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Explicit solutions to the one-phase Stefan problem with temperature-dependent thermal conductivity and a convective term

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Abstract

A one-phase Stefan problem for a semi-infinite material with temperature-dependent thermal conductivity and convective term with a constant temperature or a heat flux condition of the type $-q_0/\sqrt{t}$ $(q_0 > 0)$ at the fixed face $x = 0$ is studied. For both cases a parametric representation of the solution of the similarity type is also obtained.

AMS: 35R35; 80A22; 35C05
Keywords: Stefan problem; Free boundary problem; Moving boundary problem; Phase-change process; Nonlinear thermal conductivity; Fusion; Solidification; Similarity solution

1. Introduction

We consider a Stefan problem for a semi-infinite region $x > 0$ with phase-change temperature θ_f [18]. It is required to determine the evolution of the moving phase separation $x = s(t)$ and the temperature distribution $\theta(x, t)$. The modeling of this kind of systems is a problem with a great mathematical and industrial significance. Phase-change problems appear frequently in industrial processes an other problems of technological interest [1,2,9–11,13–16,19,30]. A large bibliography on the subject was given in [29].

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Here, we consider a one phase-change process (one-phase Stefan problem) for a nonlinear heat conduction equation with a convective term. Owing to [26] we consider the following free boundary (fusion process) problem:

$$\rho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta, x)\frac{\partial \theta}{\partial x}\right) - v(\theta)\frac{\partial \theta}{\partial x}, \quad 0 < x < s(t), \; t > 0 \tag{1}$$

$$k(\theta(0,t),0)\frac{\partial \theta}{\partial x}(0,t) = -\frac{q_0}{\sqrt{t}}, \quad q_0 > 0, \; t > 0 \tag{2}$$

$$k(\theta(s(t),t), \; s(t))\frac{\partial \theta}{\partial x}(s(t),t) = -\rho l \dot{s}(t), \quad t > 0 \tag{3}$$

$$\theta(s(t),t) = \theta_{\mathrm{f}}, \quad t > 0 \tag{4}$$

$$s(0) = 0 \tag{5}$$

where the thermal conductivity $k(\theta, x)$ and the velocity $v(\theta)$ are given by

$$v(\theta) = \rho c \frac{d}{2(a + b\theta)^2} \tag{6}$$

$$k(\theta, x) = \rho c \frac{1 + \mathrm{d}x}{(a + b\theta)^2} \tag{7}$$

and c, ρ and l are the specific heat, the density and the latent heat of fusion of the medium respectively, all of them are assumed to be constant with positive parameters a, b, d and $a + b\theta_{\mathrm{f}} > 0$. The last condition guaranteed that v and k are well defined by maximal principle. This kind of nonlinear thermal conductivity or diffusion coefficients was considered in numerous papers, e.g. [4,7,8,17,21,23,27]. The nonlinear transport equation (1) arises in connection with unsaturated flow in heterogeneous porous media. If we set $d = 0$ and $b = 0$ in the free boundary problem (1)–(7) then we retrieve the classical one-phase Lamé–Clapeyron–Stefan problem. The first explicit solution for the one-phase Stefan problem was given in [18]. Here $-q_0/\sqrt{t}$ denotes the prescribed flux on the boundary $x = 0$ which is of the type imposed in [28], where it was proven that the heat flux condition (2) on the fixed face $x = 0$ is equivalent to the constant temperature boundary condition (43) for the two-phase Stefan problem for a semi-infinite material with constant thermal coefficients in both phases. This kind of heat flux at the fixed boundary $x = 0$ was also considered in several applied problems, e.g. [3,12,22].

The goal of this paper is to determine which conditions on the parameters of the problem (in particular q_0) must be satisfied in order to have an instantaneous phase-change process. In Section 2 we follow and improve [26] in the sense that the existence of an explicit solution of the problem (1)–(5) with nonlinear coefficients (6) and (7) is obtained for all data $q_0, \rho, c, l, \theta_{\mathrm{f}}, a, b, d$ with the restriction $a + b\theta_{\mathrm{f}} > 0$. Moreover, this solution is given as a function of a parameter γ^* which is given the unique solution of the transcendental equation (35). Although, if we replace the flux condition (2) for a new temperature boundary condition (43) on the fixed face $x = 0$ we also obtain in Section 3 for all data the existence of solution which is given as a function of a parameter β which we prove it is the unique solution of the transcendental equation (71). This section is new with respect to [26]. In both cases, the explicit solution is given by a parametric representation of the similarity type. Other problems with nonlinear thermal coefficients in this subject are also given in [5,6,20,24,25].

