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GENERALIZED LAME-CLAPEYRON SOLUTION FOR A ONE-PHASE SOURCE STEFAN PROBLEM

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ABSTRACT: *In this paper we obtain a generalized Lamé-Clapeyron solution for a one-phase Stefan problem with a particular type of sources. Necessary and sufficient conditions are given in order to characterize the source term which provides a unique solution. Some estimates on the free boundary and the temperature are presented. In particular, asymptotic expansions are given for small Stefan number and source.*

KEY WORDS: Stefan problem, similarity variable, Lamé-Clapeyron solution, phase-change problem, free boundary problems, exact solutions, asymptotic expansions, quasi-steady-state method.

RESUMO: *SOLUÇÃO DE LAME-CLAPEYRON PARA O PROBLEMA DE STEFAN COM FONTE MONOFÁSICA. Obtemos uma solução de Lamé–Clapeyron generalizada para o problema monofásico de Stefan com tipo de fonte particular. Apresenta-se condições necessárias e suficientes para caracterizar o termo de fonte que garante solução única. Algumas estimativas na fronteira livre e na temperatura são também apresentadas. Em particular, expanções assintóticas são dadas para o caso quando o número de Stefan e a fonte são pequenos.*

PALAVRAS-CHAVE: Problema de Stefan; variável de similaridade; solução de Lamé–Clapeyron; problema de mudança de fase; soluções exatas; expançõoes assintóticas; método de regime quasi-estacionário.

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1. INTRODUCTION

An explicit solution for the one-phase Stefan problem corresponding to a semi-infinite material with constant thermal coefficients is well known and referred to as the classic Lamé-Clapeyron solution [1], [7], [13]. Without loss of generality, we suppose the phase-change temperature is $0°C$. We consider the following fusion problem: the material is initially in solid phase at the melting temperature and for all possible instant we have a constant temperature $B > 0$ on the fixed face $x = 0$. The problem consists in finding the liquid-solid interface (free boundary) $x = s(t) > 0$, defined for $t > 0$ with $s(0) = 0$, and the liquid temperature $\theta = \theta(x,t) > 0$, defined for $0 < x < s(t)$, $t > 0$, such that they satisfy the following conditions:

$$\begin{array}{ll} (i) & \rho c\theta_t - k\theta_{xx} = 0, 0 < x < s(t), t > 0, \\ (ii) & \theta(0,t) = B > 0, t > 0, \\ (iii) & \theta(s(t),t) = 0, t > 0, \\ (iv) & k\theta_x(s(t),t) = -\rho\ell\dot{s}(t), t > 0, \\ (v) & s(0) = 0, \end{array} \tag{1}$$

where $k > 0$ is the thermal conductivity, $\rho > 0$ the mass density, $c > 0$ the specific heat and $\ell > 0$ the latent heat of fusion.

We denote by $\alpha = \frac{k}{\rho c} > 0$ the diffusion coefficient and Ste $= \frac{Bc}{\ell} > 0$ the Stefan number. We shall denote $a = \sqrt{\alpha} = \sqrt{\frac{k}{\rho c}} > 0$, for convenience in the notation.

The solution of problem (1) is given by [1], [2], [7], [9], [12]:

$$s_0(t) = 2a\xi_0\sqrt{t}, \quad \theta_0(x,t) = B[1 - \frac{1}{\mathrm{erf}(\xi_0)}\mathrm{erf}(\frac{x}{2a\sqrt{t}})], \tag{2}$$

where $\xi_0 > 0$ is the unique solution of the following equation:

$$F_0(x) = \frac{\mathrm{Ste}}{\sqrt{\pi}}, \quad x > 0 \tag{3}$$

with the notation

$$\begin{array}{l} F_0(x) = x\ \exp(x^2)\ \mathrm{erf}(x), \\ \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-u^2)du \ \ \text{(error function)}. \end{array} \tag{4}$$

In the present paper, we consider a generalization of the equation (1-i) by

$$\rho c\theta_t - k\theta_{xx} = g(x,t), \quad 0 < x < s(t), \quad t > 0, \tag{1-i-bis}$$

where $g = g(x,t)$ represents a source term for the heat equation. We shall denote with (P) the problem corresponding to conditions (1-i-bis) and (1-ii $\cdots$ v). In many physical applications a volumetric heating/cooling term is considered. For the singular case

$$g(x,t) = \rho\ell\frac{1}{t}\beta(\frac{x}{2a\sqrt{t}}), \tag{5}$$

where $\beta = \beta(\eta)$ is any function with appropriate regularity properties (for instance, $\beta \in$ either $C^0(\Re^+)$ or $C^1(\Re^+)$), we prove necessary and sufficient conditions for the function β to have an exact solution for problem (P). This exact solution can be considered as a generalized Lamé–Clapeyron solution for the one-phase Stefan problem (P) with $\beta \neq 0$. Because of the singularity of the source, (P) can be regarded as an ill-posed problem. For a general theory on one-phase Stefan problem we refer to [1], [4], [5], [9].

We shall denote by $(N-n)$ the formula (n) of Section N and we shall omit N in the same paragraph, similarly, for theorems, lemmas, corollaries, remarks and notes.

In Section 2, we characterize the set of function β for which a unique generalized Lamé–Clapeyron solution exists, given by (2-10), (2-11) and (2-12). To that purpose, we define a function Z by (2-30) or (2-33) to obtain a unique solution, and either several solutions or no solutions at all for problem (1-i-bis). Similar situations have been recently found in other similarity solutions for the Stefan problem [6].

In Section 3, we consider the problem for small Stefan number Ste $\ll 1$ which is characterized for the quasi-steady-state solution when function β is constant and given by $\beta(\eta) = \beta(0) < 1$.

In Section 4, we give several estimates for the temperature and free boundary.

