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Explicit solutions for one-dimensional two-phase free boundary problems with either shrinkage or expansion

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ABSTRACT

We consider a one-dimensional solidification of a pure substance which is initially in liquid state in a bounded interval $[0, l]$. Initially, the liquid is above the freezing temperature, and cooling is applied at $x = 0$ while the other end $x = l$ is kept adiabatic. At the time $t = 0$, the temperature of the liquid at $x = 0$ comes down to the freezing point and solidification begins, where $x = s(t)$ is the position of the solid–liquid interface. As the liquid solidifies, it shrinks $(0 < r < 1)$ or expands $(r < 0)$ and appears a region between $x = 0$ and $x = rs(t)$, with $r < 1$. Temperature distributions of the solid and liquid phases and the position of the two free boundaries ($x = rs(t)$ and $x = s(t)$) in the solidification process are studied. For three different cases, changing the condition on the free boundary $x = rs(t)$ (temperature boundary condition, heat flux boundary condition and convective boundary condition) an explicit solution is obtained. Moreover, the solution of each problem is given as a function of a parameter which is the unique solution of a transcendental equation and for two of the three cases a condition on the parameter must be verified by data of the problem in order to have an instantaneous phase-change process. In all the cases, the explicit solution is given by a representation of the similarity type.

1. Introduction

Heat transfer problems involving a change of phase due to melting or freezing processes are very important in science and technology [1–5]. These kinds of problem are generally referred to as moving-free boundary problems which have been the subject of numerous theoretical, numerical and experimental investigations. A lot of work have been done to investigate the phase-change phenomenon in almost all its aspects. We can see a large bibliography on the subject given in [6]. Particularly, the density difference between phases has more and more experimental and analytical research interest [7–9]. For example, the difference in densities is of importance in static casting processes for an overall design of the size and shape of the mold, because phase-change materials with a few exceptions normally shrink during solidification and hence the detachment of the liquid from the mold wall results in a concave shrinkage surface due to gravity.

In the following we take into consideration a similar model as the one presented in [10,11] in order to obtain explicit solutions for a free boundary problem in which shrinkage or expansion occurs. We consider a one-dimensional solidification of a pure substance which is initially in liquid state in a bounded interval $[0, l]$. Initially, the liquid is above the freezing temperature, and cooling is applied at $x = 0$ while the other end $x = l$ is kept adiabatic. At the time $t = 0$, the temperature

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of the liquid at $x = 0$ comes down to the freezing point and solidification begins, where $x = s(t)$ is the position of the solid–liquid interface. As the liquid solidifies, it shrinks or expands and appears a region between $x = 0$ and $x = rs(t)$, where $r = \frac{\rho_1-\rho_2}{\rho_1} = 1 - \frac{\rho_2}{\rho_1}$ is a parameter, and ρ_i is the density of the region i ($i = 1$: solid; $i = 2$: liquid). Observe that $r < 1$ and it could be possible for this parameter to take negative values (if $r \in (0, 1)$ the substance shrinks; if $r < 0$ it expands). Eventually all the liquid solidifies, which is in accordance with some experiments. We consider the temperature distribution of the liquid and solid, and the position of the two free boundaries in the solidification process. Furthermore, we suppose that the pre-existing phase is at rest and the forming phase moves.

The governing equations are

$$\alpha_1 \frac{\partial^2 u_1}{\partial x^2}(x,t) = \frac{\partial u_1}{\partial t}(x,t) + r\dot{s}(t)\frac{\partial u_1}{\partial x}(x,t), \quad rs(t) < x < s(t), \ t > 0. \tag{1.1}$$

$$\alpha_2 \frac{\partial^2 u_2}{\partial x^2}(x,t) = \frac{\partial u_2}{\partial t}(x,t), \quad s(t) < x, \ t > 0. \tag{1.2}$$

where $u_i(x, t)$ is the temperature, α_i is thermal diffusivity, k_i is thermal conductivity, ρ_i is the density in region i, ($i = 1, 2$), and r is the parameter presented above.

The boundary and initial conditions are

$$u_2(+\infty, t) = u_2(x, 0) = B > u^*, \tag{1.3}$$

$$u_1(s(t), t) = u_2(s(t), t) = u^*, \tag{1.4}$$

$$k_1 u_{1x}(s(t), t) - k_2 u_{2x}(s(t), t) = \rho_1 L \dot{s}(t) \tag{1.5}$$

where u^* is the freezing temperature, L is the latent heat of fusion by unit of mass.