2. Solution of the free boundary problem with heat flux condition on the fixed face

We consider the problem (1)–(5). Taking into account (6) and (7) we can put our problem as

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1 + \mathrm{d}x}{(a + b\theta)^2}\frac{\partial\theta}{\partial x} + \frac{d}{2b(a + b\theta)}\right), \quad 0 < x < s(t), \ \ t > 0 \tag{8}$$

$$\frac{1}{(a + b\theta(0,t))^2}\frac{\partial\theta}{\partial x}(0,t) = -\frac{q_0^*}{\sqrt{t}}, \quad t > 0 \tag{9}$$

$$\frac{1 + \mathrm{d}s(t)}{(a + b\theta(s(t),t))^2}\frac{\partial\theta}{\partial x}(s(t),t) = -\alpha\dot{s}(t), \quad t > 0 \tag{10}$$

$$\theta(s(t),t) = \theta_{\mathrm{f}}, \quad t > 0 \tag{11}$$

$$s(0) = 0 \tag{12}$$

where $\alpha = l/c$ and $q_0^* = q_0/\rho c$.

We define the following transformations in the same way as in [26]:

$$\begin{cases} y = \frac{2}{d}\left[(1 + \mathrm{d}x)^{\frac{1}{2}} - 1\right] \\ S(t) = \frac{2}{d}\left[(1 + \mathrm{d}s(t))^{\frac{1}{2}} - 1\right] \\ \overline{\theta}(y,t) = \theta(x,t) \end{cases} \tag{13}$$

and

$$\begin{cases} y^* = y^*(y,t) = \int_{S(t)}^{y}(a + b\overline{\theta}(\sigma,t))\,\mathrm{d}\sigma + (\alpha b + a)S(t) \\ t^* = t \\ \theta^* = \frac{1}{a + b\overline{\theta}} \\ S^* = y^*\Big|_{y=S} = (\alpha b + a)S(t) \end{cases} \tag{14}$$

In order to obtain an alternative expression for y^* we compute

$$\begin{aligned}
\frac{\partial y^*}{\partial t} &= -(a+b\overline{\theta}(S(t),t))\dot{S}(t) + \int_{S(t)}^{y} b\frac{\partial\overline{\theta}}{\partial t}(\sigma,t)\,\mathrm{d}\sigma + (\alpha b + a)\dot{S}(t) \\
&= \alpha b\dot{S}(t) + \int_{S(t)}^{y} b\frac{\partial}{\partial\sigma}\left(\frac{1}{(a+b\overline{\theta}(\sigma,t))^2}\frac{\partial\overline{\theta}}{\partial\sigma}\right)\mathrm{d}\sigma \\
&= \alpha b\dot{S}(t) + b\left(\frac{1}{(a+b\overline{\theta}(y,t))^2}\frac{\partial\overline{\theta}}{\partial y}(y,t) - \frac{1}{\left(a+b\overline{\theta}(S(t),t)\right)^2}\frac{\partial\overline{\theta}}{\partial y}(S(t),t)\right) \\
&= \frac{b}{(a+b\overline{\theta}(y,t))^2}\frac{\partial\overline{\theta}}{\partial y}(y,t) \\
&= \int_0^y \frac{\partial}{\partial\sigma}\left(\frac{b}{(a+b\overline{\theta}(\sigma,t))^2}\frac{\partial\overline{\theta}}{\partial\sigma}(\sigma,t)\right)\mathrm{d}\sigma + \frac{b}{(a+b\overline{\theta}(0,t))^2}\frac{\partial\overline{\theta}}{\partial y}(0,t) \qquad (15) \\
&= b\int_0^y \frac{\partial\overline{\theta}}{\partial t}(\sigma,t)\,\mathrm{d}\sigma + \frac{bq_0^*}{\sqrt{t}} \qquad (16)
\end{aligned}$$

then

$$\begin{aligned}
y^*(y,t) &= \int_0^t\left(\int_0^y \frac{\partial}{\partial\sigma}(a+b\overline{\theta}(\sigma,t))\,\mathrm{d}\sigma + \frac{bq_0^*}{\sqrt{\tau}}\right)\mathrm{d}\tau + \int_0^y (a+b\overline{\theta}(\sigma,0))\,\mathrm{d}\sigma \\
&= \int_0^y (a+b\overline{\theta}(\sigma,t))\,\mathrm{d}\sigma + 2bq_0^*\sqrt{t} \qquad (17)
\end{aligned}$$