2. GENERALIZED LAMÉ–CLAPEYRON SOLUTION

We consider the following free boundary problem for the heat equation: Find the free boundary $x = s(t) > 0$, defined for $t > 0$ and $s(0) = 0$, and the temperature $\theta = \theta(x,t) > 0$, defined for $0 < x < s(t)$, $t > 0$, such that they satisfy the following conditions:

$$(P) \qquad \begin{aligned} &\rho c\theta_t - k\theta_{xx} = g(x,t), \;\; 0 < x < s(t), \;\; t > 0, \\ &\theta(0,t) = B > 0, \;\; t > 0, \\ &\theta(s(t),t) = 0, \;\; t > 0, \\ &k\theta_x(s(t),t) = -\rho\ell\dot{s}(t), \;\; t > 0, \quad s(0) = 0, \end{aligned}$$

for a given source function $g = g(x,t)$, fixed face temperature $B > 0$ and constant thermal coefficients $k, \rho, c, \ell > 0$.

Since our interest is finding solutions of the Lamé-Clapeyron type for problem (**P**), we apply the immobilization domain method [3], [8], [12], that is, we are looking for solutions of the following type

$$\theta(x,t) = T(y) \tag{1}$$

where the new independent spatial variable y is defined by

$$y = \frac{x}{s(t)}. \tag{2}$$

In this case, the problem (P) is transformed as follows: *Find functions* $T = T(y)$, *defined for* $0 < y < 1$, *and* $s = s(t)$, *defined for* $t > 0$ *such that*

$$s(t)\dot{s}(t)\, y\, T'(y) + a^2 T''(y) = -\frac{s^2(t)}{\rho c} g(y\, s(t), t),\ 0 < y < 1,\ t > 0,$$
$$T(0) = B > 0, \quad T(1) = 0,\ T'(1) = -\frac{\rho \ell}{k} s(t)\dot{s}(t),\ t > 0, \quad s(0) = 0. \tag{3}$$

We must have necessarily that $s(t)\dot{s}(t) = Const.$, i.e.

$$s(t) = 2a\xi\sqrt{t}, \tag{4}$$

where the dimensionless parameter $\xi > 0$ is unknown. Then, from (3) and (4), and for the source term g, given by expression (1-5), we obtain the following problem:

$$T''(y) + 2\xi^2 y T'(y) = -\frac{4\xi^2 \ell}{c}\beta(\xi y),\ \ 0 < y < 1,$$
$$T(0) = B, \quad T(1) = 0,\ \ T'(1) = -\frac{2\ell}{c}\xi^2. \tag{5}$$

If we define

$$R(\eta) = T(\frac{\eta}{\xi}) \quad (\text{or } T(y) = R(\xi y)), \quad \eta = \xi y, \tag{6}$$

problem (5) is equivalent to

$$R''(\eta) + 2\eta R'(\eta) = -\frac{4\ell}{c}\beta(\eta),\ \ 0 < \eta < \xi,$$
$$R(0) = B, \quad R(\xi) = 0,\ \ R'(\xi) = -\frac{2\ell}{c}\xi. \tag{7}$$

After some elementary computations the solution $R = R(\eta)$ and $\xi > 0$ of problem (7) is given explicitly by

$$R(\eta) = B - \frac{\ell\sqrt{\pi}}{c}\xi \exp(\xi^2)\,\mathrm{erf}(\eta)$$
$$+ \frac{4\ell}{c}\int_0^\eta [\int_r^\xi \beta(y)\exp(y^2)dy]\exp(-r^2)dr,\ \ 0 < \eta < \xi, \tag{8}$$

where $\xi > 0$ must verify the condition

$$R(\xi) = 0. \tag{9}$$

Taking into account (1-4), we can do an integration into condition $R(\xi) = 0$ and we obtain that the number $x = \xi > 0$ must satisfy the equation:

$$F(x, \beta) = \frac{\mathrm{Ste}}{\sqrt{\pi}},\ \ x > 0, \tag{10}$$

where function $F = F(x) = F(x, \beta)$ is defined for $x > 0$ and $\beta = \beta(r)$ by the expression

$$F(x, \beta) = F_0(x) - 2\int_0^x \exp(r^2)\,\mathrm{erf}(r)\beta(r)dr. \tag{11}$$

Note 1. From now on we shall note with $x > 0$ the dimensionless first variable of function F and the spatial variable for the temperature θ.

Therefore, we have obtained the following abstract theorem in the case where equation (10) has at least one solution $\xi > 0$.

Theorem 1. *An explicit solution of problem (P) with the source term g, defined by (1-5), as a function of β, is given by:*

$$\begin{aligned} \theta(x,t) &= B\{1 - \frac{\sqrt{\pi}}{Ste}\xi \exp(\xi^2)\ erf(\eta) + \\ &+ \frac{4}{Ste}\int_0^\eta [\int_r^\xi \beta(y)\exp(y^2)dy]\exp(-r^2)dr\}, \\ s(t) &= 2a\xi\sqrt{t}, \quad \eta = \frac{x}{2a\sqrt{t}} \in (0,\xi), \end{aligned} \tag{12}$$

where the number $\xi > 0$ is a solution of equation (10).

Remark 1. Function F satisfies the following properties:

$$F(0^+, \beta) = 0, \tag{13}$$

$$\frac{\partial F}{\partial x}(x,\beta) = \frac{2x}{\sqrt{\pi}} + \exp(x^2)\,\mathrm{erf}(x)[1 + 2x^2 - 2\beta(x)], \tag{14}$$

$$\begin{aligned} \frac{\partial^2 F}{\partial x^2}(x,\beta) &= \frac{4}{\sqrt{\pi}}(1 + x^2) + 2F_0(x)(3 + 2x^2) \\ &- 4[\frac{1}{\sqrt{\pi}} + F_0(x)]\beta(x) - 2\exp(x^2)\,\mathrm{erf}(x)\beta'(x). \end{aligned} \tag{15}$$

It is clear that (14) and (15) are meaningful when β is continuous and continuous differentiable, respectively. The space where $\beta(\cdot)$ will be considered has not been yet defined.