In Section 2, we consider the problem (1.1)–(1.5) plus the following temperature condition on the left boundary given by:

$$u_1(rs(t), t) = A, \quad t > 0 \tag{1.6}$$

where A is a constant such that $A < u^* < B$.

In Section 3, we take the problem (1.1)–(1.5) plus a heat-flux condition on the left boundary given by

$$k_1 \frac{\partial u_1}{\partial x}(rs(t), t) = \frac{q_0}{\sqrt{t}}. \tag{1.7}$$

Here $\frac{q_0}{\sqrt{t}}$ denotes the prescribed heat flux on the boundary $x = rs(t)$ which is a condition of the type imposed in [12], where it was proven that the heat flux condition (1.7) on the fixed face $x = 0$ is equivalent to the constant temperature boundary condition (1.6) for the two-phase Stefan problem for a semi-infinite material with constant thermal coefficients in both phases. This kind of heat flux at the fixed boundary $x = 0$ was also considered in several applied problems, e.g. [13–15]. We will prove in Section 4 that the free boundary problems (1.1)–(1.5), (1.6) and (1.1)–(1.5), (1.7) are also equivalent.

In Section 5, we study the problem (1.1)–(1.5) plus a convective cooling condition on the left boundary given by

$$k_1 \frac{\partial u_1}{\partial x}(rs(t), t) = \frac{h_0}{\sqrt{t}}\left(u_1(rs(t), t) - u_0\right), \tag{1.8}$$

where h_0 is the coefficient which characterizes the dependent-time heat transfer coefficient given by $\frac{h_0}{\sqrt{t}}$ and u_0 is the external temperature ($u_0 < u^*$). The boundary condition (1.8) was considered in [16,9,17] for the classical Stefan problem.

In [10] a similarity solution was obtained by assuming the void thermal resistance proportional to its size. In [18], the solution of the two-phase problem was obtained by using the method of matched asymptotics not including the effect of shrinkage or expansion. In [19] the numerical solution of a similar problem was obtained, with a convective condition by applying the method of perturbation analysis. In [20] it was studied a one-phase free boundary problem for a PDE system in a domain which shrinkage occurs.

Finally, in Section 6 we find the equivalence between problems (1.1)–(1.5), (1.6) and (1.1)–(1.5), (1.8), and the equivalence between problems (1.1)–(1.5), (1.7) and (1.1)–(1.5), (1.8).

In the text we will use the error function *erf* defined by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-z^2\right) dz, \quad x > 0 \tag{1.9}$$

and the complementary error function *erfc* which is defined by $erfc(x) = 1 - erf(x)$.

The goal of this paper is to determine the temperature distribution of the solid and liquid regions and the position of the two free boundaries in the solidification process with either shrinkage or expansion, in $rs(t) < x < s(t)$, $t > 0$, $0 < r < 1$ or $r < 0$ respectively, in three different cases, changing the condition on the boundary $x = rs(t)$. Moreover, the solution of each problem is given as a function of a certain parameter which is given as the unique solution of a transcendental equation. Furthermore, we study which conditions must be satisfied on the parameters of each problem in order to have an instantaneous phase-change process. In all the cases, the explicit solution is given by a parametric representation of the similarity type. Other problems in this subject are also given in [21–25].

2. Solidification of a pure substance with a temperature condition at $x = rs(t)$

We consider the problem (1.1)–(1.5) plus the temperature boundary condition (1.6). The partial differential equations, Eqs. (1.1) and (1.2), can easily be transformed into ordinary differential equations [26], since a non-dimensional similarity coordinate y can be found:

$$y = \frac{x}{2\lambda\sqrt{\alpha_2 t}}, \tag{2.1}$$

where α_2 is the thermal diffusivity of liquid region and λ is a dimensionless unknown coefficient to be determined.