Now, applying (13) and (14) the problem (8)–(12) is transformed in a classical Stefan problem given by

$$\frac{\partial\theta^*}{\partial t^*} = \frac{\partial^2\theta^*}{\partial y^{*2}}, \quad 2bq_0^*\sqrt{t^*} < y^* < S^*(t^*), \quad t^* > 0 \qquad (18)$$

$$\frac{\partial\theta^*}{\partial y^*}(2bq_0^*\sqrt{t^*},t^*) = \frac{q_0^* b}{\sqrt{t^*}}\theta^*(2bq_0^*\sqrt{t^*},t^*), \quad t^* > 0 \qquad (19)$$

$$\frac{\partial\theta^*}{\partial y^*}(S^*(t^*),t^*) = \alpha^*\dot{S}^*(t^*), \quad t^* > 0 \qquad (20)$$

$$\theta^*(S^*(t^*),t^*) = \theta_{\mathrm{f}}^*, \quad t^* > 0 \qquad (21)$$

$$S^*(0) = 0 \qquad (22)$$

where

$$\alpha^* = \frac{\alpha b}{(\alpha b + a)(a+b\theta_{\mathrm{f}})}, \quad \theta_{\mathrm{f}}^* = \frac{1}{a+b\theta_{\mathrm{f}}} \qquad (23)$$

Then, if we introduce the similarity variable:

$$\xi^* = \frac{y^*}{\sqrt{2\gamma^* t^*}} \tag{24}$$

where γ^* is a dimensionless positive constant to be determined, and the solution is sought of the type

$$\begin{cases} \theta^*(y^*, t^*) = \Theta^*(\xi^*) \\ S^*(t^*) = \sqrt{2\gamma^* t^*} \end{cases} \tag{25}$$

then, we get that (18)–(22) yields

$$\frac{\mathrm{d}^2\Theta^*}{\mathrm{d}\xi^{*2}} + \gamma^*\xi^*\frac{\mathrm{d}\Theta^*}{\mathrm{d}\xi^*} = 0, \quad bq_0^*\sqrt{\frac{2}{\gamma^*}} < \xi^* < 1 \tag{26}$$

$$\frac{\mathrm{d}\Theta^*}{\mathrm{d}\xi^*}\left(bq_0^*\sqrt{\frac{2}{\gamma^*}}\right) = \sqrt{2\gamma^*}q_0^* b\Theta^*\left(bq_0^*\sqrt{\frac{2}{\gamma^*}}\right) \tag{27}$$

$$\frac{\mathrm{d}\Theta^*}{\mathrm{d}\xi^*}(1) = \alpha^*\gamma^* \tag{28}$$

$$\Theta^*(1) = \theta_\mathrm{f}^* \tag{29}$$

The solution of the differential equation (26) is given by

$$\Theta^*(\xi^*) = A\,\mathrm{erf}\left(\sqrt{\frac{\gamma^*}{2}}\xi^*\right) + B \tag{30}$$

where A and B are two unknown coefficients. From (27) and (29) we get

$$A = \frac{-\theta_\mathrm{f}^*}{g\left(bq_0^*, -\frac{1}{\sqrt{\pi}}\right) - \mathrm{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right)} \tag{31}$$

$$B = \frac{\theta_\mathrm{f}^* g\left(bq_0^*, -\frac{1}{\sqrt{\pi}}\right)}{g\left(bq_0^*, -\frac{1}{\sqrt{\pi}}\right) - \mathrm{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right)} \tag{32}$$

where

$$\begin{aligned} & g(x,p) = \mathrm{erf}(x) + pR(x), \quad x > 0, \ p \in \mathbb{R} \\ & R(x) = \frac{\exp(-x^2)}{x}, \quad x > 0 \end{aligned} \tag{33}$$

and

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \exp(-u^2)\,\mathrm{d}u \tag{34}$$

