J. Phys. A: Math. Gen. **33** (2000) 395–404. Printed in the UK PII: S0305-4470(00)03068-7

Explicit solutions to the two-phase Stefan problem for Storm-type materials

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Received 30 March 1999

Abstract. A reciprocal transformation is employed to reduce a two-phase Stefan problem in nonlinear heat conduction into a form which admits a class of exact solutions analogous to the classical Neumann solution. The problem is considered for materials of Storm type (Rogers 1985 *J. Phys. A: Math. Gen.* **18** 105–9). Two related cases are considered, one of them has a flux condition of the type $-q_0/\sqrt{t}$ ($q_0 > 0$) and the existence and uniqueness of the solution is proved when q_0 satisfies a certain inequality which generalizes the work of Tarzia (1981 *Q. Appl. Math.* **39** 491–7), obtained for constant thermal coefficients, the other one has a temperature condition on the fixed face and the existence and uniqueness is proved for all data.

1. Introduction

We consider a two-phase Stefan problem for a semi-infinite region $x > 0$ with phase change temperature T_f. It is required to determine the evolution of the moving phase separation boundary $x = X(t)$ and the temperature distribution. The modelling of this type of system is a problem of great mathematical and industrial significance. Phase-change problems appear frequently in industrial processes and other problems of technological interest [1, 6–9, 11, 14, 25]. An extensive bibliography on the subject was given in [21].

Here, we consider a phase-change process (Stefan problem) for a nonlinear heat conduction equation which admits a class of exact solutions analogous to the classical Neumann solution [13]. In this paper we shall use the similarity method in order to find an exact solution to a free-boundary problem. This methodology has been used successfully in many problems, for example, [3–5, 10, 12, 17, 18, 22–24]. In all of these cases, this methodology has led to important physical consequences.

In [15] the following free-boundary (fusion process) problem was considered:

$$\rho c_{p_1}(T_1)\frac{\partial T_1}{\partial t} = \frac{\partial}{\partial x}\left(k_1(T_1)\frac{\partial T_1}{\partial x}\right) \qquad X(t) < x < \infty \quad t > 0 \tag{1}$$

$$k_1(T_1)\frac{\partial T_1}{\partial x} - k_2(T_2)\frac{\partial T_2}{\partial x} = L\rho \dot{X} \qquad x = X(t) \tag{2}$$

$$T_1 = T_2 = T_f \qquad x = X(t) \tag{3}$$

$$\rho c_{p_2}(T_2)\frac{\partial T_2}{\partial t} = \frac{\partial}{\partial x}\left(k_2(T_2)\frac{\partial T_2}{\partial x}\right) \qquad 0 < x < X(t) \quad t > 0 \tag{4}$$

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$$T_1(x,0) = T_0 < T_f \tag{5}$$

$$X(0) = 0 \tag{6}$$

$$k_2(T_2(0,t))\frac{\partial T_2}{\partial x}(0,t) = -\frac{q_0}{\sqrt{t}} \qquad q_0 > 0 \quad t > 0. \tag{7}$$

In the above, $T_i(x,t)$, $c_{p_i}(T_i)$, $k_i(T_i)$, $i = 1,2$ represent in turn the temperature distribution, specific heat and thermal conductivity in the two phases, solid and liquid, respectively. The density ρ of the medium is assumed to be constant and L denotes the latent heat of fusion of the medium. Here $-q_0/\sqrt{t}$ denotes the prescribed flux on the boundary $x = 0$, while T_0 represents the initial temperature of the medium. It is noted that the two-phase Stefan problem in linear heat conduction with constant thermal coefficients and a heat flux of the type (7) was investigated in [20]. It was proved that a necessary and sufficient condition in order to have an instantaneous phase-change process is that an inequality for the coefficient q_0 should be verified.