Lemma 2. *The function F satisfies the following properties:*

(i) $\frac{\partial F}{\partial x}(x,\beta) > 0 \Longleftrightarrow \beta(x) < \psi_0(x)$

where function $\psi_0 = \psi_0(x)$ is defined by

$$\psi_0(x) = \frac{1}{2} + x^2 + \frac{x}{\sqrt{\pi}}G(x), \quad G(x) = \frac{\exp(-x^2)}{erf(x)}, \quad x > 0. \tag{16}$$

(ii) $\frac{\partial^2 F}{\partial x^2}(x,\beta) \geq 0, \ \forall x > 0$ *if and only if* β *verifies the following differential inequality:*

$$\beta'(x) + M'(x)\beta(x) \leq N(x), \ \forall x > 0, \tag{17}$$

where functions M *and* N *are defined by*

$$\begin{aligned} M(x) &= x^2 + \log(erf(x)), \ x > 0, \\ N(x) &= x(3 + 2x^2) + \frac{2}{\sqrt{\pi}}(1 + x^2)G(x), \ x > 0. \end{aligned} \tag{18}$$

(iii) *The differential inequality (17) implies that function* β *necessarily satisfies the inequality* $\beta(x) \leq \psi_{C_\beta}(x), \ \forall x > 0$*, where function* ψ_C *is defined by*

$$\psi_C(x) = \psi_0(x) + CG(x), \ x > 0 (C \in \Re), \tag{19}$$

and C_β *is supposed to be a real number (finite) which is defined by* $C_\beta = \lim_{x \to 0^+}[\beta(x) \ erf(x)]$ *as a function of* β.

Proof. By means of the following expressions,

$$\begin{aligned} &M(x) = 2\int [x + \frac{1}{\sqrt{\pi}}G(x)]dx, \exp(-M(x)) = G(x), \\ &\int x \exp(x^2) \operatorname{erf}(x) dx = \frac{1}{2}\exp(x^2)\operatorname{erf}(x) - \frac{x}{\sqrt{\pi}}, \\ &\int x^3 \exp(x^2)\operatorname{erf}(x)dx = \frac{x}{\sqrt{\pi}} - \frac{x^3}{3\sqrt{\pi}} + \frac{(x^2-1)}{2}\exp(x^2)\operatorname{erf}(x), \\ &\int N(x)\exp(M(x))dx = \frac{x}{\sqrt{\pi}} + (\frac{1}{2} + x^2)\exp(x^2)\operatorname{erf}(x), \\ &\exp(-M(x))\int N(x)\exp(M(x))dx = \psi_0(x), \end{aligned} \tag{20}$$

and with some elementary computation, the lemma can be established.

Lemma 3. *If the function* β *verifies the inequality*

$$\beta(x) \leq \psi_C(x), \ \forall x > 0, \tag{21}$$

for some constant $C \in \Re$*, then we have the following estimates:*

$$\begin{aligned} &F(x,\beta) \geq F(x,\psi_C) = -2Cx, \ \forall x > 0, \\ &\frac{\partial F}{\partial x}(x,\beta) \geq \frac{\partial F}{\partial x}(x,\psi_C) = -2C, \ \forall x > 0. \end{aligned} \tag{22}$$

Proof. From elementary computations and taking into account the following expressions:

$$\begin{aligned} &\int \operatorname{erf}(x)dx = x\operatorname{erf}(x) + \frac{1}{\sqrt{\pi}}\exp(-x^2), \\ &\int (\frac{1}{2} + x^2)\exp(x^2)\operatorname{erf}(x)dx = \frac{F_0(x)}{2} - \frac{x^2}{2\sqrt{\pi}}, \end{aligned} \tag{23}$$

we can prove the lemma.

Remark 2. We have the following real functions

$$
\begin{aligned}
U(x) &= \frac{1}{2} + x^2 + \frac{1}{2}\exp(-x^2), \quad x > 0, \\
V(x) &= \frac{1}{x}\exp(x^2)\operatorname{erf}(x) = \frac{1}{xG(x)}, \quad x > 0, \\
\varphi_C(x) &= \frac{1}{2} + x^2 + CG(x) = \psi_C(x) - \frac{xG(x)}{\sqrt{\pi}}, \quad x > 0, (C \in \Re).
\end{aligned} \tag{24}
$$

In what follows, we will use the properties:

$$
\begin{aligned}
& G(0^+) = +\infty, \ \ G(+\infty) = 0, \ \ G' < 0 \text{ in } \Re^+, \\
& V(0^+) = \frac{2}{\sqrt{\pi}}, \ \ V(+\infty) = +\infty, \ \ V' > 0 \text{ in } \Re^+, \\
& U(0^+) = 1, \ \ U(+\infty) = +\infty, \ \ U'(0^+) = 0, \ \ U' > 0 \text{ in } \Re^+, \\
& \psi_0(0^+) = 1, \ \ \psi_0(+\infty) = +\infty, \ \ \psi_0'(0^+) = 0, \ \ \psi_0' > 0 \text{ in } \Re^+, \\
& \frac{1}{2} + x^2 < U(x) < \psi_0(x), \ \ \forall x > 0,
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
& \varphi_C(0^+) = +\infty, \ \ \varphi_C(+\infty) = +\infty, \\
& \lim_{C \to +\infty} [\min_{x>0} \varphi_C(x)] = +\infty \ \ (C > 0), \\
& \varphi_C(0^+) = -\infty, \ \ \varphi_C(+\infty) = +\infty, \ \ {\varphi'}_C > 0 \text{ in } \Re^+ \ \ (C < 0), \\
& \frac{1}{2} + x^2 < \varphi_C(x) < \psi_0(x), \ \ \forall x > C\sqrt{\pi} \ \ (C > 0), \\
& \lim_{x \to +\infty} [\psi_0(x) - (\frac{1}{2} + x^2)] = 0.
\end{aligned} \tag{26}
$$

Now we can prove the following first existence theorem.

Theorem 4. *Let β be a real continuous function in $\Re_0^+$.*

(i) *If β verifies the inequality*

$$\beta(x) < \psi_0(x) - \varepsilon, \ \ \forall x \geq 0 \ \textit{for some} \ \varepsilon > 0, \tag{27}$$

then there exists a unique number $\xi = \xi(\,Ste) > 0$ which is the solution of the equation (10) for each $Ste > 0$.