The unknown constant γ^* is determined by the remaining boundary condition (28) which yields the following equation:

$$\alpha^*\sqrt{\gamma^*}=-\sqrt{\frac{2}{\pi}}\frac{\theta_{\mathrm{f}}^*}{g\left(bq_0^*,-\frac{1}{\sqrt{\pi}}\right)-\operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right)}\exp\left(\frac{\gamma^*}{2}\right),\quad \gamma^*>0 \tag{35}$$

Theorem 1. *The free boundary problem* (1)–(5) *has a unique solution of the similarity type for all data* $q_0,\rho,c,l,\theta_{\mathrm{f}},a,b,d$. *Moreover, the solution is given by*

$$\begin{aligned}\theta(\xi)&=\frac{1}{b}\left[\frac{1}{A\operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\xi^*\right)+B}-a\right]\\ \xi&=\frac{y}{\sqrt{2\gamma t}}=\frac{\frac{2}{d}\left[(1+\mathrm{d}x)^{\frac{1}{2}}-1\right]}{\sqrt{2\gamma t}}\\ s(t)&=\frac{1}{d}\left[\left(1+\frac{d}{2}\sqrt{2\gamma t}\right)^2-1\right]\end{aligned} \tag{36}$$

where

$$\xi=(\alpha b+a)\int_{\sqrt{\frac{2}{\gamma^*}}bq_0^*}^{\xi^*}\left[A\operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\sigma\right)+B\right]\mathrm{d}\sigma \tag{37}$$

and A,B *and* γ *are given by* (31), (32) *and* (42) *respectively.*

Proof. First, we have to study the existence and uniqueness of the Eq. (35) If we define $\mu=\sqrt{\gamma^*/2}$, then μ must be the solution of the following equation:

$$g\left(bq_0^*,-\frac{1}{\sqrt{\pi}}\right)=g\left(x,-\frac{1}{\sqrt{\pi}}\frac{\theta_{\mathrm{f}}^*}{\alpha^*}\right),\quad x>0 \tag{38}$$

It is easy to see that $g(0,p)=-\infty$, $g(+\infty,p)=1$ and $(\partial g/\partial x)(x,p)>0$, $\forall x>0$, $\forall p<0$. Moreover we have

$$\frac{\theta_{\mathrm{f}}^*}{\alpha^*}=\frac{1/(a+b\theta_{\mathrm{f}})}{\alpha b/(\alpha b+a)(a+b\theta_{\mathrm{f}})}=1+\frac{a}{\alpha b} \tag{39}$$

so the Eq. (38) can be written as

$$g\left(bq_0^*, -\frac{1}{\sqrt{\pi}}\right) = g\left(x, -\frac{1}{\sqrt{\pi}}\left(1+\frac{a}{\alpha b}\right)\right), \quad x > 0 \tag{40}$$

Therefore, taking into account that $g\left(bq_0^*, -1/\sqrt{\pi}\right) < 1$ then Eq. (40) has a unique solution for all data. From (25) we obtain the expression of $S(t)$ given by

$$S(t) = \sqrt{2\gamma t} \tag{41}$$

where γ is given by

$$\gamma = \frac{\gamma^*}{(\alpha b + a)^2} \tag{42}$$

From (24), (25) and (30) we can obtain the parametric solution of the problem (1)–(5) given by (36) and (37). Note that

$$\xi^* \mid_{\xi=0} = \frac{2bq_0^*\sqrt{t^*}}{\sqrt{2\gamma^* t^*}} = \sqrt{\frac{2}{\gamma^*}}bq_0^*.$$

We remark that the temperature at the fixed face $x = 0$ is constant in time which origins the following section. □

3. Solution of the free boundary problem with temperature boundary condition on the fixed face

Now, we consider the problem (1)–(5) but the condition (2) will be replaced by the following temperature boundary condition ($\theta_0 > \theta_f$):

$$\theta(0,t) = \theta_0, \quad t > 0 \tag{43}$$

We can define the same transformations (13) and (14) as were done for the previous problem but now we get

$$\frac{\partial y^*}{\partial t} = b\int_0^y \frac{\partial\overline{\theta}}{\partial t}(\sigma,t)\,\mathrm{d}\sigma + \frac{b}{\left(a+b\overline{\theta}(0,t)\right)^2}\frac{\partial\overline{\theta}}{\partial y}(0,t) = b\int_0^y \frac{\partial\overline{\theta}}{\partial t}(\sigma,t)\,\mathrm{d}\sigma + \frac{b}{(a+b\theta_0)^2}\frac{\partial\overline{\theta}}{\partial y}(0,t).$$