Our investigation is henceforth confined to materials for which

$$K_i\frac{\Phi_i'}{\Phi_i^2} = k_i(T_i) \qquad i = 1,2 \quad K_i > 0 \tag{8}$$

where

$$\Phi_i(T_i) = \int_{T_{0i}}^{T_i} S_i(\sigma)\,\mathrm{d}\sigma \qquad S_i(T_i) = \rho c_{p_i}(T_i) \quad i = 1,2. \tag{9}$$

The goal of this paper is to complement [15] and to prove in section 2 the existence and uniqueness of the solution of the problem (1)–(7) if and only if the positive constant q_0 is large enough, i.e.

$$q_0 > \sqrt{K_2}G^{-1}\left(\sqrt{\frac{K_2}{K_1}}\frac{1}{(\Phi_1(T_f)/\Phi_1(T_0) - 1)}\right) \tag{10}$$

where $G^{-1} : (1, +\infty) \longmapsto (0, +\infty)$ is the inverse function of G with

$$G(x) = \operatorname{erf}(x) + \frac{1}{\sqrt{\pi}}\frac{\exp(-x^2)}{x} \qquad x > 0. \tag{11}$$

The function G was defined in [2] and it was proved that

$$G(0^+) = +\infty \qquad G(+\infty) = 1 \qquad \text{and} \qquad G'(x) < 0 \quad \forall\, x > 0. \tag{12}$$

The inequality (10) for the coefficient q_0 generalizes the corresponding inequality which has been obtained for phase-change materials with constant thermal coefficients [20]

In section 3 we consider the problem (1)–(6) and the flux condition (7) will be replaced by the following temperature condition:

$$T_2(0,t) = T_m > T_f \tag{13}$$

on the fixed face. We can remark that there exists a relationship between both condition (7) and (13) on the fixed face $x = 0$ which is given by (46).

We prove the existence and uniqueness of the solution of problem (1)–(6) and (13) for all thermal conditions.

2. Existence and uniqueness of the solution of the free-boundary problem with a flux condition on the fixed face

We consider the problem (1)–(7). If we now set

$$\overline{T_i} = \Phi_i(T_i) = \int_{T_{0i}}^{T_i} \rho c_{pi}(\sigma)\,\mathrm{d}\sigma \qquad i = 1,2 \tag{14}$$

then (1) and (4) become

$$\frac{\partial \overline{T_i}}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k_i(T_i)}{\Phi_i'(T_i)}\frac{\partial \overline{T_i}}{\partial x}\right) \qquad i = 1,2. \tag{15}$$

We remark that if (8) is true then $k_i(T_i)$ and $S_i(T_i)$ verify the Storm relation [16]

$$\frac{1}{\sqrt{k_i(T_i)\,S_i(T_i)}}\frac{\mathrm{d}}{\mathrm{d}T}\left(\log\sqrt{\frac{S_i(T_i)}{k_i(T_i)}}\right) = \frac{1}{\sqrt{K_i}} \qquad i = 1,2. \tag{16}$$

The above condition was originally obtained by Storm [19] in an investigation of heat conduction in simple monatomic metals. There, the validity of the approximation (16) was examined for aluminium, silver, sodium, cadmium, zinc, copper and lead.

Using (8) in (15) reduces the heat conduction equations in the two phases to the form

$$\frac{\partial \overline{T_i}}{\partial t} - k_i\frac{\partial}{\partial x}\left(\frac{1}{\overline{T_i}^2}\frac{\partial \overline{T_i}}{\partial x}\right) = 0 \qquad i = 1,2. \tag{17}$$

Now we introduce the similarity variable

$$\xi = \frac{x}{X(t)} \qquad X(t) = \sqrt{2\gamma t} \quad \gamma > 0 \tag{18}$$

and solutions of (17) are sought of the type

$$\overline{T}_i(x,t) = \varphi_i\left(\frac{x}{\sqrt{2\gamma t}}\right) \qquad i = 1,2. \tag{19}$$

Using the reciprocal transformation

$$\mathrm{d}\xi = \Phi_i^*\,\mathrm{d}\xi_i^* \qquad \Phi_i^* = \Phi_i^{-1} \tag{20}$$

and after several calculations we obtain that the required temperature distributions T_1 and T_2 are given parametrically by

$$\begin{aligned} T_1 &= \Phi_1^{-1}\left\{A_1\,\mathrm{erf}\left[\left(\frac{\gamma}{2K_1}\right)^{1/2}\xi_1^*\right] + B_1\right\}^{-1} \\ \xi_1 &= 1 + \int_{\lambda_1}^{\xi_1^*}\left\{A_1\,\mathrm{erf}\left[\left(\frac{\gamma}{2K_1}\right)^{1/2}\sigma\right] + B_1\right\}\mathrm{d}\sigma \end{aligned} \tag{21}$$

