MAT

Serie A

Conferencias, seminarios y trabajos de Matemática

ISSN:1515-4904

7

Primeras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico

Primera Parte

Departamento de Matemática, Rosario, Argentina 2004

UNIVERSIDAD AUSTRAL
FACULTAD DE CIENCIAS EMPRESARIALES

ISSN 1515-4904

MAT

SERIE A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMÁTICA

No. 7

PRIMERAS JORNADAS SOBRE ECUACIONES DIFERENCIALES, OPTIMIZACIÓN Y ANÁLISIS NUMÉRICO

Primera Parte

Domingo A. Tarzia (Ed.)

INDICE

Rosario, Octubre 2004

AN INTEGRAL EQUATION IN ORDER TO SOLVE A ONE-PHASE STEFAN PROBLEM WITH NONLINEAR THERMAL CONDUCTIVITY *

María F. Natale (1) - Domingo A. Tarzia (1) (2)

(1) Depto. Matemática, FCE, Universidad Austral
Paraguay 1950, S2000FZF Rosario, Argentina
(2) and CONICET, Argentina.
E-mail: Maria.Natale@fce.austral.edu.ar
Domingo.Tarzia@fce.austral.edu.ar

Abstract

We study a one-phase Stefan problem for a semi-infinite material with temperature-dependent thermal conductivity with a constant temperature or a heat flux condition of the type $-q_0/\sqrt{t}$ ($q_0 > 0$) at the fixed face $x = 0$. We obtain in both cases sufficient conditions for data in order to have a parametric representation of the solution of the similarity type for $t \geq t_0 > 0$ with t_0 an arbitrary positive time. These explicit solutions are obtained through the unique solution of an integral equation with the time as a parameter.

Key words : Stefan problem, free boundary problem, moving boundary problem, phase-change process, nonlinear thermal conductivity, fusion, solidification, similarity solution.

2000 AMS Subject Classification: 35R35, 80A22, 35C05

I. Introduction. We will consider a phase-change problem (Stefan problem) for a non-linear heat conduction equation for a semi-infinite region $x > 0$ with a nonlinear thermal conductivity $k(\theta)$ given by

$$k(\theta) = \frac{\rho c}{(a + b\theta)^2} \tag{1}$$

and phase change temperature θ_f. This kind of thermal conductivity or diffusion coefficient was considered in [3, 5, 6, 14, 16, 18, 21, 23, 28]. The modeling of this type of systems is a great mathematical and industrial significance problem. Phase-change problems appear frequently in industrial processes and other problems of technological interest [1, 7, 8, 9, 11, 12, 13, 15, 17]. A recent large bibliography on the subject was given recently in [27].

*MAT - Serie A, 7 (2004), 15-24.

The mathematical formulation of our free boundary (fusion process) problem consists in determining the evolution of the moving phase separation $x = s(t)$ and the temperature distribution $\theta = \theta(x,t)$ satisfying the conditions

$$\rho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta)\frac{\partial \theta}{\partial x}\right) \ , \ 0 < x < s(t) \ , \ t > 0 \tag{2}$$

$$k(\theta(0,t))\frac{\partial \theta}{\partial x}(0,t) = -\frac{q_0}{\sqrt{t}} \ , \ q_0 > 0 \ , \ t > 0 \tag{3}$$

$$k\left(\theta\left(s(t),t\right)\right)\frac{\partial \theta}{\partial x}\left(s(t),t\right) = -\rho l \ \dot{s}(t) \ , \ t > 0 \tag{4}$$

$$\theta\left(s(t),t\right) = \theta_f \ , \ t > 0 \tag{5}$$

$$s(0) = 0 \tag{6}$$

where $a + b\theta_f > 0$, in order to guarantee that k is well defined. Here $-q_0/\sqrt{t}$ denotes the prescribed heat flux on the boundary $x = 0$ which is of the type imposed in [26]. This kind of heat flux condition (3) was also considered in numerous papers, e.g. [2, 10, 22]. Other problems in this subject are [4, 19, 23, 24].

