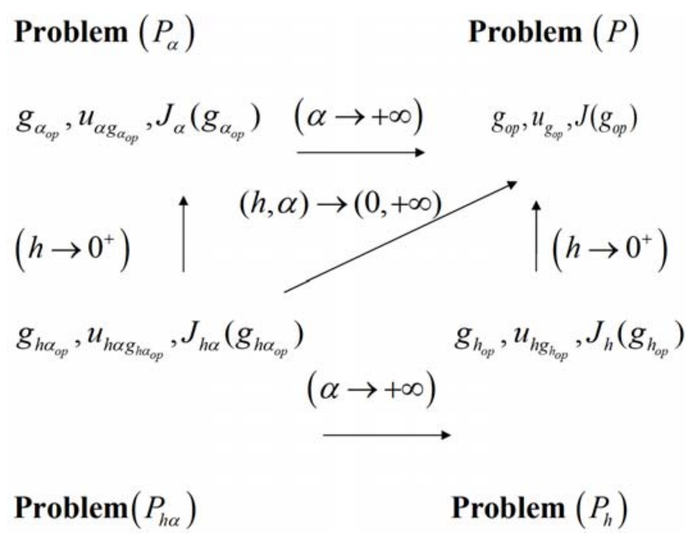

Figure 1. Complete diagram of convergence.

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