(ii) *If β verifies the inequality $\beta \leq \psi_C$ in $\Re^+$ for some constant $C < 0$ then we obtain the same conclusion of part (i).*

(iii) *If β verifies the inequality $\beta \leq \varphi_C$ in $\Re^+$ for some constant $C > 0$ then there exist a number $\xi = \xi\ (\,Ste) > 2C\sqrt{\pi}$ which is a solution of the equation (10) for all $Ste > 0$.*

(iv) *If $\beta \equiv \varphi_C$ in part (iii) we can also deduce the uniqueness of the number ξ.*

Proof. (i) It is enough to apply part (i) of Lemma 2, (13) and the fact that $F(+\infty, \beta) = +\infty$ because

$$F(x,\beta) \geq F(x, \psi_0 - \varepsilon) = 2\varepsilon \int_0^x \exp(r^2)\,\mathrm{erf}(r)dr. \tag{28}$$

(ii) It is a consequence of Lemma 3.

(iii) We have

$$F(x,\beta) \geq \frac{1}{\sqrt{\pi}} x(x - 2C\sqrt{\pi}), \forall x > 0, \tag{29}$$

that is, we obtain at least a solution $\xi > 2C\sqrt{\pi}$ of the equation (10). This procedure can be applied for all Ste > 0.

(iv) Since we have the equality in (29), we can also obtain the uniqueness of ξ.

Corollary 5. (i) *The inequality $\beta(x) < \frac{1}{2} + x^2, \forall x > 0$, is a sufficient condition to have a unique number $\xi > 0$, solution of the equation (10).*

(ii) *If β is real function bounded from above by a constant M_0, then there exists at least an element $\xi > 0$ solution of the equation (10).*

Proof. In view of (26) it is sufficient to choose $C > 0$ so large such that $\min_{x>0} \varphi_C(x) \geq M_0$ and then we can apply part (iii) of Theorem 4.

Corollary 6. *For the particular and interesting case $\beta \leq 0$ in $\Re_0^+$ we have a unique element $\xi > 0$, the solution of the equation (10).*

Remark 3. In parts (iii) of Theorem 4 and (ii) of Corollary 5 we cannot affirm the uniqueness of the element ξ.

Remark 4. If β verifies condition (27) we have that $\beta(0^+) < 1$ and the existence and uniqueness of number ξ. We can also have $\beta(0^+) \geq 1$ by choosing a suitable constant $C > 0$ (part (iii) of Theorem 4) but in this case we cannot assure the uniqueness of ξ.

We give now a general theorem which gives sufficient conditions on function β to have a unique number $\xi > 0$, the solution of equation (10).

Theorem 7. *Let β be a continuous real function on $\Re^+$ such that $x\beta(x)$ is locally integrable on $\Re^+$. Define the function Z by*

$$Z = Z_\beta(x) = \exp(x^2)\ erf(x)[\psi_0(x) - \beta(x)], x > 0, \tag{30}$$

which is continuous and locally integrable on $\Re^+$. If the function Z satisfies the following conditions

$$Z > 0 \ on\ (\nu, +\infty), \quad \int_0^{+\infty} Z(t)dt = +\infty, \tag{31}$$

where $\nu \geq 0$ is defined by

$$\nu = inf\{x \geq 0 / \int_0^x Z(t)dt > 0\} \ \ (\equiv \nu_Z), \tag{32}$$

then for any Ste > 0, there exists a unique number $\xi = \xi(Ste) > 0$ which is the solution of equation (10) for the given function β. Conversely, if for the given function β, equation (10) has a unique root $\xi = \xi(Ste) > 0$ for any Ste > 0 then there exists a continuous and locally integrable function Z on $\Re^+$ satisfying (31) and (32) such that

$$\beta(x) = \psi_0(x) - Z(x)G(x) \ \ (\equiv \beta_Z(x)), \ \ x > 0, \tag{33}$$

where functions ψ_0 and G are defined by (16). Moreover in any case the root $\xi > \nu$.

Proof. Due to the definition of functions ψ_0 and G, and the second equality in (23), we obtain

$$F(x, \beta_Z) = F_0(x) - 2\int_0^x \exp(r^2)\operatorname{erf}(r)[\psi_0(r) - Z(r)G(r)]dr =$$
$$= 2\int_0^x Z(r)dr, \ \ \forall x \geq 0. \tag{34}$$

From elementary consideration on function Z, the proof is achieved.

Conversely, if we define

$$\nu = \inf\{x \geq 0 / F(x, \beta) > 0\}, \quad Z(x) = \frac{1}{2}\frac{\partial F}{\partial x}(x, \beta), \ \ x \geq 0, \tag{35}$$

we get the thesis because of the relation (14), F is a strictly increasing function over $(\nu, +\infty)$ and verifies $F(+\infty, \beta) = +\infty$ (by uniqueness of number ξ, solution of the equation (10), for all Ste > 0). By definition of number ν in (35) we have $\xi > \nu$.

Remark 5. Let Z be the function in Theorem 7. If $Z(0^+) > 0$ (or $Z(0^+) < 0$) then $\beta_Z(0^+) = -\infty$ (or $\beta_Z(0^+) = +\infty$). On the other hand, if $Z(0^+) = 0$ then

$$\beta_Z(0^+) = 1 - \frac{\sqrt{\pi}}{2}\lim_{x\to 0^+}\frac{Z(x)}{x} = 1 - \frac{\sqrt{\pi}}{2}Z'(0^+), \tag{36}$$

when $Z'(0^+)$ exists.

Remark 6. We can improve Remark 4 by using appropriate conditions for the function Z in Theorem 7. We have $\beta(0^+) = +\infty$ by taking $Z(0^+) < 0$ and $\beta(0^+) = 1 - \frac{\sqrt{\pi}}{2}Z'(0^+) > 0$ by taking $Z(0^+) = 0$ and $Z'(0^+) < 0$. We have also improved condition (27) to have a unique $\xi > 0$ because it is possible to have $\beta > \psi_0$ in the internal $(0, \nu)$ by choosing an appropriate function Z such that $Z < 0$ in $(0, \nu)$.

Remark 7.(i) If we choose function Z like $Z(x) = Z_C(x) = \frac{x}{\sqrt{\pi}} - C \ \ (C > 0)$, which satisfies the hypotheses of Theorem 7, then we obtain $\beta_{Z_C} \equiv \varphi_C$ and $\nu = 2C\sqrt{\pi}$ (c.f. Theorem 4).