The solutions are sought of the type:

$$u_1(x,t) = \Theta_1(y), \quad r < y < 1 \tag{2.2}$$

$$u_2(x,t) = \Theta_2(y), \quad y > 1 \tag{2.3}$$

then the free boundary $s(t)$ of the problem (1.1)–(1.6) must be of the type

$$s(t) = 2\lambda\sqrt{\alpha_2 t}; \quad t > 0 \tag{2.4}$$

and the problem (1.1)–(1.6) yields

$$\frac{\alpha_1}{2\alpha_2\lambda^2}\Theta_1'' + (y-r)\Theta_1' = 0, \quad r < y < 1 \tag{2.5}$$

$$\Theta_2'' + 2\lambda^2 y\,\Theta_2' = 0, \quad y > 1 \tag{2.6}$$

$$\Theta_2(+\infty) = B \tag{2.7}$$

$$\Theta_1(r) = A \tag{2.8}$$

$$\Theta_1(1) = \Theta_2(1) = u^* \tag{2.9}$$

$$k_1\Theta_1'(1) - k_2\Theta_2'(1) = 2\lambda^2\alpha_2\rho_1 L. \tag{2.10}$$

It is easy to see that from (2.5)–(2.9) we can obtain

$$\Theta_1(y) = A + \left(u^* - A\right)\frac{erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\lambda(y-r)\right)}{erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\lambda(1-r)\right)}, \quad r < y < 1 \tag{2.11}$$

$$\Theta_2(y) = B - \left(B - u^*\right)\frac{erfc\,(\lambda y)}{erfc\,(\lambda)}, \quad y > 1. \tag{2.12}$$

Now, we have to consider here the condition (2.10) which implies that λ must be the solution of the following equation:

$$\Psi_1(x) = \Phi(x), \quad x > 0 \tag{2.13}$$

where

$$\Psi_1(x) = \frac{k_1\,(u^* - A)}{\rho_1 L}\sqrt{\frac{1}{\pi\alpha_1\alpha_2}}F_2\left(\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)x\right), \quad x > 0 \tag{2.14}$$

$$\Phi(x) = x + \frac{k_2\,(B - u^*)}{\alpha_2\rho_1 L\sqrt{\pi}}F_1(x), \quad x > 0 \tag{2.15}$$

$$F_1(x) = \frac{\exp(-x^2)}{erfc(x)}, \qquad F_2(x) = \frac{\exp(-x^2)}{erf(x)}, \quad x > 0. \tag{2.16}$$

In [27] it has been proved that

$$F_1(0^+) = 1, \qquad F_1(+\infty) = +\infty, \qquad F_1'(x) > 0, \quad x > 0$$

$$F_2(0^+) = +\infty, \qquad F_2(+\infty) = 0, \qquad F_2'(x) < 0, \quad x > 0$$

so we have that Ψ_1 is a strictly decreasing function for $x > 0$, with the properties $\Psi_1(0^+) = +\infty$, $\Psi_1(+\infty) = 0$; and Φ is a strictly increasing function for $x > 0$, with the properties $\Phi(0^+) = \frac{k_2(B-u^*)}{\alpha_2\rho_1 L\sqrt{\pi}}$, $\Phi(+\infty) = +\infty$. Then, there exists a unique solution λ of the Eq. (2.13) and then we have the following theorem:

Theorem 2.1. *Eq. (2.13) has a unique solution $\lambda > 0$. Moreover, the free boundary problem (1.1)–(1.6) has an explicit solution given by*

$$u_1(x,t) = A + \left(u^* - A\right) \frac{erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\lambda\left(\frac{x}{2\lambda\sqrt{\alpha_2 t}} - r\right)\right)}{erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\lambda(1-r)\right)}, \quad rs(t) < x < s(t),\ t > 0 \tag{2.17}$$

$$u_2(x,t) = B - \left(B - u^*\right) \frac{erfc\left(\frac{x}{2\sqrt{\alpha_2 t}}\right)}{erfc\ (\lambda)}, \quad x > s(t),\ t > 0 \tag{2.18}$$

and the free boundary $s(t)$ is given by (2.4) where the coefficient λ is the unique solution of Eq. (2.13).

Remark 2.1. In the particular case $r = 0$ we have the classical Neumann solution [2] for the two-phase Stefan problem.

3. Solidification of a pure substance with a heat-flux condition at $x = rs(t)$

In this section we consider problem (1.1)–(1.5) with the heat-flux boundary condition (1.7) [24,25,12]. We can work as before, transforming this free boundary problem into a system of ordinary differential equations through a non-dimensional similarity coordinate $y = \frac{x}{2\mu\sqrt{\alpha_2 t}}$. It is easy to see that the free boundary must be of the type

$$s(t) = 2\mu\sqrt{\alpha_2 t} \tag{3.1}$$

where μ is a dimensionless constant to be determined. We obtain:

$$\frac{\alpha_1}{2\alpha_2\mu^2}\,\Theta_1'' + (y - r)\,\Theta_1' = 0, \qquad r < y < 1 \tag{3.2}$$