and

$$\begin{aligned} T_2 &= \Phi_2^{-1}\left\{A_2\,\mathrm{erf}\left[\left(\frac{\gamma}{2K_2}\right)^{1/2}\xi_2^*\right] + B_2\right\}^{-1} \\ \xi_2 &= \int_{-\sqrt{2/\gamma}q_0}^{\xi_2^*}\left\{A_2\,\mathrm{erf}\left[\left(\frac{\gamma}{2K_2}\right)^{1/2}\sigma\right] + B_2\right\}\mathrm{d}\sigma \end{aligned} \tag{22}$$

where the unknowns γ, A_i, B_i, λ_i $(i = 1, 2)$ must satisfy the following system (cf [15]):

$$A_1 \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_1}}\lambda_1\right) + B_1 = \frac{1}{\Phi_1(T_f)} \tag{23}$$

$$A_1 + B_1 = \frac{1}{\Phi_1(T_0)} \tag{24}$$

$$A_2 \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\lambda_2\right) + B_2 = \frac{1}{\Phi_2(T_f)} \tag{25}$$

$$\sqrt{\frac{K_2}{\pi}} A_2 \exp\left(\frac{-q_0^2}{K_2}\right) = q_0\left(A_2 \operatorname{erf}\left(\frac{-q_0}{\sqrt{K_2}}\right) + B_2\right) \tag{26}$$

$$\lambda_1 = L\rho + \Phi_1(T_f) - \Phi_2(T_f) + \lambda_2 \tag{27}$$

$$1 = \int_{-\sqrt{2/\gamma}q_0}^{\lambda_2} A_2 \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\sigma\right) \mathrm{d}\sigma + B_2\left(\lambda_2 + \sqrt{\frac{2}{\gamma}}q_0\right) \tag{28}$$

$$-A_1\Phi_1(T_f)\sqrt{\frac{2K_1}{\pi}} \exp\left(-\frac{\gamma}{2K_1}\lambda_1^2\right) + A_2\Phi_2(T_f)\sqrt{\frac{2K_2}{\pi}} \exp\left(-\frac{\gamma}{2K_2}\lambda_2^2\right) = L\rho\sqrt{\gamma} \tag{29}$$

where

$$\lambda_1 = \xi_1^*|_{\xi=1} \qquad \text{and} \qquad \lambda_2 = \xi_2^*|_{\xi=1} \tag{30}$$

and

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)\,\mathrm{d}u. \tag{31}$$

From (23)–(26) we obtain

$$\begin{aligned} A_1 &= \frac{1}{1 - \operatorname{erf}(\sqrt{\gamma/2K_1}\,\lambda_1)}\left(\frac{1}{\Phi_1(T_0)} - \frac{1}{\Phi_1(T_f)}\right) \\ B_1 &= \frac{1}{1 - \operatorname{erf}(\sqrt{\gamma/2K_1}\,\lambda_1)}\left(\frac{1}{\Phi_1(T_f)} - \frac{\operatorname{erf}(\sqrt{\gamma/2K_1}\,\lambda_1)}{\Phi_1(T_0)}\right) \end{aligned} \tag{32}$$

$$\begin{aligned} A_2 &= \frac{1}{\Phi_2(T_f)(G(q_0/\sqrt{K_2}) + \operatorname{erf}(\sqrt{\gamma/2K_2}\,\lambda_2))} \\ B_2 &= \frac{G(q_0/\sqrt{K_2})}{\Phi_2(T_f)(G(q_0/\sqrt{K_2}) + \operatorname{erf}(\sqrt{\gamma/2K_2}\,\lambda_2))}. \end{aligned} \tag{33}$$

Taking into account (32) and (33), equation (28) by integration becomes

$$F(\gamma, \lambda_2) = 0 \tag{34}$$

where $F = F(\gamma, \lambda_2)$ is defined by

$$F(\gamma, \lambda_2) = -1 + \frac{\lambda_2}{\Phi_2(T_f)} \frac{G(u(\gamma, \lambda_2)) + m}{m + \operatorname{erf}(u(\gamma, \lambda_2))} \qquad \gamma > 0 \quad \lambda_2 > 0 \tag{35}$$

with

$$m = G\left(\frac{q_0}{\sqrt{K_2}}\right) > 1 \qquad \text{and} \qquad u(\gamma, \lambda_2) = \sqrt{\frac{\gamma}{2K_2}}\lambda_2. \tag{36}$$

Now, we just have to solve the system (34) and (29) to complete the solution, where λ_1 is given by (27). First, we study equation (34).