(ii) On the other hand, if $Z(x) = \varepsilon > 0$, $\forall x > 0$, then we get $\beta_Z(x) = \psi_0(x) - \varepsilon G(x)$, $\forall x > 0$, and $\nu = 0$.

(iii) Similarly, if $Z(x) = \varepsilon \exp(x^2)\operatorname{erf}(x)$, $\forall x > 0$, for some constant $\varepsilon > 0$, then we have $\beta_Z(x) = \psi_0(x) - \varepsilon, \forall x > 0$, and $\nu = 0$ (c.f. Theorem 4).

Remark 8. If $Z \leq 0$ ($Z > 0$) in a right neighborhood of $x = 0$ under the hypotheses of Theorem 7, then we have $\nu \geq 0$ ($\nu = 0$).

Remark 9. Under relations (30) and (33) we can deduce the following equivalences:

$$(i) \quad \beta < \psi_0 \text{ in } \Re^+ \Longleftrightarrow Z > 0 \text{ in } \Re^+.$$
$$(ii) \quad \beta(x) < \psi_0(x) - \varepsilon,\ \forall x > 0 \Longleftrightarrow Z(x) > \varepsilon \exp(x^2)\,\mathrm{erf}(x),\ \forall x > 0 \tag{37}$$

where ε is any positive real number.

Remark 10. (Example of the nonexistence solution of the equation (10)). We consider the case $Z(x) = \exp(-x), x > 0$, which gives

$$\beta(x) = \frac{1}{2} + x^2 + [\frac{x}{\sqrt{\pi}} - \exp(-x)]\frac{\exp(-x^2)}{\mathrm{erf}(x)},\ \ x > 0,$$
$$\nu = 0,\ \text{and } F(x, \beta) = 2[1 - \exp(-x)],\ \ x > 0. \tag{38}$$

Therefore, for the physical situation $\frac{\text{Ste}}{\sqrt{\pi}} = \frac{Bc}{\ell\sqrt{\pi}} \geq 2$, we have no explicit solution of the type (12). This occurs because function Z does not verify the limit condition (31). Moreover, others examples of nonexistence solution of the equation (10) can be constructed by choosing functions Z with the property $\int_0^{+\infty} Z(x)dx < +\infty$.

Remark 11. (Example of multiple solutions of equation (10)). We consider the case $Z(x) = 3(x-1)(x-3), x > 0$, which gives

$$\beta(x) = \frac{1}{2} + x^2 + \frac{\exp(-x^2)}{\mathrm{erf}(x)}(-3x^2 + (12 + \frac{1}{\sqrt{\pi}})x - 9),$$
$$F(x, \beta) = 2x(x-3)^2,\ \ x > 0,\ \text{and } \nu = 0. \tag{39}$$

Because the function F has a relative maximum $F(1) = 8$ in $x = 1$, we deduce that the equation (10) has: three Solutions if $0 <$ Ste $< 8\sqrt{\pi}$, two Solutions if Ste $= 8\sqrt{\pi}$, one Solution if Ste $> 8\sqrt{\pi}$. This occurs because function Z is not always positive over the interval $(\nu, +\infty)$ (Z is negative over the interval (1,3)). Moreover, others examples of multiple solutions of the equation (10) can be constructed by choosing suitable functions Z.

Remark 12. (Chaotic example). We consider the case

$$Z(x) = \cos\ x,\quad F(x, \beta) = 2\sin\ x,\ \ x > 0,$$
$$\beta(x) = \frac{1}{2} + x^2 + [\frac{x}{\sqrt{\pi}} - \cos\ x]\frac{\exp(-x^2)}{\mathrm{erf}(x)},\ \text{and } \nu = 0, \tag{40}$$

which gives for the equation (10) an infinity countable set of solutions if $0 <$ Ste $\leq 2\sqrt{\pi}$, and the nonexistence of solutions if Ste $> 2\sqrt{\pi}$.

Note 2. It is important to remark that Theorem 7 is a constructive theorem to obtain a large family of functions β, that is a family of the source term g, which has an exact solution of the Lamé–Clapeyron type (12) (c.f. Theorem 1). Moreover, the hypotheses for the function Z in Theorem 7 are optimal in order to have a unique solution ξ of equation (10) (c.f. the counterexamples given in the last remarks). On the other hand, the relation (30) can be useful in several problems.

3. SOLUTION FOR A SMALL STEFAN NUMBER

We shall consider the behavior of the solution (2-10), (2-12) when the Stefan number is small (classic approximation [3], [10], [11]), i.e. $0 < \text{Ste} = \frac{Bc}{\ell} \ll 1$. If we suppose that function β verifies the conditions (2-27) and (2-37i) then we have $Z_\beta > 0$ in $\Re^+$, i.e. $\nu = 0$. Therefore, we can use the following first order approximations: $\exp(x^2) \approx 1$, $\operatorname{erf}(x) \approx \frac{2}{\sqrt{\pi}}x$, $\beta(x) \approx \beta(0) = \beta_0 < 1 - \varepsilon < 1$, in the definition (2-11) of function F to obtain the first order approximation of the solution ξ of the equation (2-10), i.e.

$$\xi_{ap} = \sqrt{\frac{\text{Ste}}{2(1-\beta_0)}}. \tag{1}$$

Remark 1. It is important to remark that the first order approximation is only true when the number ν is zero. Moreover, we can also suppose that $\beta_0 < 1$ to obtain (1).

We shall interpret the meaning of the formula (1).

Theorem 1. *The solution of the quasi-steady state free boundary problem (P) where* $g(x,t) = \dfrac{\rho\ell\beta_0}{t}$ *with* $\beta_0 < 1$, *i.e.*

$$\begin{aligned} &-k\theta_{xx} = \frac{1}{t}\rho\ell\beta_0, \quad 0 < x < s(t), \quad t > 0, \\ &\theta(0,t) = B > 0, \quad t > 0, \\ &\theta(s(t),t) = 0, \quad k\theta_x(s(t),t) = -\rho\ell\dot{s}(t), \quad t > 0, \\ &s(0) = 0 \end{aligned} \tag{2}$$

is given by

$$\theta(x,t) = \frac{B}{1-\beta_0}\left(1 - \frac{x}{s(t)}\right)\left[1 - \beta_0\left(1 - \frac{x}{s(t)}\right)\right], \quad s(t) = 2a\xi_{ap}\sqrt{t} \tag{3}$$

where ξ_{ap} *is given by (1).*

Proof. We propose a quadratic function in variable x for the temperature, that is $\theta(x,t) = C_1(1 - \frac{x}{s(t)}) + C_2(1 - \frac{x}{s(t)})^2$, where C_1 and C_2 are two unknown constants. Solving all conditions given in (2), we obtain an ordinary differential equation for $s = s(t)$, whose solution is given by (3) and (1). Moreover, we have $C_1 = \frac{B}{1-\beta_0}$, $C_2 = \frac{-B\beta_0}{1-\beta_0}$.

