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Domingo A. Tarzia (Ed.)

INDICE

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An Exact Solution for a One-Phase Stefan Problem with Nonlinear Thermal Coefficients *

Ma. Fernanda NATALE (1) **and Domingo A. TARZIA** (1) (2)

(1) Depto. Matemática, FCE, Universidad Austral,
Paraguay 1950, S2000FZF Rosario, ARGENTINA.
(2) CONICET, ARGENTINA.
E-mail: Maria.Natale@fce.austral.edu.ar;
Domingo.Tarzia@fce.austral.edu.ar

Abstract

We study a one-phase Lamé-Clapeyron-Stefan problem with nonlinear thermal coefficients. We proof the existence of solution by using an integral equation of Volterra type when the diffusion coefficient $k(T)/\rho(T)c(T)$ is constant. We also give the corresponding explicit solution.

Resumen: Se estudia el problema de Lamé-Clapeyron-Stefan a una fase con coeficientes térmicos no lineales. Se prueba la existencia de una solución usando una ecuación integral de tipo Volterra cuando el coeficiente de difusión $k(T)/\rho(T)c(T)$ es constante. Se da también la correspondiente solución explícita.

Key words: Stefan problem, Free boundary problem, Nonlinear thermal coefficients, Explicit solution.

Palabras claves: Problema de Stefan, Problemas de frontera libre, Coeficientes térmicos no lineales, Solución explícita.

AMS Subject classification: 35R35, 80A22, 35C05.

I. Introduction.

The Lamé-Clapeyron-Stefan problem is nonlinear even in its simplest form because some boundary conditions are given on a free boundary whose law of motion is unknown beforehand and has to be determined while solving the problem. It becomes "doubly" nonlinear if the variability of the thermal parameters with temperature is also taken into consideration. It is essential to take into account the parametric nonlinearity when studying the problems related to the surface melting bodies. The present study provides the existence of an exact solutions to one such nonlinear unidimensional problem. We consider the following melting problem for a semi-infinite material:

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$$\rho(T)c(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial y}\left(k(T)\frac{\partial T}{\partial y}\right) \quad , \quad 0 < y < s(t) \tag{1}$$

$$T(0,t) = T_b \quad , \quad T(s(t),t) = T_m < T_b \tag{2}$$

$$k\left(T(s(t),t)\right)\frac{\partial T}{\partial y}(s(t),t) = -\rho_0\, l\, s'(t) \quad , \quad s(0) = 0 \tag{3}$$

where $T = T(y,t)$ is the temperature of the solid phase, $\rho(T), c(T), k(T)$ are the density, specific heat, and thermal conductivity, respectively, T_m is the phase-change temperature, T_b is the temperature on the fixed face $y = 0$, $\rho_0 > 0$ is the constant density of mass at the melting temperature, $l > 0$ is the latent heat of fusion by unity of mass and $s(t)$ is the position of phase change location. This problem was first considered in [4]. We will prove an existence theorem considering the case $k(T)/\rho(T)c(T) = \text{constant} > 0$ through an integral equation equivalent to $(1)-(3)$.

II. The one-phase Stefan problem with nonlinear coefficients

If we define the following transformation

$$\theta(y,t) = \frac{T(y,t) - T_m}{T_b - T_m} \qquad (T(y,t) = T_m + (T_b - T_m)\theta(y,t)) \tag{4}$$

and we assume a similarity solution of the type

$$\theta(y,t) = f(\eta) \quad , \quad \eta = \frac{y}{2\sqrt{\alpha_0 t}} \tag{5}$$

we obtain the following free boundary problem:

$$[L(f)f'(\eta)]' + 2\eta N(f)f'(\eta) = 0 \quad , \quad 0 < \eta < \eta_0 \tag{6}$$

$$f(0) = 1 \quad , \quad f(\eta_0) = 0 \tag{7}$$

$$f'(\eta_0) = -\frac{2\eta_0\alpha_0\rho_0 l}{k(T_m)(T_b - T_m)}. \tag{8}$$

where $N(\theta) = \rho(T)c(T)/\rho_0 c_0$, $L(\theta) = k(T)/k_0$ and k_0, ρ_0, c_0 and $\alpha_0 = k_0/\rho_0 c_0$ are the reference thermal conductivity, density of mass, specific heat and thermal diffusivity respectively and the free boundary $s(t)$ must be of the type

$$s(t) = 2\eta_0\sqrt{\alpha_0 t} \tag{9}$$

where η_0 is also a positive parameter to be determined. We have that the problem $(6)-(7)$ is equivalent to the following nonlinear integral equation of Volterra type:

$$f(\eta) = 1 - \frac{\Phi\left[\eta, L(f), N(f)\right]}{\Phi\left[\eta_0, L(f), N(f)\right]} \tag{10}$$

where Φ is given by

$$\Phi\left[\eta, L(f), N(f)\right] = \frac{2}{\sqrt{\pi}}\int_0^\eta \frac{1}{L(f)(t)}\exp\left(-2\int_0^t \frac{N(f(s))}{L(f(s))}s\,ds\right)dt \tag{11}$$

The condition (8) becomes

$$\frac{1}{\Phi\left[\eta_0, L(f), N(f)\right]} = \frac{\eta_0 l\sqrt{\pi}}{c_0(T_b - T_m)} \exp\left(2\int_0^{\eta_0} \frac{N(f(s))}{L(f(s))} s\, ds\right) \tag{12}$$

and then the following lemma holds.