Lemma 1. *Equation (34) defines implicitly an increasing function* $\lambda_2 = \lambda_2(\gamma)$ *such as* $F(\gamma, \lambda_2(\gamma)) = 0$. *Moreover, we have* $\lambda_2(0^+) = 0$ *and* $\lambda_2(+\infty) = \Phi_2(T_f)$.

Proof. It is sufficient to apply Dini's theorem by proving that $\frac{\partial F}{\partial \lambda_2}(\gamma, \lambda_2) \neq 0$ for all γ and λ_2. We have

$$\frac{\partial F}{\partial \lambda_2}(\gamma, \lambda_2) = \frac{1}{\Phi_2(T_f)} W(u(\gamma, \lambda_2), m) \tag{37}$$

where

$$W(u, m) = 1 - \frac{2u \exp(-u^2)}{\sqrt{\pi}(m + \operatorname{erf}(u))} - \frac{2\exp(-2u^2)}{\pi(m + \operatorname{erf}(u))^2}. \tag{38}$$

In order to prove that $\partial F/\partial \lambda_2 > 0$ for all γ and λ_2, we note that

$$\frac{\partial W}{\partial m}(u, m) = \frac{2u \exp(-u^2)}{\sqrt{\pi}(m + \operatorname{erf}(u))} + 4\frac{\exp(-2u^2)}{\pi(m + \operatorname{erf}(u))^3} > 0. \tag{39}$$

Then $W(u, m) > W(u, 1)$ for all $u \geqslant 0$.

Therefore, it is sufficient to demonstrate that $W(u, 1) > 0$.

We obtain $W(0^+, 1) = 1 - 2/\pi$, $W(\infty, 1) = 1$, and after many tedious manipulations we also obtain $(\partial W/\partial u)(u, 1) \neq 0, \forall u \geqslant 0$. So, we have $0 < 1 - 2/\pi < W(u, 1) < 1, \forall u \geqslant 0$ and then $W(u, m) > W(u, 1) > 0$ for all $u \geqslant 0$ and finally we have that $\partial F/\partial \lambda_2 > 0$ for all γ and λ_2.

Furthermore, taking into account (35) and (36) we obtain

$$\frac{\partial F}{\partial \gamma}(\gamma, \lambda_2) = \frac{-u^2}{\Phi_2(T_f)\gamma(m + \operatorname{erf}(u))}\sqrt{\frac{2K_2}{\gamma\pi}}\left(\frac{1}{2u^2} + 1 + \frac{\exp(-u^2)}{\sqrt{\pi}u(m + \operatorname{erf}(u))}\right) < 0. \tag{40}$$

Owing to $\partial F/\partial \lambda_2 > 0$ and $\partial F/\partial \gamma < 0$ by Dini's theorem, it results that there exists an implicit function $\lambda_2 = \lambda_2(\gamma)$ such that $F(\gamma, \lambda_2(\gamma)) = 0$ for all γ and its derivative is given by $\lambda_2'(\gamma) = -(\partial F/\partial \gamma)/(\partial F/\partial \lambda_2) > 0$, for all γ. □

Now, replacing $\lambda_2 = \lambda_2(\gamma)$ in (29), we have the following theorem:

Theorem 1. *The free-boundary problem (23)–(29) has a unique solution if and only if* q_0 *verifies the inequality (10).*

Proof. In lemma 1 we found that $\lambda_2 = \lambda_2(\gamma)$ is an increasing function, then for (27) $\lambda_1 = \lambda_1(\gamma)$ is an increasing function too, with the properties $\lambda_1(0^+) = L\rho + \Phi_1(T_f) - \Phi_2(T_f)$ and $\lambda_1(+\infty) = L\rho + \Phi_1(T_f)$. Finally, we have to study the existence and uniqueness of equation (29). Taking into account (32) and (33), equation (29) becomes

$$\Psi(\gamma) = L\rho\sqrt{\gamma} \qquad \gamma > 0 \tag{41}$$

where

$$\Psi(\gamma) = -\sqrt{\frac{2K_1}{\pi}}\frac{\Phi_1(T_f) - \Phi_1(T_0)}{\Phi_1(T_0)}\frac{\exp(-(\gamma/2K_1)\lambda_1^2(\gamma))}{1 - \operatorname{erf}(\sqrt{\gamma/2K_1}\,\lambda_1(\gamma))}$$
$$+\sqrt{\frac{2K_2}{\pi}}\frac{\exp(-(\gamma/2K_2)\lambda_2^2(\gamma))}{m + \operatorname{erf}(\sqrt{\gamma/2K_2}\,\lambda_2(\gamma))}. \tag{42}$$