Then

$$\begin{aligned} y^*(y,t) &= \int_0^t \left(\int_0^y \frac{\partial}{\partial\tau}(a+b\overline{\theta}(\sigma,\tau))\,\mathrm{d}\sigma + \frac{b}{(a+b\theta_0)^2}\frac{\partial\overline{\theta}}{\partial y}(0,\tau)\right)\mathrm{d}\tau + \int_0^y \left(a+b\overline{\theta}(\sigma,0)\right)\mathrm{d}\sigma \\ &= \int_0^y (a+b\overline{\theta}(\sigma,t))\,\mathrm{d}\sigma + \frac{b}{(a+b\theta_0)^2}\int_0^t \frac{\partial\overline{\theta}}{\partial y}(0,\tau)\,\mathrm{d}\tau \end{aligned} \tag{44}$$

Our problem becomes (20)–(22) and

$$\frac{\partial\theta^*}{\partial t}=\frac{\partial^2\theta^*}{\partial y^*},\quad \frac{b}{(a+b\theta_0)^2}\int_0^{t^*}\frac{\partial\overline{\theta}}{\partial y}(0,\tau)\,d\tau<y^*<S^*(t^*),\quad t^*>0 \tag{45}$$

$$\theta^*\left(\frac{b}{(a+b\theta_0)^2}\int_0^{t^*}\frac{\partial\overline{\theta}}{\partial y}(0,\tau)\,d\tau,t^*\right)=\theta_0^* \tag{46}$$

where θ_f^* and α^* are given by (23) and

$$\theta_0^*=\frac{1}{a+b\overline{\theta}(0,t)}=\frac{1}{a+b\theta(0,t)}=\frac{1}{a+b\theta_0} \tag{47}$$

It easy to see that we have a classical Stefan problem so that the free boundary must be of the type

$$S^*(t^*)=\sqrt{2\gamma^*t^*}\quad\left(S(t)=\sqrt{2\gamma t}\right) \tag{48}$$

where γ^* (i.e. γ) is a dimensionless constant to be determined.

If we introduce the similarity variable (24) and we propose the solution of the type (25) then the problem (20)–(22) and (45) and (46) yields (28), (29) and

$$\frac{d^2\Theta^*}{d\xi^{*2}}+\gamma^*\xi^*\frac{d\Theta^*}{d\xi^*}=0,\quad \frac{b}{(a+b\theta_0)^2\sqrt{2\gamma^*t^*}}\int_0^{t^*}\frac{\partial\overline{\theta}}{\partial y}(0,\tau)\,d\tau<\xi^*<1,\quad t^*>0 \tag{49}$$

$$\Theta^*\left(\frac{b}{(a+b\theta_0)^2\sqrt{2\gamma^*t^*}}\int_0^{t^*}\frac{\partial\overline{\theta}}{\partial y}(0,\tau)\,d\tau\right)=\theta_0^*,\quad t^*>0 \tag{50}$$

From (50) we must necessarily have that there exist a constant ξ_0^* such that

$$\int_0^{t^*}\frac{\partial\overline{\theta}}{\partial y}(0,\tau)\,d\tau=\xi_0^*\frac{(a+b\theta_0)^2}{b}\sqrt{2\gamma^*t^*} \tag{51}$$

Therefore (49) and (50) can be written as

$$\frac{d^2\Theta^*}{d\xi^{*2}}+\gamma^*\xi^*\frac{d\Theta^*}{d\xi^*}=0,\quad \xi_0^*<\xi^*<1 \tag{52}$$

$$\Theta^*(\xi_0^*)=\theta_0^*,\quad t^*>0 \tag{53}$$

The solution of the problem (28), (29), (52), (53) is given by

$$\Theta^*(\xi^*) = A'\operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\xi^*\right) + B', \quad \xi_0^* < \xi^* < 1 \tag{54}$$

where the unknown coefficients ξ_0^*, A', B' and γ^* must satisfy the following equations:

$$A'\operatorname{erf}\left(\xi_0^*\sqrt{\frac{\gamma^*}{2}}\right) + B' = \theta_0^*, \quad t > 0 \tag{55}$$

$$\sqrt{\frac{2}{\gamma^*}}\exp\left(-\frac{\gamma^*}{2}\right) = \frac{\alpha^*}{A'}\sqrt{\pi} \tag{56}$$

$$A'\operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right) + B' = \theta_{\mathrm{f}}^* \tag{57}$$

Firstly, we shall obtain the following preliminary result.