The free boundary problem (2) – (6) with $k(\theta)$ defined by (1) is a particular case of one studied in [20, 25] by taking the parameter $d = 0$ for the following equation

$$\rho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta)\frac{\partial \theta}{\partial x}\right) - v(\theta)\frac{\partial \theta}{\partial x} \ , \ 0 < x < s(t) \ , \ t > 0 \tag{7}$$

where the thermal conductivity $k(\theta)$ and the velocity term $v(\theta)$ are given by (1) and

$$v(\theta) = \rho c \frac{d}{2\left(a + b\theta\right)^2} \tag{8}$$

respectively, and c, ρ and l are the specific heat, the density and the latent heat of fusion of the medium respectively, all of them are assumed to be constant with positive parameters a, b and d.

In those papers temperature and flux type conditions on the fixed face $x = 0$ were studied. Furthermore, necessary and sufficient conditions for the existence of an explicit solution was found in [20]. Here we study the case without the velocity term, i.e. $d = 0$ in the differential equation (7) which cannot be obtained from what it was previously done in [20, 25] for the case $d \neq 0$. In those papers it was defined the transformation

$$y = \frac{2}{d}\left[(1 + dx)^{\frac{1}{2}} - 1\right] \tag{9}$$

which is the identity if we take $d \to 0$ since

$$\lim_{d \to 0} \frac{2}{d}\left[(1 + dx)^{\frac{1}{2}} - 1\right] = x \ , \ \forall \, x > 0.$$

Then, the case $d = 0$ must be solved by using other techniques which will be the goal of this study.

In Section II we prove the existence and uniqueness of an explicit solution of the similarity type of the free boundary problem $(2)-(6)$ for $t \geq t_0 > 0$ with t_0 an arbitrary positive time when data satisfy condition $a + b\theta_f \geq bl/c$. The solution is explicitly given by $(41), (47)$ and (48), and by $(50), (51)$ for the cases $a + b\theta_f > bl/c$ and $a + b\theta_f = bl/c$ respectively. The explicit solution for the two cases is obtained through the unique solution of an integral equation in which time is a parameter.

Besides, there does not exist any solution of the similarity type to the free boundary problem $(2)-(6)$ for the case $a + b\theta_f < bl/c$.

II. Existence and uniqueness of solution of the free boundary problem with flux boundary condition on the fixed face.

We consider the free boundary problem $(2)-(6)$ with the parameters a, b and the coefficients l, c satisfy the following condition

$$a + b\theta_f > \frac{bl}{c} \quad . \tag{10}$$

If we define

$$\Theta = \frac{1}{a + b\theta} , \tag{11}$$

the problem $(2)-(6)$ becomes

$$\frac{\partial \Theta}{\partial t} = \Theta^2 \frac{\partial^2 \Theta}{\partial x^2} , \ 0 < x < s(t) , \ t > 0 \tag{12}$$

$$\frac{\partial \Theta}{\partial x}(0,t) = \frac{w}{\sqrt{t}} , \ t > 0 \tag{13}$$

$$\frac{\partial \Theta}{\partial x}(s(t),t) = \frac{bl}{c} \dot{s}(t) , \ t > 0 \tag{14}$$

$$\Theta(s(t),t) = \frac{1}{a + b\theta_f} , \ t > 0 \tag{15}$$

$$s(0) = 0 \tag{16}$$

where w is a constant defined by

$$w = \frac{bq_0}{\rho c}. \tag{17}$$

Let us perform the transformation

$$\begin{gathered} \chi(x,t) = \int_0^x \frac{d\eta}{\Theta(\eta,t)} \\ \Psi(\chi,t) = \Theta(x,t) \end{gathered} \tag{18}$$

and

$$S(t) = \chi(s(t),t) \quad . \tag{19}$$

The problem $(12)-(16)$ becomes

$$\frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial \chi^2} - \frac{w}{\sqrt{t}} \frac{\partial \Psi}{\partial \chi} , \ 0 < \chi < S(t) , \ t > 0 \tag{20}$$

$$\frac{\partial \Psi}{\partial \chi}(0,t) = \frac{w}{\sqrt{t}} \Psi\left(0,t\right) \ , \ t > 0 \tag{21}$$

$$\frac{\partial \Psi}{\partial \chi}(S(t),t) = \frac{1}{\left(a + b\theta_f\right)\left(\frac{c}{bl}\left(a + b\theta_f\right) - 1\right)}\left(\overset{\bullet}{S}\left(t\right) - \frac{w}{\sqrt{t}}\right) \ , \ t > 0 \tag{22}$$

$$\Psi(S(t),t) = \frac{1}{a + b\theta_f} \ , \ t > 0 \tag{23}$$

$$S(0) = 0 \tag{24}$$

where

$$\overset{\bullet}{S}\left(t\right) = \left(a + b\theta_f - \frac{bl}{c}\right) \overset{\bullet}{s}\left(t\right) + \frac{w}{\sqrt{t}} \,. \tag{25}$$