It easy to see that Ψ is a decreasing function such that $\Psi(+\infty) = \infty$. Now, it is necessary to know the sign of $\Psi(0^+)$ where

$$\Psi(0^+) = \sqrt{\frac{2K_1}{\pi}}\left(1 - \frac{\Phi_1(T_f)}{\Phi_1(T_0)}\right) + \sqrt{\frac{2K_2}{\pi}}\frac{1}{m}. \tag{43}$$

From (12) [2] we have

$$\begin{aligned}\Psi(0^+) > 0 \quad &\Longleftrightarrow \quad \sqrt{\frac{K_2}{K_1}}\frac{1}{(\Phi_1(T_f)/\Phi_1(T_0) - 1)} > m = G\left(\frac{q_0}{\sqrt{K_2}}\right)\\ &\Longleftrightarrow \quad G^{-1}\left(\sqrt{\frac{K_2}{K_1}}\frac{1}{(\Phi_1(T_f)/\Phi_1(T_0) - 1)}\right) < G^{-1}(m) = \frac{q_0}{\sqrt{K_2}}\\ &\Longleftrightarrow \quad q_0 > \sqrt{K_2}G^{-1}\left(\sqrt{\frac{K_2}{K_1}}\frac{1}{(\Phi_1(T_f)/\Phi_1(T_0) - 1)}\right)\end{aligned} \tag{44}$$

that is inequality (10). To summarize, if the condition (10) is verified, Ψ is a decreasing function such that $\Psi(0^+) > 0$ and $\Psi(+\infty) = -\infty$, so there exists a unique value γ which satisfies the transcendental equation (41). □

Then we have the following theorem:

Theorem 2. *The problem (1)–(7) has a Neumann-type unique solution if and only if the coefficient q_0 verifies the inequality (10). In this case the solution is given by (18), (21), (22), (32) and (33), $\lambda_2 = \lambda_2(\gamma)$ is given by lemma 1, $\lambda_1 = \lambda_1(\gamma)$ is given by (27) and γ is the unique solution of equation (41).*

Theorem 2 shows us that when the thermal heat flux input coefficient q_0 has a lower bound of the type (10) we obtain an instantaneous phase-change process.

In contrast, if q_0 does not verify (10) then we only have a heat conduction problem for the initial solid phase.

In the case where q_0 verifies the inequality (10), we can compute the temperature on the fixed face $x = 0$. This temperature is given by

$$T_2(0,t) = \Phi_2^{-1}\left(\frac{q_0\sqrt{\pi}\,\Phi_2(T_f)}{\sqrt{K_2}}\frac{\operatorname{erf}(\sqrt{\gamma/2K_2}\,\lambda_2(\gamma)) + G(q_0/\sqrt{K_2})}{\exp(-q_0^2/K_2)}\right) \tag{45}$$

which satisfies the condition $T_2(0,t) > T_f, \forall t > 0$.

Therefore, we can consider the problem (1)–(6) and (13). In the next section we shall prove that this new mathematical problem has a similarity solution for all data, including $T_m > T_f$.

Remark 1. From condition (7), an imposed heat flux proportional to the $-\frac{1}{2}$ power of t, we can obtain condition (13), a constant-temperature boundary condition, through the following relationship:

$$T_m = \Phi_2^{-1}\left(\frac{q_0\sqrt{\pi}\,\Phi_2(T_f)}{\sqrt{K_2}}\frac{\operatorname{erf}(\sqrt{\gamma/2K_2}\,\lambda_2(\gamma)) + G(q_0/\sqrt{K_2})}{\exp(-q_0^2/K_2)}\right) \tag{46}$$

where γ is the unique solution of equation (41).

This was previously observed in [20] for constant thermal coefficients in both phases.

Remark 2. For the solidification process with an imposed heat flux proportional to the $-\frac{1}{2}$ power of t we can obtain a similar result to theorem 2 for the fusion process.