We suppose that function β (or its corresponding function Z) verifies the hypotheses (e.g. Theorem 2-7) to have a unique solution ξ of the equation (2-10). We shall give a general method to obtain ξ_{ap} as function of $Z = Z(x)$ and $\nu > 0$ when $Z(\nu) > 0$. Indeed, in this case, we have $F(\nu, \beta) = 0$, $\frac{\partial F}{\partial x}(\nu, \beta) = 2Z(\nu) > 0$. Therefore, if we consider the first order approximation in equation (2-10) then we obtain that $\xi > \nu > 0$ is the solution of the following linear equation

$$2Z(\nu)(x - \nu) = \frac{\text{Ste}}{\sqrt{\pi}}, \quad x > \nu, \tag{4}$$

which is given by

$$\xi_{ap} = \nu + \frac{\text{Ste}}{2\sqrt{\pi}Z(\nu)}. \tag{5}$$

Thus, we have obtained the following lemma.

Lemma 2. *If function Z verifies the hypotheses of Theorem 2-7 and $Z(\nu) > 0$ then we obtain that the first order approximation ξ_{ap} of the solution ξ of equation (2-10) for a small Stefan number Ste is given by the expression (5).*

Remark 2. If we consider the case $Z = Z_C$, where $C > 0$ is a given constant, then we get (c.f. also Remark 2-7)

$$F(x, \beta) = \frac{x}{\sqrt{\pi}}(x - 2C\sqrt{\pi}), \;\; \beta(x) = \varphi_C(x), \;\; x > 0, \quad \nu = 2C\sqrt{\pi} > 0, \tag{6}$$

and therefore, the exact solution of the equation (2-10) is given by

$$\xi_C = C\sqrt{\pi}[1 + \sqrt{1 + \frac{\text{Ste}}{\pi C^2}}]. \tag{7}$$

Remark 3. If we use the approximation $\sqrt{1+x} \approx 1 + \frac{x}{2}$, $|x| \ll 1$, in (7) when Ste $\ll 1$, we obtain that $\xi_C \approx \xi_{ap} = 2C\sqrt{\pi} + \frac{\text{Ste}}{2C\sqrt{\pi}}$, where we can verify that ξ_{ap} coincides with the expression given by (5) because $\nu = 2C\sqrt{\pi} > 0$ and $Z(\nu) = C > 0$.

Remark 4. When $\nu > 0$ and $Z(\nu) = 0$ we replace the linear equation (4) by

$$2\, Z^{(n)}(\nu)\frac{(x - \nu)^n}{n!} = \frac{\text{Ste}}{\sqrt{\pi}}, \quad x > \nu, \tag{8}$$

where $n > 1$ is the first order such that the n-derivative of function Z at the point ν is different from 0.

4. TEMPERATURE AND FREE BOUNDARY ESTIMATES

In this paragraph we shall give some estimates of the temperature θ and the free boundary $s(t)$, given by (2-12), in particular for the coefficient ξ (solution of equation (2-10)).

First, we can obtain the following result for the particular case $\beta \leq 0$ over $\Re_0^+$.

Lemma 1. *If $\beta \leq 0$ over $\Re_0^+$ then we have*

$$\begin{aligned}&(\mathrm{i})0 < \xi \leq \xi_0 \ \ and\ \ 0 < s(t) \leq s_0(t), t > 0,\\ &(\mathrm{ii})\theta(x,t) \leq \theta_0(x,t), 0 \leq x \leq s(t), t > 0,\end{aligned} \tag{1}$$

where θ_0, s_0, ξ_0 represents the solution (I-2)-(I-3) corresponding to the case $\beta \equiv 0$.

Proof. Using Corollary (2-6) and the fact that $F(x,\beta) \geq F(x,0) = F_0(x)$, we obtain $\xi \leq \xi_0$, that is (1-i). Therefore, in the domain $0 < x < s(t), t > 0$ we can apply the maximum principle in order to obtain (1-ii).

Remark 1. We cannot obtain the inequality (1-ii) by using directly the definition of θ, given by the expression (2-12) (c.f. Corollary 4 for an improvement of (1-ii)).

Let $\theta_{\beta_i} = \theta_i, s_{\beta_i} = s_i, Z_{\beta_i} = Z_i, \nu_{Z_i} = \nu_i \geq 0$ and $\xi(\beta i) \geq 0$ be the temperature (defined by (2-12)), the free boundary (defined by (2-12)), Z-function (defined by (2-30)), the numbers ν (defined by (2-32)) and ξ (solution of equation (2-10)) corresponding to the data β_i for $i = 1, 2$. We have the following comparison result:

Theorem 2. (a) *We have the following equivalence:*

$$\beta_2 \leq \beta_1 \ \ over\ \Re^+ \iff Z_1 \leq Z_2 \ \ over\ \Re^+. \tag{2}$$

(b) *If $\beta_2 \leq \beta_1$ over $\Re^+$ then we obtain the following properties.*

$$(\mathrm{i})\ \nu_2 \leq \nu_1, \qquad (\mathrm{ii})\ \nu_1 = 0 \Rightarrow \nu_2 = 0. \tag{3}$$

On the other hand, if Z_1 verifies the hypotheses of Theorem (2-7) and $Z_2 > 0$ on $(\nu_2, \nu_1]$, then Z_2 also verifies the hypotheses of Theorem (2-7). Moreover, we obtain that there exists a unique $\xi(\beta_2)$ which satisfies the inequalities

$$\xi(\beta_2) \leq \xi(\beta_1), \ \ s_2(t) \leq s_1(t), \ \ t \geq 0. \tag{4}$$