If we introduce the similarity variable

$$\xi = \frac{\chi}{2\sqrt{t}} \qquad , \tag{26}$$

and the solution is sought of type

$$\Psi\left(\chi,t\right) = \varphi\left(\xi\right) = \varphi\left(\frac{\chi}{2\sqrt{t}}\right) \tag{27}$$

then the free boundary $S(t)$ of the problem $(20)-(24)$ must be of the type

$$S(t) = 2\Lambda_1\sqrt{t} \ , \ t > 0 \tag{28}$$

with $\Lambda_1 > 0$ an unknown coefficient to be determined and the problem $(20)-(24)$ yields

$$\varphi''\left(\xi\right) + 2\varphi'\left(\xi\right)\left(\xi - w\right) = 0 \ , \ 0 < \xi < \Lambda_1 \tag{29}$$

$$\varphi'\left(0\right) = 2w\varphi\left(0\right) \tag{30}$$

$$\varphi\left(\Lambda_1\right) = \frac{1}{a + b\theta_f} \tag{31}$$

$$\varphi'\left(\Lambda_1\right) = \frac{2}{\left(a + b\theta_f\right)\left(\frac{c}{bl}\left(a + b\theta_f\right) - 1\right)}\left(\Lambda_1 - w\right). \tag{32}$$

Taking into account the expression (25) we have

$$s(t) = 2\lambda_1\sqrt{t} \tag{33}$$

with

$$\lambda_1 = \frac{\Lambda_1 - w}{a + b\theta_f - \frac{bl}{c}} \quad . \tag{34}$$

If we integrate (29) we obtain

$$\varphi(\xi) = D_1 \operatorname{erf}(\xi - w) + C_1 \tag{35}$$

where D_1 and C_1 are two constants of integration which can be determined from (30) and (31)

$$D_1 = \frac{\sqrt{\pi} w \exp(w^2)}{(a + b\theta_f)\left[1 + \sqrt{\pi} w \exp(w^2)\left(\operatorname{erf}(\Lambda_1 - w) + \operatorname{erf}(w)\right)\right]} \tag{36}$$

$$C_1 = \frac{1 + \sqrt{\pi} w \exp(w^2) \operatorname{erf}(w)}{(a + b\theta_f)\left(1 + \sqrt{\pi} w \exp(w^2)\left(\operatorname{erf}(\Lambda_1 - w) + \operatorname{erf}(w)\right)\right)} \tag{37}$$

Now, we have to consider here the condition (32) which implies that Λ_1 must be the solution of the following equation

$$W_1(x) = W_2(x) \quad , \quad x > w \tag{38}$$

where

$$W_1(x) = \frac{w \exp(w^2) \exp\left[-(x - w)^2\right]}{1 + w \exp(w^2) \sqrt{\pi}\left(\operatorname{erf}(x - w) + \operatorname{erf}(w)\right)} \tag{39}$$

and

$$W_2(x) = \frac{bl}{c(a + b\theta_f) - bl}(x - w) \quad . \tag{40}$$

It is easy to prove that $W_1(0) = w > 0, W_1(+\infty) = 0$, and W_1 is a decreasing function, and $W_2(w) = 0, W_2(+\infty) = +\infty$ and W_2 is an increasing function because condition (10). So, there exists a unique solution Λ_1 of the equation (38) and then we have the following theorem.

Theorem 1.- Let us consider the hypothesis (10).