3. Existence and uniqueness of a solution of the free-boundary problem with a temperature condition on the fixed face

Now, we consider the problem (1)–(6) and the temperature condition (13) on the fixed face $x = 0$. Using (8), (14), (18) and

$$\xi_1 = 1 + \int_{\xi_1^*|_{\xi_1=1}}^{\xi_1^*} \varphi_1^* \, d\xi_1^* \qquad \text{with} \quad \varphi_1^* = \frac{1}{\varphi_1} \tag{47}$$

and

$$\xi_2 = \int_0^{\xi_2^*} \varphi_2^* \, d\xi_2^* \qquad \text{with} \quad \varphi_2^* = \frac{1}{\varphi_2} \tag{48}$$

the required temperature distributions T_1 and T_2 of the problem (1)–(6), (13) are given parametrically by

$$\begin{aligned} T_1 &= \Phi_1^{-1}\left\{A_1 \operatorname{erf}\left[\left(\frac{\gamma}{2K_1}\right)^{1/2} \xi_1^*\right] + B_1\right\}^{-1} \\ \xi_1 &= \int_{\lambda_1}^{\xi_1^*} \left\{A_1 \operatorname{erf}\left[\left(\frac{\gamma}{2K_1}\right)^{1/2} \sigma\right] + B_1\right\} d\sigma + 1 \end{aligned} \tag{49}$$

and

$$\begin{aligned} T_2 &= \Phi_2^{-1}\left\{A_2 \operatorname{erf}\left[\left(\frac{\gamma}{2K_2}\right)^{1/2} \xi_2^*\right] + B_2\right\}^{-1} \\ \xi_2 &= \int_0^{\xi_2^*} \left\{A_2 \operatorname{erf}\left[\left(\frac{\gamma}{2K_2}\right)^{1/2} \sigma\right] + B_2\right\} d\sigma \end{aligned} \tag{50}$$

where the unknowns $\gamma, A_i, B_i, \lambda_i$ $(i = 1, 2)$ must satisfy the following system:

$$\frac{1}{\Phi_1(T_f)} = A_1 \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_1}}\lambda_1\right) + B_1 \tag{51}$$

$$\frac{1}{\Phi_2(T_f)} = A_2 \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\lambda_2\right) + B_2 \tag{52}$$

$$\frac{1}{\Phi_1(T_0)} = A_1 + B_1 \tag{53}$$

$$\frac{1}{\Phi_2(T_m)} = B_2 \tag{54}$$

$$\lambda_1 = \lambda_2 + L\rho + \Phi_1(T_f) - \Phi_2(T_f) \tag{55}$$

$$1 = \int_0^{\lambda_2} A_2 \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\sigma\right) d\sigma + B_2\lambda_2 \tag{56}$$

$$-A_1\Phi_1(T_f)\sqrt{\frac{2K_1}{\pi}} \exp\left(-\frac{\gamma}{2K_1}\lambda_1^2\right) + A_2\Phi_2(T_f)\sqrt{\frac{2K_2}{\pi}} \exp\left(-\frac{\gamma}{2K_2}\lambda_2^2\right) = L\rho\sqrt{\gamma} \tag{57}$$

where λ_1 and λ_2 are given by (30).

From (51) and (54) we obtain

$$A_1 = \frac{1}{1-\operatorname{erf}(\sqrt{\gamma/2K_1}\,\lambda_1)}\left(\frac{1}{\Phi_1(T_0)} - \frac{1}{\Phi_1(T_f)}\right)$$
$$A_2 = \frac{1}{(1/\Phi_2(T_f) - 1/\Phi_2(T_m))\operatorname{erf}(\sqrt{\gamma/2K_2}\,\lambda_2)} \tag{58}$$

$$B_1 = \frac{1}{1-\operatorname{erf}(\sqrt{\gamma/2K_1}\,\lambda_1)}\left(\frac{1}{\Phi_1(T_f)} - \frac{\operatorname{erf}(\sqrt{\gamma/2K_1}\,\lambda_1)}{\Phi_1(T_0)}\right) \qquad B_2 = \frac{1}{\Phi_2(T_m)}. \tag{59}$$

Taking into account (58) and (59), equation (56) becomes

$$H(\gamma, \lambda_2) = 0 \tag{60}$$

where

$$H(\gamma, \lambda_2) = -1 + A_2(\gamma, \lambda_2)\int_0^{\lambda_2} \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\sigma\right) d\sigma + B_2\lambda_2. \tag{61}$$

Then we just have to solve the system (57) and (60) with unknowns γ and λ_2 to complete the solution. First, in analogous form developed in section 2, we study equation (60).

Lemma 2. *There exists an increasing function* $\lambda_2 = \lambda_2(\gamma)$ *such as* $H(\gamma, \lambda_2(\gamma)) = 0$ *for all* γ.