(c) *If in addition to the hypotheses of part (b) we have $\beta_1 \geq 0$, over $\Re^+$, then we get*

$$\theta_2(x,t) \leq \theta_1(x,t), \ \ 0 \leq x \leq s_2(t), \ \ t > 0. \tag{5}$$

Proof. The equivalence (2) follows from the definitions (2-30) and (2-33). We use the fact $\int_0^x Z_2(t)dt \geq \int_0^x Z_1(t)dt > 0, \forall x > \nu_1$, to get that $\nu_2 \leq \nu_1$, i.e. (3). By using (a), we obtain

$\int_0^{+\infty} Z_2(t)dt = +\infty$, $\nu_2 \le \nu_1$, $Z_2 > 0$ over $(\nu_1, +\infty)$ and $F(x,\beta_1) \le F(x,\beta_2)$ over $\Re^+$, that is (4). We obtain (5) by means of $\beta_1 \ge 0$ and the maximum principle in the domain $0 < x < s_2(t)$, $t > 0$.

Remark 2. The same consequence (4) of part (b) of Theorem 2 holds true if we replace the condition "$Z_2 > 0$ over $(\nu_2, \nu_1]$" by "$\nu_1 = 0$". This assertion makes use of (3-ii).

In order to obtain some others estimates of the temperature we can modify the third term on the right hand size in (2-12) to have the following results.

Theorem 3. *Let β be a real continuous function over $\Re^+$ which satisfies the hypotheses of Theorem 2-7 relative to the unique solution $\xi > 0$ of equation (2-10). Then we obtain:*

(i) *The temperature θ can be expressed by*

$$\theta(x,t) = \theta(\eta) = B\{1 - \frac{\sqrt{\pi}}{Ste}\xi \exp(\xi^2) \;\; erf(\eta) + \\ + \frac{2\sqrt{\pi}}{Ste}[\int_0^{\eta} \beta(r)\exp(r^2) \;\; erf(r)dr \\ + erf(\eta)\int_{\eta}^{\xi} \beta(r)\exp(r^2)dr]\}, \; for\; \eta = \frac{x}{2a\sqrt{t}} \in (0,\xi). \tag{6}$$

(ii) *If $\eta = \eta^\star$ is a critical point of the temperature $\theta = \theta(\eta)$ (that is $\frac{d\theta}{d\eta}(\eta^\star) = 0$, i.e. $\eta^\star$ is a maximum, minimum or inflection point) then $\eta = \eta^\star$ must satisfy the following equation:*

$$\int_{\eta}^{\xi} \beta(r)\exp(r^2)dr = \frac{1}{2}\xi\exp(\xi^2), \;\; 0 < \eta < \xi. \tag{7}$$

Moreover, in this case, we have

$$\theta(\eta^\star) = B[1 + \frac{2\sqrt{\pi}}{Ste}\int_0^{\eta^\star} \beta(r)\exp(r^2) \;\; erf(r)dr] \\ = B\{1 + \frac{\sqrt{\pi}}{Ste}[F_0(\eta^\star) - F(\eta^\star,\beta)]\}, \tag{8}$$

$$\frac{d^2\theta}{d\eta^2}(\eta^\star) = \frac{-4B}{Ste}\beta(\eta^\star). \tag{9}$$

Proof. (i) The third term on the right hand side for the temperature θ in (2-14) is a double integral over a domain which can be expressed by the union of two subdomains as follows $(0 < \eta < \xi)$: $\{(r,y)/r < y < \xi, 0 < r < \xi\} = \{(r,y)/0 < r < \eta, \eta < y < \xi\} \bigcup \{(r,y)/0 < r < \eta, 0 < y < \eta\}$. Therefore, we can deduce (6) after exchanging the order of integration in the triangle.

(ii) It follows from the expressions $(0 < \eta < \xi)$:

$$\frac{d\theta}{d\eta}(\eta) = \frac{2B}{\text{Ste}} \exp(-\eta^2)[2 \int_{\eta}^{\xi} \beta(r) \exp(r^2)\, dr - \xi \exp(\xi^2)], \tag{10}$$

$$\frac{d^2\theta}{d\eta^2}(\eta) = -2\eta \frac{d\theta}{d\eta}(\eta) - \frac{4B}{\text{Ste}}\beta(\eta). \tag{11}$$

Remark 3. If $\eta = \eta^*$ is a critical point of $\theta = \theta(\eta)$ then we have a relative

$$\text{maximum if } \beta(\eta^*) > 0, \quad \text{minimum if } \beta(\eta^*) < 0. \tag{12}$$

As a direct consequence of the above Theorem we have:

Corollary 4. (i) *If $\beta \leq 0$ over $\Re^+$ then the temperature θ cannot have a critical point. Moreover, we can improve (1-ii) by*

$$0 \leq \theta(x,t) \leq \theta_0(x,t), \;\; 0 \leq x \leq s(t), \;\; t > 0. \tag{13}$$

(ii) *Suppose $\beta > 0$ over $\Re^+$ and satisfies the hypotheses in order that equation (2-10) has a unique solution. If the equation (7) has at least a solution then the critical point $\eta = \eta^*$ is unique and we have $0 \leq \theta(\eta) \leq \theta(\eta^*)$, $0 < \eta < \xi$, with $\theta(\eta^*) > B$, where $\theta(\eta^*)$ is given by (8). Moreover, if only $\beta \geq 0$ over $\Re^+$ then we may have an whole interval of critical points, but in any case we have $\theta(\eta) \geq 0$, $0 \leq \eta \leq \xi$.*

Lemma 5. *If $\beta \geq 0$ over $\Re^+$ and Z satisfies the hypotheses of Theorem 2-7, then we have*

$$\begin{gathered} \xi \geq \xi_0, \quad s(t) \geq s_0(t), \;\; t \geq 0, \\ \theta(x,t) \geq \theta_0(x,t), \;\; 0 \leq x \leq s_0(t), \; t \geq 0. \end{gathered} \tag{14}$$

Proof. It follows from Theorem 2 and the maximum principle.