(i) If (Θ, s) is a solution of the free boundary problem $(12) - (16)$ then $\Theta = \Theta(x, t)$ is a solution, in variable x, of the integral equation:

$$\Theta(x,t) = C_1 + D_1 \operatorname{erf}\left(\frac{\int_0^x \frac{d\eta}{\Theta(\eta,t)}}{2\sqrt{t}} - w\right) \ , \ 0 \leq x \leq s(t) \ , \tag{41}$$

where $t > 0$ is a parameter and w, D_1 and C_1 are defined by $(17), (36)$ and (37) respectively, and $s(t)$ is given by (33) and Λ_1 is the unique solution of the Eq. (38). Moreover, function $Y(x,t)$ defined by

$$Y(x,t) = \frac{1}{2\sqrt{t}} \int_0^x \frac{d\eta}{\Theta(\eta,t)} - w \quad , \quad 0 \leq x \leq s(t) \ , \ t > 0 \tag{42}$$

satisfies the conditions

$$\frac{\partial Y}{\partial x}(x,t) = \frac{1}{2\sqrt{t}} \frac{1}{\Theta(x,t)} \ , \ 0 < x < s(t) \ , \ t > 0 \tag{43}$$

$$Y(0,t) = -w \ , \ t > 0 \tag{44}$$

$$\frac{\partial Y}{\partial t}(x,t) = -\frac{1}{2t}\left(Y(x,t) + \frac{D_1}{\sqrt{\pi}}\frac{\exp\left(-Y^2(x,t)\right)}{\Theta(x,t)}\right) \ , \ 0 < x < s(t) \ , \ t > 0 \tag{45}$$

$$Y\left(s(t),t\right) = \Lambda_1 - w \ , \ t > 0 \tag{46}$$

(ii) Conversely, if Θ is a solution of the integral equation (41) with s given by (33) and function Y, defined by (42) satisfies the conditions $(43) - (46)$, and w, D_1 and C_1 are defined by $(17), (36)$ and (37) respectively, and Λ_1 is the unique solution of the Eq. (38) then (Θ, s) is a solution of the free boundary problem $(12) - (16)$.

(iii) The integral equation (41) has a unique solution for $t \geq t_0 > 0$ with t_0 is an arbitrary positive time.

(iv) The free boundary problem $(2) - (6)$ satisfying the hypothesis (10) has a unique similarity type solution (θ, s) for $t \geq t_0 > 0$ (with t_0 an arbitrary positive time) which is given by

$$\theta(x,t) = \frac{1}{b}\left[\frac{1}{\Theta(x,t)} - a\right], \quad 0 < x < s(t), \quad t \geq t_0 > 0 \tag{47}$$

$$s(t) = \frac{2\left(\Lambda_1 - w\right)}{a + b\theta_f - \dfrac{bl}{c}}\sqrt{t} \ , \ t \geq t_0 > 0 \tag{48}$$

where Θ is the unique solution of the integral Eq. (41) where Λ_1 is the unique solution of the Eq. (38), and w, D_1 and C_1 are defined by $(17), (36)$ and (37) respectively.

Proof.

(i) From the previous computation we have

$$\Theta(x,t) = \varphi(\xi) = C_1 + D_1 \operatorname{erf}\left(\xi - w\right) = C_1 + D_1 \operatorname{erf}\left(\frac{\int_0^x \frac{d\eta}{\Theta(\eta,t)}}{2\sqrt{t}} - w\right)$$

that is Θ is a solution of the integral equation (41). Function Y, defined by (42), satisfies the conditions $(43), (44)$ by elementary computations, and

$$\frac{\partial Y}{\partial t}(x,t) = -\frac{1}{4t\sqrt{t}}\int_0^x \frac{d\eta}{\Theta(\eta,t)} - \frac{1}{2\sqrt{t}}\int_0^x \Theta_{xx}(\eta,t)d\eta =$$

$$= -\frac{1}{2\sqrt{t}}\left(\frac{Y(x,t)}{\sqrt{t}} + \Theta_x(x,t)\right) = -\frac{1}{2\sqrt{t}}\left(\frac{Y(x,t)}{\sqrt{t}} + \frac{D_1}{\sqrt{\pi t}}\frac{\exp\left(-Y^2(x,t)\right)}{\Theta(x,t)}\right)$$

that is (45). Finally we get

$$Y\left(s(t),t\right) = \frac{1}{2\sqrt{t}}\int_0^{s(t)} \frac{d\eta}{\Theta(\eta,t)} - w = \frac{\chi\left(s(t),t\right)}{2\sqrt{t}} - w = \frac{S\left(t\right)}{2\sqrt{t}} - w = \Lambda_1 - w$$

that is (46).