$$\Theta_2'' + 2\mu^2 y\,\Theta_2' = 0, \qquad y > 1 \tag{3.3}$$

$$\Theta_2(+\infty) = B \tag{3.4}$$

$$\Theta_1'(r) = 2\frac{q_0\sqrt{\alpha_2}}{k_1}\mu. \tag{3.5}$$

$$\Theta_1(1) = \Theta_2(1) = u^* \tag{3.6}$$

$$k_1\Theta_1'(1) - k_2\Theta_2'(1) = 2\mu^2\alpha_2\rho_1 L. \tag{3.7}$$

From these equations we get

$$\Theta_1(y) = u^* - \frac{q_0\sqrt{\pi\alpha_1}}{k_1}\left(erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\mu(1-r)\right) - erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\mu(y-r)\right)\right), \quad r < y < 1 \tag{3.8}$$

$$\Theta_2(y) = B - \left(B - u^*\right)\frac{erfc\ (\mu y)}{erfc\ (\mu)}, \quad y > 1, \tag{3.9}$$

where μ must be the solution of the following equation:

$$\Psi_2(x) = \Phi(x), \quad x > 0, \tag{3.10}$$

with $\Phi(x)$ is given by (2.15), and

$$\Psi_2(x) = \frac{q_0}{\rho_1 L\sqrt{\alpha_2}}\exp\left(-\frac{\alpha_2}{\alpha_1}(1-r)^2x^2\right), \quad x > 0. \tag{3.11}$$

Taking into account that Ψ_2 is a strictly decreasing function for $x > 0$, with the properties $\Psi_2(0^+) = \frac{q_0}{\rho_1 L\sqrt{\alpha_2}}$, $\Psi_2(+\infty) = 0$; then Eq. (3.10) has a unique solution μ if and only if $\frac{q_0}{\rho_1 L\sqrt{\alpha_2}} > \frac{k_2(B-u^*)}{\alpha_2\rho_1 L\sqrt{\pi}}$, which can be summarized in:

Theorem 3.1. *If*

$$q_0 > \frac{k_2\,(B - u^*)}{\sqrt{\pi\alpha_2}} \tag{3.12}$$

holds, then Eq. (3.10) has a unique solution $\mu > 0$. Moreover, the free boundary problem (1.1)–(1.5), (1.7) has an explicit solution given by

$$u_1(x,t) = u^* - \frac{q_0\sqrt{\pi\alpha_1}}{k_1}\left(erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\mu(1-r)\right)\right) - erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\mu\left(\frac{x}{2\mu\sqrt{\alpha_2 t}} - r\right)\right), \quad rs(t) < x < s(t), \quad t > 0 \tag{3.13}$$

$$u_2(x,t) = B - \left(B - u^*\right) \frac{erfc\left(\frac{x}{2\sqrt{\alpha_2 t}}\right)}{erfc\left(\mu\right)}, \quad x > s(t),\ t > 0 \tag{3.14}$$

and the free boundary $s(t)$ is given by (3.1) where the coefficient μ is the unique solution of Eq. (3.10).

Remark 3.1. It is important to remark that the inequality (3.12) does not depends on $r < 1 (0 < r < 1$ or $r < 0)$. Furthermore, in the particular case $r = 0$ we have the results obtained in [12].

4. Relationship between heat transfer problems with temperature and heat flux at $x = rs(t)$

Now, we consider the solution $u_1(x,t)$ of the problem (1.1)–(1.5), (1.7) given by (3.13). Computing $u_1(rs(t), t)$ and we have:

$$u_1(rs(t), t) = u^* - \frac{q_0\sqrt{\pi\alpha_1}}{k_1} erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\mu(1-r)\right) = A_0(\mu) < u^* \tag{4.1}$$

which is constant in time.