Lemma 1. *There exists a constant $q_0^* > 0$ such that*

$$\frac{\partial\overline{\theta}}{\partial y}(0,t) = -\frac{q_0^*}{\sqrt{t}}, \quad \forall t > 0 \tag{58}$$

where

$$q_0^* = \frac{A'}{b\sqrt{\pi}\theta_0^{*3}}\exp\left(-\frac{\gamma^*\xi_0^{*2}}{2}\right) \tag{59}$$

Proof. From (14), (44), (25) and (54) we have

$$\frac{\partial\overline{\theta}}{\partial y}(0,t) = \frac{-A'(a+b\theta_0)}{b\sqrt{\pi t}\left(A'\operatorname{erf}\left(\xi_0^*\sqrt{\frac{\gamma^*}{2}}\right) + B'\right)^2}\exp\left(-\frac{\gamma^*\xi_0^{*2}}{2}\right), \quad t > 0 \tag{60}$$

then we find that

$$q_0^* = \frac{A'(a+b\theta_0)}{b\sqrt{\pi}\left(A'\operatorname{erf}\left(\xi_0^*\sqrt{\frac{\gamma^*}{2}}\right) + B'\right)^2}\exp\left(-\frac{\gamma^*\xi_0^{*2}}{2}\right) \tag{61}$$

or equivalently

$$q_0^* = \frac{A'}{b\sqrt{\pi}\theta_0^{*3}} \exp\left(-\frac{\gamma^* \xi_0^{*2}}{2} \right) \tag{62}$$

Moreover, from (56) we have that $A' > 0$, then $q_0^* > 0$. □

Remark 1. We have that

$$\xi_0^* = \frac{b\theta_0^{*2}}{\sqrt{2\gamma^* t}} \int_0^t \frac{\partial \overline{\theta}}{\partial y}(0, \tau)\, d\tau$$

and then for Lemma 1, we obtain

$$\xi_0^* = -bq_0^*\theta_0^{*2}\sqrt{\frac{2}{\gamma^*}} \tag{63}$$

Now we have to solve the system (55)–(57), (62) and (63) where A', B', γ^* and ξ_0^*. From (55) and (57) we have

$$A' = \frac{\theta_0^* - \theta_f^*}{\mathrm{erf}\left(\xi_0^*\sqrt{\frac{\gamma^*}{2}}\right) - \mathrm{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right)} \tag{64}$$

$$B' = \frac{\theta_f^* \mathrm{erf}\left(\xi_0^*\sqrt{\frac{\gamma^*}{2}}\right) - \theta_0^* \mathrm{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right)}{\mathrm{erf}\left(\xi_0^*\sqrt{\frac{\gamma^*}{2}}\right) - \mathrm{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right)} \tag{65}$$

Now, we have to solve the following system:

$$\frac{\exp(-\beta^2)}{\beta} = \alpha_1(\mathrm{erf}(z) - \mathrm{erf}(\beta)) \tag{66}$$

$$\frac{\exp(-z^2)}{z} = \alpha_2(\mathrm{erf}(z) - \mathrm{erf}(\beta)) \tag{67}$$

where

$$\beta = \sqrt{\frac{\gamma^*}{2}}, \quad z = \xi_0^*\sqrt{\frac{\gamma^*}{2}},$$

$$\alpha_1 = \frac{\alpha^*\sqrt{\pi}}{\theta_0^* - \theta_f^*} = \frac{-\alpha\sqrt{\pi}(a + b\theta_0)}{(\alpha b + a)(\theta_0 - \theta_f)} < 0 \tag{68}$$

$$\alpha_2 = \frac{\theta_0^* \sqrt{\pi}}{\theta_0^* - \theta_{\mathrm{f}}^*} = \frac{\sqrt{\pi}(a + b\theta_{\mathrm{f}})}{b(\theta_0 - \theta_{\mathrm{f}})} > 0 \tag{69}$$