(ii) In order to proof that (Θ, s) is a solution of the free boundary problem $(12) - (16)$ we get:

a)

$$\Theta_{xx}(x,t) = \left(\frac{D_1}{\sqrt{\pi t}} \frac{\exp\left(-Y^2(x,t)\right)}{\Theta(x,t)} \right)_x =$$

$$= -\frac{D_1}{\sqrt{\pi t}} \frac{\exp\left(-Y^2(x,t)\right)}{\Theta^2(x,t)} \left(Y(x,t) + \frac{D_1}{\sqrt{\pi}} \frac{\exp\left(-Y^2(x,t)\right)}{\Theta(x,t)} \right);$$

b)

$$\Theta_t(x,t) = \frac{2D_1}{\sqrt{\pi}} \exp\left(-Y^2(x,t)\right) Y_t(x,t) =$$

$$= -\frac{D_1}{\sqrt{\pi t}} \exp\left(-Y^2(x,t)\right) \left(Y(x,t) + \frac{D_1}{\sqrt{\pi}} \frac{\exp\left(-Y^2(x,t)\right)}{\Theta(x,t)} \right)$$

that is Eq. (12);

c)

$$\Theta(0,t) = C_1 - D_1 \operatorname{erf}(w) = \frac{D_1}{\sqrt{\pi} w \exp(w^2)};$$

d)

$$\Theta_x(0,t) = \frac{D_1}{\sqrt{\pi t}} \frac{\exp\left(-Y^2(0,t)\right)}{\Theta(0,t)} = \frac{w}{\sqrt{t}}, \text{ that is (13);}$$

e)

$$\Theta(s(t),t) = C_1 + D_1 \operatorname{erf}(\Lambda_1 - w) = \frac{1}{a + b\theta_f}, \text{ that is (15);}$$

f)

$$\Theta_x(s(t),t) = \frac{D_1}{\sqrt{\pi t}} \frac{\exp\left(-Y^2(s(t),t)\right)}{\Theta(s(t),t)} = \frac{(a+b\theta_f)\, D_1}{\sqrt{\pi t}} \exp\left(-(\Lambda_1 - w)^2\right) =$$

$$= \frac{1}{\sqrt{t}} W_1(\Lambda_1) = \frac{1}{\sqrt{t}} W_2(\Lambda_1) =$$

$$= \frac{1}{\sqrt{t}} \frac{bl}{c(a+b\theta_f) - bl} (\Lambda_1 - w) = \frac{bl\lambda_1}{c\sqrt{t}} = \frac{bl}{c} \dot{s}(t), \text{ that is (14)}$$

(iii) Now in order to complete the proof, we just have to proof the existence of a solution of the integral equation (41). If we define $Y(x,t)$ by (42) then, Eq. (41) is equivalent to the following Cauchy differential problem

$$\begin{aligned} &\frac{\partial Y}{\partial x}(x,t) = \frac{1}{2\sqrt{t}} \frac{1}{(C_1 + D_1 \operatorname{erf}(Y(x,t)))} \equiv G_1(x,t,Y(x,t)) \quad , \quad 0 < x < s(t) \ , \ t > 0 \\ &Y(0,t) = -w \ , \end{aligned} \tag{49}$$

with a positive parameter $t > 0$. We have $\left| \frac{\partial G_1}{\partial Y} \right| \leq \frac{D_1}{C_1^2 \sqrt{\pi t}}$ which is bounded for all

$t \geqslant t_0 > 0$, $0 \leq x \leq s(t)$, for an arbitrary positive time t_0. Then, problem (49) (i.e. the integral Eq. (41)) has a unique solution for $t \geq t_0 > 0$, for an arbitrary positive time t_0.

(iv) It follows from elementary but tedious computation.■

Remark 1. $Y(x,t)$ dos not possess a limit at $(0,0)$ because $Y(0,t) = -w = -\frac{bq_0}{\rho c} < 0$ for $t > 0$ and $\lim\limits_{t \mapsto 0} Y(s(t),t) = \Lambda_1 - w > 0$ for all $t > 0$.