If we replace A by $A_0(\mu)$ in condition (1.6) and we solve the free boundary problem (1.1)–(1.6) we obtain the following similarity solutions:

$$\tilde{u}_1(x,t) = A_0(\mu) + \left(u^* - A_0(\mu)\right) \frac{erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\lambda\left(\frac{x}{2\lambda\sqrt{\alpha_2 t}} - r\right)\right)}{erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\lambda(1-r)\right)}, \quad rs(t) < x < s(t),\ t > 0 \tag{4.2}$$

$$\tilde{u}_2(x,t) = B - \left(B - u^*\right) \frac{erfc\left(\frac{x}{2\sqrt{\alpha_2 t}}\right)}{erfc\left(\mu\right)}, \quad x > s(t),\ t > 0 \tag{4.3}$$

and $s(t) = 2\lambda\sqrt{\alpha_2 t}$ is the free boundary. The unknown coefficient λ must be the solution of the following equation:

$$\sqrt{\frac{\alpha_2}{\alpha_1\pi}} k_1 \left(u^* - A_0(\mu)\right) F_2\left(\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\lambda\right) = \alpha_2\rho_1 L\lambda + \frac{k_2\left(B - u^*\right)}{\sqrt{\pi}} F_1(\lambda). \tag{4.4}$$

Theorem 4.1. *Under the hypotheses (3.12) the solution μ of Eq. (3.10) is also solution of Eq. (4.4), i.e., $\mu = \lambda$*

Proof. We have:

$$\begin{aligned}
&\mu \text{is a solution of Eq. (4.4))} \Leftrightarrow \\
&\Leftrightarrow \sqrt{\frac{1}{\alpha_1\alpha_2\pi}}\frac{k_1}{\rho_1 L}\left(u^* - A_0(\mu)\right) F_2\left(\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\lambda\right) = \lambda + \frac{k_2\left(B-u^*\right)}{\alpha_2\rho_1 L\sqrt{\pi}} F_1(\lambda) \Leftrightarrow \\
&\Leftrightarrow \frac{q_0}{\rho_1 L\sqrt{\alpha_2}} erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\mu(1-r)\right) F_2\left(\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\lambda\right) = \lambda + \frac{k_2\left(B-u^*\right)}{\alpha_2\rho_1 L\sqrt{\pi}} F_1(\lambda) \Leftrightarrow \\
&\Leftrightarrow \Psi_2(\mu) = \Phi(\mu) \Leftrightarrow \mu \text{is a solution of Eq. (3.10)}
\end{aligned}$$

i.e, $\mu = \lambda$. □

As a consequence of Theorem 4.1, we can translate inequality (3.12) for q_0 for the free boundary problem (1.1)–(1.5), (1.7) to an inequality for λ for the free boundary problem (1.1)–(1.6), that is to say,

$$q_0 = \frac{k_1\left(u^* - A\right)}{\sqrt{\pi\alpha_1} erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\lambda\right)} > \frac{k_2\left(B - u^*\right)}{\sqrt{\pi\alpha_2}} \tag{4.5}$$

that is the inequality

$$erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)\lambda\right) < \frac{k_1\left(u^* - A\right)}{k_2\left(B - u^*\right)}\sqrt{\frac{\alpha_2}{\alpha_1}} \tag{4.6}$$

which is valid for problem (1.1)–(1.6).

This quotation makes sense when the right hand side of the equation is less than one, that is to say:

Corollary 4.1. *When data for the free boundary problem (1.1)–(1.6) verifies the inequality*

$$\frac{k_1\left(u^* - A\right)}{k_2\left(B - u^*\right)}\sqrt{\frac{\alpha_2}{\alpha_1}} < 1 \tag{4.7}$$

then the coefficient λ *of the free boundary* (2.4) *satisfies the inequality*

$$\lambda < \frac{1}{1-r}\sqrt{\frac{\alpha_1}{\alpha_2}}erf^{-1}\left(\frac{k_1\,(u^*-A)}{k_2\,(B-u^*)}\sqrt{\frac{\alpha_2}{\alpha_1}}\right). \tag{4.8}$$

5. Solidification of a pure substance with a convective condition at $x = rs(t)$

In this section we consider the free boundary problem (1.1)–(1.5) plus a convective cooling condition given by (1.8) where h_0 is the coefficient which characterizes the dependent-time heat transfer coefficient [16,9]. If we define

$$y = \frac{x}{2\xi\sqrt{\alpha_2 t}} \tag{5.1}$$

where α_2 is the thermal diffusivity of liquid region, we obtain

$$u_1(x,t) = u_0 + \left(u^* - u_0\right)\frac{k_1 + \sqrt{\pi\alpha_1}\,h_0\,erf\left(\frac{x}{2\sqrt{\alpha_1 t}} - r\xi\sqrt{\frac{\alpha_2}{\alpha_1}}\right)}{k_1 + \sqrt{\pi\alpha_1}\,h_0\,erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\xi(1-r)\right)}, \quad rs(t) < x < s(t),\ t > 0 \tag{5.2}$$