Then, we have now to solve (66) and

$$\beta \exp(\beta^2) = \frac{\alpha_2}{\alpha_1} z \exp(z^2) \tag{70}$$

From (66) we have $z = \mathrm{erf}^{-1}(W(\beta))$ where function W is defined by

$$W(x) = \frac{1}{\alpha_1 x} \exp(-x^2) + \mathrm{erf}(x)$$

with $W(x_1) = -1$ and $\beta > x_1$. Then we can write (67) in the following way

$$\beta \exp(\beta^2) = B(\beta), \quad \beta > x_1 \tag{71}$$

where function B is defined by

$$B(x) = \frac{\alpha_2}{\alpha_1} \mathrm{erf}^{-1}(W(x)) \exp((\mathrm{erf}^{-1}(W(x)))^2) \quad \text{for } x > x_1.$$

Taking into account that B is a decreasing function, $B(x_1) = +\infty$, $B(+\infty) = -\infty$, then there exists a unique solution $\beta > x_1$ of Eq. (71) and then we have $z = \mathrm{erf}^{-1}(W(\beta))$.

Finally, resuming the previous results we have the following theorem.

Theorem 2. *The free boundary problem* (1), (43), (3)–(5) *has a unique solution of a similarity type for all data* $q_0, \rho, c, l, \theta_{\mathrm{f}}, a, b, d$. *Moreover, the solution is given by*

$$\begin{aligned}
\theta(\xi) &= \frac{1}{b}\left[\frac{1}{A' \,\mathrm{erf}\left(\sqrt{\frac{\gamma^*}{2}}\xi^*\right) + B'} - a\right] \\
\xi &= \frac{y}{\sqrt{2\gamma t}} = \frac{\frac{2}{d}\left[(1 + \mathrm{d}x)^{\frac{1}{2}} - 1\right]}{\sqrt{2\gamma t}} \\
s(t) &= \frac{1}{d}\left[\left(1 + d\sqrt{\frac{\gamma t}{2}}\right)^2 - 1\right]
\end{aligned} \tag{72}$$

where

$$\xi = (\alpha b + a)\int_{\xi^*|_{\xi=0}}^{\xi^*}\left[A'\operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\sigma\right) + B'\right]\mathrm{d}\sigma$$
$$\gamma = \frac{\gamma^*}{(\alpha b + a)^2} \tag{73}$$

and the coefficients A' and B' are given by

$$A' = \frac{b(\theta_{\mathrm{f}} - \theta_0)}{(a + b\theta_{\mathrm{f}})(a + b\theta_0)\left(\operatorname{erf}\left(\xi_0^*\sqrt{\frac{\gamma^*}{2}}\right) - \operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right)\right)} \tag{74}$$

$$B' = \frac{(a + b\theta_0)\operatorname{erf}\left(\xi_0^*\sqrt{\frac{\gamma^*}{2}}\right) - (a + b\theta_{\mathrm{f}})\operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right)}{(a + b\theta_{\mathrm{f}})(a + b\theta_0)\left(\operatorname{erf}\left(\xi_0^*\sqrt{\frac{\gamma^*}{2}}\right) - \operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right)\right)} \tag{75}$$

where $\gamma^* = 2\beta^2$, $\xi_0^* = \sqrt{2/\gamma^*}\operatorname{erf}(W(\beta))$ and β is the unique solution of the Eq. (71).

Remark 2. The case without the convective term in (1), that is $d = 0$, it cannot be obtained from what we did previously for the case $d \neq 0$ and taking $d \to 0$, because the transformation $x \to y$ through (13) is the identity since

$$\lim_{d\to 0}\frac{2}{d}\left[(1 + \mathrm{d}x)^{\frac{1}{2}} - 1\right] = x, \quad \forall x > 0.$$

Then, the case $d = 0$ must be solved by using other techniques which will be developed in a forthcoming paper.

4. Conclusion

For a one-phase Stefan problem for a semi-infinite material with temperature-dependent thermal conductivity (7) and a convective term (6) with a heat flux (2) or a constant temperature boundary condition (43) a parametric representation of the similarity type is obtained. For the heat flux boundary condition we have followed and improved [26] obtaining the existence theorem in Section 2. Moreover a new temperature boundary condition is also studied obtaining the corresponding existence theorem in Section 3. For all cases their explicit solutions are given as a functions of a parameter which is defined as the unique solution of a transcendental equation.

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