If Θ is the solution of the integral equation (41) then Θ is strictly monotone in variable x. We obtain that $\theta(x,t) = (1/\Theta(x,t) - a)/b$ does not have limit when $(x,t) \to (0,0)$ but $\theta(x,t)$ is bounded in a neighborhood of $(0,0)$ checking that

$$\begin{aligned} \theta_f &= \lim_{(\eta,\tau)\mapsto(0,0)} \inf \theta(\eta,\tau) \leq \theta(x,t) \leq \lim_{(\eta,\tau)\mapsto(0,0)} \sup \theta(\eta,\tau) = \\ &= \theta_f + \frac{a+b\theta_f}{b}\sqrt{\pi} w \exp(w^2)\left(\operatorname{erf}(w) + \operatorname{erf}(\Lambda_1 - w)\right) \end{aligned}$$

When the hypothesis (10) is not satisfied we can follow an analogous method to the one described before in order to obtain the following result.

Theorem 2.

(i) The result of the Theorem 1 is also true if we replace the condition (10) by $a + b\theta_f = \frac{bl}{c}$. Furthermore, in this case, the solution of the free boundary problem $(2) - (6)$ is given by

$$\theta(x,t) = \frac{1}{b}\left[\frac{1}{\Theta(x,t)} - a\right] \quad , \quad s(t) = 2D_0\sqrt{\frac{t}{\pi}} \tag{50}$$

where Θ is the unique solution of the following integral equation

$$\Theta(x,t) = D_0 \operatorname{erf}\left(\frac{\int_0^x \frac{d\eta}{\Theta(\eta,t)}}{2\sqrt{t}} - w\right) + \frac{c}{bl} \ , \ 0 \leq x \leq s(t) \ , \tag{51}$$

with

$$D_0 = \frac{q_0\sqrt{\pi}\exp(w^2)}{\rho l\left(1 + \sqrt{\pi} w \exp(w^2)\operatorname{erf}(w)\right)}$$

for $t \geq t_0 > 0$, $0 \leq x \leq s(t)$ for any arbitrary positive time t_0 and w defined by (17).

(ii) There does not exist any solution to the free boundary problem $(2) - (6)$ for the case $a + b\theta_f < \frac{bl}{c}$.■

A more complete version of these results and the corresponding study for the analogous problem with a temperature condition on the fixed face $x = 0$ instead of the heat flux condition (3) will be given in a forthcoming paper.

Acknowledgments. This paper has been partially sponsored by the project PIP # 2000/2000 from CONICET-UA, Rosario (Argentina), proyect # 13900/4 from Fundación Antorchas (Argentina) and ANPCYT PICT #03-11165 from Agencia (Argentina).

References

[1] V. Alexiades and A.D. Solomon, *Mathematical modeling of melting and freezing processes,* Hemisphere - Taylor & Francis, Washington (1983).

[2] J. R. Barber, *An asymptotic solution for short-time transient heat conduction between two similar contacting bodies,* Int. J. Heat Mass Transfer *32, No 5,* 943-949 (1989).

[3] G. Bluman, S. Kumei, *On the remarkable nonlinear diffusion equation,* J. Math Phys. *21,* 1019-1023 (1980).

[4] A. C. Briozzo, M. F. Natale and D. A. Tarzia, *Determination of unknown thermal coefficients for Storm's-type materials through a phase-change process,* Int. J. Non-Linear Mech. *34,* 324-340 (1999).

[5] P. Broadbridge, *Non-integrability of non-linear diffusion-convection equations in two spatial dimensions,* J. Phys. A: Math. Gen *19,* 1245-1257 (1986).

[6] P. Broadbridge, *Integrable forms of the one-dimensional flow equation for unsaturated heterogeneous porous media,* J. Math. Phys. *29,* 622-627 (1988).

[7] J. R. Cannon, *The one-dimensional heat equation,* Addison - Wesley, Menlo Park (1984).

[8] H. S. Carslaw and J. C. Jaeger, *Conduction of heat in solids,* Oxford University Press, London (1959).