Proof. It is sufficient to apply Dini's theorem by proving that $\frac{\partial H}{\partial \lambda_2}(\gamma, \lambda_2) \neq 0$ for all γ and λ_2. We have

$$\frac{\partial H}{\partial \lambda_2}(\gamma, \lambda_2) = A_2(\gamma, \lambda_2)\operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\lambda_2\right) + \frac{\partial A_2}{\partial \lambda_2}(\gamma, \lambda_2)\int_0^{\lambda_2} \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\sigma\right) d\sigma + B_2$$
$$= \frac{1}{\Phi_2(T_f)}\left(1 - \frac{\beta\exp(-u^2)}{\operatorname{erf}(u)}\left(u + \frac{\exp(-u^2) - 1}{\sqrt{\pi}\operatorname{erf}(u)}\right)\right) \tag{62}$$

where u is given by (36),

$$\beta = \frac{2}{\sqrt{\pi}}\left(1 - \frac{\Phi_2(T_f)}{\Phi_2(T_m)}\right) > 0 \qquad \text{and} \qquad \beta_1 = \frac{1}{\sqrt{\pi}}\left(\frac{1}{\Phi_2(T_f)} - \frac{1}{\Phi_2(T_m)}\right). \tag{63}$$

Then we obtain

$$\frac{\partial H}{\partial \lambda_2}(\gamma, \lambda_2) > 0 \iff \frac{\sqrt{\pi}}{\beta} > M(u) \tag{64}$$

where the function M is defined by

$$M(x) = \frac{\exp(-x^2)}{\operatorname{erf}(x)}\left(x\sqrt{\pi} + \frac{\exp(-x^2)}{\operatorname{erf}(x)} - \frac{1}{\operatorname{erf}(x)}\right) \qquad \forall\, x > 0. \tag{65}$$

Owing to $\beta < 2/\sqrt{\pi}$, $M(0) = \frac{1}{4}\pi$, $M(+\infty) = 0$ and $M'(x) < 0, \forall x > 0$, we obtain that (64) is true for all γ, λ_2.

Furthermore, we find

$$\frac{\partial H}{\partial \gamma}(\gamma, \lambda_2) = -\frac{\beta_1\lambda_2}{\sqrt{\pi}\gamma\operatorname{erf}(u)}\left((1 - \exp(-u^2))\left(\frac{1}{2u} - \frac{\exp(-u^2)}{\sqrt{\pi}\operatorname{erf}(u)}\right) + u\exp(-u^2)\right) < 0. \tag{66}$$

Then for (64) and (66), we have that $\lambda_2'(\gamma) = -\frac{\partial H}{\partial \gamma}(\gamma, \lambda_2) / \frac{\partial H}{\partial \lambda_2}(\gamma, \lambda_2) > 0$. □

Theorem 3. *The free-boundary problem (51)–(57) has a unique solution for all data.*

Proof. The coefficients A_i, B_i $(i = 1, 2)$ are given by (58) and (59), respectively. In lemma 2 we found that $\lambda_2 = \lambda_2(\gamma)$ is an increasing function, then for (55) we obtain that $\lambda_1 = \lambda_1(\gamma)$ is an increasing function too. Finally, we have to study the existence and uniqueness of equation (57). Taking into account (58) and (59), equation (57) becomes

$$\Lambda(\gamma) = L\rho\sqrt{\gamma} \qquad \gamma > 0 \tag{67}$$

where function Λ is defined by

$$\Lambda(\gamma) = -\sqrt{\frac{2K_1}{\pi}}\left(\frac{\Phi_1(T_f) - \Phi_1(T_0)}{\Phi_1(T_0)}\right)\frac{\exp(-(\gamma/2K_1)\lambda_1^2(\gamma))}{(1 - \operatorname{erf}(\sqrt{\gamma/2K_1}\,\lambda_1(\gamma)))} + \sqrt{\frac{2K_2}{\pi}}\left(\frac{\Phi_2(T_m) - \Phi_2(T_f)}{\Phi_2(T_m)}\right)\frac{\exp(-(\gamma/2K_2)\lambda_2^2(\gamma))}{\operatorname{erf}(\sqrt{\gamma/2K_2}\,\lambda_2(\gamma))} \qquad \gamma > 0 \tag{68}$$

and it satisfies that Λ is a decreasing function with $\Lambda(+\infty) = -\infty$ and $\Lambda(0^+) = +\infty$. Then there exists a unique γ which is a solution of the transcendental equation (67). □

Conclusion

We have obtained a similarity solution, analogous to the classical Neumann solution, corresponding to the fusion process with nonlinear thermal coefficients for Storm-type materials and a constant initial temperature T_0 of less than the melting temperature T_f.