Lemma 1. The solution of the free boundary problem $(1)-(3)$ is given by (9) and $T(y,t) = T_m + (T_b - T_m) f(\eta), \eta = y/2\sqrt{\alpha_0 t}$ where the function $f = f(\eta)$ and the coefficient $\eta_0 > 0$ must satisfy the nonlinear integral equation (10) and the condition (12).■

From now on we suppose the particular case $N(f)/L(f) = \chi_0 > 0$ and then function Φ is now defined by

$$\Phi\left[\eta, L(f)\right] = \Phi\left[\eta, L(f), \chi_0 L(f)\right] = \frac{2}{\sqrt{\pi}} \int_0^{\eta} \frac{1}{L(f(t))} \exp\left(-\eta_0 t^2\right) dt \tag{13}$$

Firstly we shall prove that the integral equation (10) has a unique solution for any given $\eta_0 > 0$. Secondly, in order to solve the problem $(1)-(3)$ we shall consider the system (10) and (12). Let $C^0\left[0, \eta_0\right]$ be the space of continuous real functions defined on $\left[0, \eta_0\right]$.

Then we have

Theorem 2. Let η_0 be a given positive real number. We suppose that the dimensionless thermal conductivity function $L(f)$ verify the properties

$$0 < L_1 \leq L(f) \leq L_2 \quad , \quad \forall f \in C^0[0, \eta_0] \tag{14}$$

$$|L(g) - L(h)| \leq L_3 \|g - h\| \quad , \quad \forall g, h \in C^0[0, \eta_0] \tag{15}$$

and the coefficients L_1, L_2 and L_3 verifies the restriction $2L_2^2 L_3/L_1^3 < 1$ then there exists a unique solution of the integral equation (10).

Proof. Let $W : C^0[0, \eta_0] \longrightarrow C^0[0, \eta_0]$ be defined by $W(f)_{(\eta)} = 1 - \Phi\left[\eta, L(f)\right] / \Phi\left[\eta_0, L(f)\right]$. The solution of the equation (10) is the fixed point of the operator W. We obtain that W is a contraction, therefore there exists a unique solution of (10)) for a given $\eta_0 > 0$.■

We remark that the solution f of the integral equation (10) depends on the real number $\eta_0 > 0$; for convenience in the notation from now on we take $f(\eta) = f_{\eta_0}(\eta) = f(\eta_0, \eta)$, $0 < \eta < \eta_0$, $\eta_0 > 0$.

Theorem 3. We suppose that the hypothesis of Theorem 2 hold and the inequality

$$2\chi_0\eta_0 > \frac{d}{d\eta_0}(\log(L(f(\eta_0, \eta)))) \quad , \quad \forall\, \eta_0 > 0 \quad , \quad \forall\, \eta \in (0, \eta_0) \tag{16}$$

is verified, then there exists a unique solution of the free boundary problem (10) and (12).

A more compete version of these results and the corresponding study for the two-phase Stefan problem will be given in a forthcoming paper. Now, we show two explicit cases which satisfy the above results.

Example 1. In the particular case $N = L = 1$, the solution of integral equation (10) is given by [1] $f(\eta) = 1 - \mathrm{erf}(\eta)/\mathrm{erf}(\eta_0), 0 < \eta < \eta_0$ where $\eta_0 > 0$ is the unique solution of the equation

$$\frac{Ste}{\sqrt{\pi}} \equiv \frac{c_0(T_b - T_m)}{l\sqrt{\pi}} = x\,\mathrm{erf}(x)\exp(x^2) \quad , \quad x > 0 \tag{17}$$

Example 2. In [2, 3] was considered $N(\theta) = 1$, and $L(\theta) = 1 + \beta\theta$. In this case the integral equation for f becomes

$$[(1 + \beta f)f'(\eta)]' + 2\eta f'(\eta) = 0 \quad , \quad 0 < \eta < \eta_0 \tag{18}$$

where η_0 is the unique solution of the following equation

$$\frac{1}{\Phi(\eta_0, 1 + \beta f, 1)} = \frac{\sqrt{\pi}}{Ste}\,\eta_0 \exp\left(2\int_0^{\eta_0} \frac{1}{1 + \beta f(s)}\, s\, ds\right) \tag{19}$$

where Ste is the Stefan number (see 17) and $\Phi(\eta_0, 1 + \beta f, 1)$ is the error modified function. Taking into account (4) the corresponding solution is given by

$$T(x,t) = T_b + (T_m - T_b)\frac{\Phi(\eta, 1 + \beta f, 1)}{\Phi(\eta_0, 1 + \beta f, 1)} \quad , \quad 0 < \eta < \eta_0 \quad , \quad \eta = \frac{y}{2\sqrt{\alpha_0 t}} \tag{20}$$

Note that the example 1 is obtained taking $\beta = 0$.

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