[9] J. M. Chadam and H. Rasmussen H (Eds.), *Free boundary problems involving solids.* Pitman Research Notes in Mathematics Series *281,* Longman, Essex (1993).

[10] M. N. Coelho Pinheiro, *Liquid phase mass transfer coefficients for bubbles growing in a pressure field: a simplified analysis,* Int. Comm. Heat Mass Transfer *27 No 1,* 99-108 (2000)

[11] J. Crank J, *Free and moving boundary problems,* Clarendon Press, Oxford (1984).

[12] J. I. Diaz, M. A. Herrero, A. Liñan and J. L. Vazquez (Eds.), *Free boundary problems: theory and applications,* Pitman Research Notes in Mathematics Series *323,* Longman, Essex (1995).

[13] A. Fasano and M. Primicerio (Eds.), *Nonlinear diffusion problems,* Lecture Notes in Math. N.1224, Springer Verlag, Berlin (1986).

[14] A. S. Fokas and Y. C. Yortsos, *On the exactly solvable equation* $S_t = \left[(\beta S+\gamma)^{-2} S_x\right]_x + \alpha(\beta S+\gamma)^{-2} S_x$ *occurring in two-phase flow in porous media,* SIAM J. Appl. Math., *42 No 2,* 318-331 (1982).

[15] N. Kenmochi (Ed.), *Free Boundary Problems: Theory and Applications, I,II,* Gakuto International Series: Mathematical Sciences and Applications, *Vol. 13, 14,* Gakkotosho, Tokyo (2000).

[16] J. H. Knight, J. R. Philip, *Exact solutions in nonlinear diffusion,* J. Engrg. Math., *8,* 219-227 (1974).

[17] G. Lamé and B. P. Clapeyron, *Memoire sur la solidification par refroidissement d'un globe liquide,* Annales Chimie Physique *47,* 250-256 (1831).

[18] A. Munier, J. R. Burgan, J. Gutierrez, E. Fijalkow and M. R. Feix, *Group transformations and the nonlinear heat diffusion equation,* SIAM J. Appl. Math., *40 No 2,* 191-207 (1981).

[19] M. F. Natale and D. A. Tarzia, *Explicit solutions to the two-phase Stefan problem for Storm-type materials,* J. Phys. A: Math. Gen *33*, 395-404 (2000).

[20] M. F. Natale and D. A. Tarzia, *Explicit solutions to the one-phase Stefan problem with temperature-dependent thermal conductivity and a convective term,* Int. J. Engng. Sci., *41,* 1685-1698 (2003).

[21] R. Philip, *General method of exact solution of the concentration-dependent diffusion equation,* Australian J. Physics, *13,* 1-12 (1960).

[22] A. D. Polyanin and V. V. Dil'man, *The method of the 'carry over' of integral transforms in non-linear mass and heat transfer problems,* Int. J. Heat Mass Transfer *33 No 1,* 175-181 (1990)

[23] C. Rogers, *Application of a reciprocal transformation to a two-phase Stefan problem,* J. Phys. A: Math. Gen *18,* 105-109 (1985).

[24] C. Rogers, *On a class of moving boundary problems in non-linear heat condition: Application of a Bäcklund transformation,* Int. J. Non-Linear Mech. *21,* 249-256 (1986).

[25] C. Rogers and P. Broadbridge, *On a nonlinear moving boundary problem with heterogeneity: application of reciprocal transformation,* Journal of Applied Mathematics and Physics (ZAMP) *39,* 122-129 (1988).

[26] D. A. Tarzia, *An inequality for the coefficient* σ of the free boundary $s(t) = 2\sigma\sqrt{t}$ of the Neumann solution for the two-phase Stefan problem, Quart. Appl. Math *39,* 491-497 (1981).

[27] D. A. Tarzia, *A bibliography on moving - free boundary problems for the heat-diffusion equation. The Stefan and related problems,* MAT-Serie A #*2* (2000) (with 5869 titles on the subject, 300 pages). See www.austral.edu.ar/MAT-SerieA/2(2000)/

[28] P. Tritscher and P. Broadbridge, *A similarity solution of a multiphase Stefan problem incorporating general non-linear heat conduction,* Int. J. Heat Mass Transfer, *37 No 14,* 2113-2121 (1994).