$$u_2(x,t) = B - \left(B - u^*\right)\frac{erfc\left(\frac{x}{2\sqrt{\alpha_2 t}}\right)}{erfc\ (\xi)}, \quad x > s(t),\ t > 0 \tag{5.3}$$

$$s(t) = 2\xi\sqrt{\alpha_2 t}, \quad t > 0 \tag{5.4}$$

where the unknown dimensionless coefficient ξ must be the solution of the equation:

$$\Psi_3(x) = \Phi(x), \quad x > 0 \tag{5.5}$$

with $\Phi(x)$ is given by (2.15), and

$$\Psi_3(x) = \frac{(u^* - u_0)\ h_0}{\rho_1 L\sqrt{\alpha_2}}\,\frac{\exp\left(-\frac{\alpha_2}{\alpha_1}(1-r)^2x^2\right)}{1 + \sqrt{\pi\alpha_1}\,\frac{h_0}{k_1}\,erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}(1-r)x\right)}, \quad x > 0. \tag{5.6}$$

So, we can have the following theorem as before:

Theorem 5.1. *If*

$$h_0 > \frac{k_2\,(B-u^*)}{\sqrt{\pi\alpha_2}\,(u^*-u_0)} \tag{5.7}$$

holds, then Eq. (5.5) *has a unique solution* $\xi > 0$. *Moreover, the free boundary problem* (1.1)–(1.5), (1.8) *has an explicit solution given by* (5.2), (5.3), *and the free boundary* $s(t)$ *is given by* (5.4) *where the coefficient* ξ *is the unique solution of Eq.* (5.5). *Plus, the temperature on the left free boundary* $x = rs(t)$ *is constant for all* $t > 0$ *and* $u_0 < u_1(rs(t), t) = Const. < u^*$.

Remark 5.1. It is important to remark that (5.7) does not depend on $r < 1$ ($0 < r < 1$ or $r < 0$). Furthermore, in the particular case $r = 0$ we have the results obtained in [16,9].

6. Relationship between heat transfer problems with a convective condition and a temperature condition or a heat flux condition at $x = rs(t)$

Summarizing, we have the following results, which may be proved in much the same way as in Section 4:

Theorem 6.1. *Under the hypothesis* (5.7) *the solution* ξ *of Eq.* (5.5) *is also solution of Eq.* (4.4), *i.e.*, $\xi = \lambda$

Corollary 6.1. *Suppose that* $u_0 < A < u^* < B$. *When data for problem* (1.1)–(1.6) *verifies the inequality*

$$\frac{k_1\sqrt{\alpha_2}\,(u^*-A)\,(u^*-u_0)}{k_2\sqrt{\alpha_1}\,(B-u^*)\,(A-u_0)} < 1 \tag{6.1}$$

then the coefficient λ *of the free boundary* (2.4) *satisfies the inequality*

$$\lambda < \frac{1}{1-r}\sqrt{\frac{\alpha_1}{\alpha_2}}erf^{-1}\left(\frac{k_1\,(u^*-A)\,(u^*-u_0)}{k_2\,(A-u_0)\,(B-u^*)}\sqrt{\frac{\alpha_2}{\alpha_1}}\right). \tag{6.2}$$

Theorem 6.2. *Under the hypotheses* (3.12) *the solution* ξ *of Eq.* (5.5) *is also solution of Eq.* (3.10), *i.e.*, $\xi = \mu$.

Corollary 6.2. *When data for problem (1.1)–(1.5), (1.8) verifies the inequality*

$$\frac{k_2\,(B-u^*)}{k_1\,(u^*-u_0)}\sqrt{\frac{\alpha_1}{\alpha_2}} < 1 \tag{6.3}$$

and h_0 verifies the inequalities

$$1 < \frac{h_0\,(u^*-u_0)\,\sqrt{\pi\alpha_2}}{k_2\,(B-u^*)} < \frac{1}{1-\frac{k_2(B-u^*)}{k_1(u^*-u_0)}\sqrt{\frac{\alpha_1}{\alpha_2}}} \tag{6.4}$$

then the coefficient ξ of the free boundary (5.4) satisfies the inequality

$$erf\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\xi(1-r)\right) < \left[\frac{h_0\,(u^*-u_0)\,\sqrt{\pi\alpha_2}}{k_2\,(B-u^*)} - 1\right]\frac{k_1}{h_0\sqrt{\pi\alpha_1}}. \tag{6.5}$$

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