We have proved that there exists an explicit solution for all data when a temperature condition (13) is imposed on the fixed face $x = 0$. If we consider condition (7), an imposed heat flux proportional to the $-\frac{1}{2}$ power of t, then the explicit solution is obtained if and only if the thermal flux input coefficient q_0 has a lower bound given by (10).

The two boundary conditions on the fixed face (7), with datum q_0, and (13), with datum T_m, are related through the relationship given by (46).

Moreover, all results obtained for the fusion process for the Storm-type materials can also be found for the solidification process with the corresponding analogous initial and boundary conditions.

Acknowledgments

This paper has been partially sponsored by project no 4798/96 '*Free Boundary Problems for the Heat-Diffusion Equation*' from CONICET-UA, Rosario (Argentina). We thank the anonymous referees whose detailed comments helped us to improve the paper.

References

[1] Alexiades V and Solomon A D 1983 *Mathematical Modeling of Melting and Freezing Processes* (Washington, DC: Hemisphere, London: Taylor and Francis)
[2] Briozzo A C, Natale M F and Tarzia D A 1999 *Int. J. Non-Linear Mechanics* **34** 324–40
[3] Briozzo A C and Tarzia D A 1999 An explicit solution for an instantaneous two-phase Stefan problem with nonlinear thermal coefficients and a particular heat flux on the fixed face, in preparation
[4] Caginalp G 1984 *Adv. Appl. Math.* **49** 476–88
[5] Caginalp G and Nishiura Y 1991 *Q. Appl. Math.* **5** 147–62
[6] Cannon J R 1984 *The One-Dimensional Heat Equation* (Menlo Park, CA: Addison-Wesley)
[7] Carslaw H S and Jaeger J C 1959 *Conduction of Heat in Solids* (London: Oxford University Press)

[8] Chadam J M and Rasmussen H (ed) 1993 *Free Boundary Problems Involving Solids (Pitman Research Notes in Mathematics Series vol 281)* (London: Longman)
[9] Crank J 1984 *Free and Moving Boundary Problems* (Oxford: Clarendon)
[10] Dewynne J N, Howison S D, Ockendon J R and Xie W 1989 *J. Austral. Math. Soc.* B **31** 81–96
[11] Diaz J I, Herrero M A, Liñan A and Vazquez J L (ed) 1995 *Free Boundary Problems: Theory and Applications (Pitman Research Notes in Mathematics Series vol 323)* (London: Longman)
[12] Fasano A, Primicerio M and Tarzia D 1999 *Math. Models Method. Appl. Sci.* **9** 1–10
[13] Lamé G and Clapeyron B P 1831 Memoire sur la solidification par refroidissement d'un globe liquide *Ann. Chim. Phys.* **47** 250–6
[14] Lunardini V J 1991 *Heat Transfer with Freezing and Thawing* (Amsterdam: Elsevier)
[15] Rogers C 1985 *J. Phys. A: Math. Gen.* **18** 105–9
[16] Rogers C 1986 *Int. J. Non-Linear Mech.* **21** 249–56
[17] Sanziel M C and Tarzia D A 1989 *Math. Notae* **33** 25–32
[18] Solomon A D, Wilson D G and Alexiades V 1983 *Q. Appl. Math.* **41** 237–43
[19] Storm M L 1951 *J. Appl. Phys.* **22** 940–51
[20] Tarzia D A 1981 *Q. Appl. Math* **39** 491–7
[21] Tarzia D A 1988 A bibliography on moving–free boundary problems for the heat diffusion equation. The Stefan problem Progetto Nazionale MPI: Equazioni di Evoluzione e Applicazioni Fisico—matematiche Firenze (with 2528 references). An up-dated version has been published in 1999 by Austral University with 5869 titles on the subject.
[22] Tritscher P and Broadbridge P 1994 *Int. J. Heat Mass Transfer* **37** 2113–21
[23] Weiner J H 1955 *Brit. J. Appl. Phys.* **6** 361–3
[24] Wilson D G 1978 *SIAM J. Appl. Math.* **35** 135–47
[25] Wrobel L C and Brebbia C A (ed) 1993 *Computational Methods for Free and Moving Boundary Problems in Heat and Fluid Flow* (Southampton: Computational Mechanics Publications)