Corollary 6.(i) *If $\beta_2 \leq \beta_1$ over $\Re^+$ and Z_1 and Z_2 verify the hypotheses of Theorem 2-7 (we use the same notation of Theorem 2), then we have*

$$0 \leq \xi(\beta_2) \leq \xi(\beta_1), \quad 0 \leq s_2(t) \leq s_1(t), \; t \geq 0. \tag{15}$$

(ii) *If $\beta(x) \leq \psi_0(x) - \varepsilon$ for some real constant $\varepsilon > 0$ then $0 < \xi(\beta) \leq H_0^{-1}(\frac{\text{Ste}}{2\varepsilon\sqrt{\pi}})$, where H_0^{-1} is the inverse function of H_0 which is defined by $H_0(x) = \int_0^x \exp(r^2)\, erf(r) dr, x > 0$, and satisfies $H_0(0^+) = 0$, $H_0(+\infty) = +\infty$, $H_0' > 0$ over $\Re^+$.*

(iii) If the function β satisfies the hypotheses of Theorem 2-7 and $\beta \leq \varphi_C$ over $\Re^+$, for some real constant $C > 0$, then we have

$$\begin{gathered} 2C\sqrt{\pi} < \xi(\beta) \leq \xi_C = C\sqrt{\pi}[1 + \sqrt{1 + \frac{\text{Ste}}{\pi C^2}}], \\ \theta(\eta) \leq \frac{B\sqrt{\pi}}{\text{Ste}} F_0(\xi_C), \; 0 \leq \eta \leq \xi. \end{gathered} \tag{16}$$

Remark 4. It is necessary to suppose that β satisfies the hypotheses of Theorem 2-7 (i.e. $Z_\beta > 0$ over $(\nu(Z_\beta), 2C\sqrt{\pi})$) in Corollary 6 part (iii) because $\nu(Z(\varphi_C)) = \nu(Z_C) = 2C\sqrt{\pi} > 0$, that is we can not apply Remark 2. Moreover, if β does not satisfy the hypotheses of Theorem 2-7, then in general we can not affirm the uniqueness of $\xi(\beta)$, but all these numbers satisfy the inequality (16) from part (ii) of Theorem 2-4 and the fact that $\frac{\text{Ste}}{\sqrt{\pi}} = F(\xi(\beta), \beta) \geq F(\xi(\beta), \varphi_C) = \xi(\beta)(\frac{\xi(\beta)}{\sqrt{\pi}} - 2C)$, that is $\xi(\beta) \leq \xi_C$.

Theorem 7. *Let Z and $Z_\star$ be two continuous real functions over $\Re_0^+$ which satisfy the hypotheses of Theorem 2-7. Let β and $\beta_\star$ be their corresponding β-functions. Let $H_\star$ be the real function defined by*

$$H_\star(x) = \frac{Ste}{2\sqrt{\pi}} + \int_0^x Z_\star(r)dr, \quad x \geq 0. \tag{17}$$

Then we have the following results:

(i) *The number $\xi(\beta) > 0$ satisfies the inequality:*

$$H_\star(\xi(\beta)) \leq F_0(\xi(\beta)). \tag{18}$$

(ii) *If, in addition, function $Z_\star$ is such that there exists at least a solution to the following equation*

$$H_\star(x) = F_0(x), \quad x > 0 \tag{19}$$

then we have $\xi(\beta) \geq \eta_0 > 0$, where $x = \eta_0 > 0$ is the first positive root of equation (19).

(iii) *If the function $Z_\star$ satisfies the inequality*

$$H_\star(x) \leq F_0(x), \quad \textit{for some } x > 0, \tag{20}$$

then there exists at least one number $\eta_0 > 0$ which solves the equation (19).

(iv) *If the function $Z_\star$ satisfies the inequality*

$$Z_\star(x) \leq \frac{x}{\sqrt{\pi}} + (\frac{1}{2} + x^2)\exp(x^2)\ \mathit{erf}(x), \quad x \geq 0, \tag{21}$$

then the condition (20) holds true. Moreover, in this case we have $\beta_\star \geq 0$ over $\Re^+$ and $Z_\star(0) \leq 0$.

Proof. (i) From (2-11) and (2-23), we get

$$\frac{\text{Ste}}{\sqrt{\pi}} = F(\xi(\beta), \beta) \leq F(\xi(\beta), -\beta_\star) = 2F_0(\xi(\beta)) - 2\int_0^{\xi(\beta)} Z_\star(r)dr, \tag{22}$$

that is (18).

(ii) It follows from (18), (19) and the definition of $\eta_0 > 0$.

(iii) This assertion follows from (19) and the fact that $F_0(0^+) = 0, \;\; H_\star(0^+) = \frac{\text{Ste}}{2\sqrt{\pi}} > 0$.

(iv) If the function $Z_\star$ satisfies the inequality (21) then, by taking into account (2-23), we obtain

$$H_\star(x) \le \frac{\text{Ste}}{2\sqrt{\pi}} + \frac{1}{2}F_0(x), \quad x \ge 0, \tag{23}$$

that is, we get (20) for all $x \ge \xi_0 > 0$. Moreover, using (2-33) we obtain $\beta_\star \ge 0$ over $\Re^+$.

Corollary 8. *If the function $Z_\star$ in Theorem 7 is given by $Z_\star(x) = \frac{x}{\sqrt{\pi}} - C,\ \ x \ge 0\ \ (C > 0)$, then we obtain*

$$\beta_\star(x) = \varphi_C(x), \quad x > 0 \quad \textit{(c.f. Remark 2-9)},$$
$$H_\star(x) = \frac{x^2}{2\sqrt{\pi}} - Cx + \frac{Ste}{2\sqrt{\pi}}, \quad x \ge 0,$$
$$H_\star(0) = \frac{Ste}{2\sqrt{\pi}} > 0, \quad H'_\star(0) = -C < 0, \quad H_\star(+\infty) = +\infty,$$
$$H_\star(x) \ge H_\star(C\sqrt{\pi}) = \frac{\sqrt{\pi}}{2}(\frac{Ste}{\pi} - C^2), \quad x \ge 0. \tag{24}$$

Moreover, there exists at least one solution of the equation (19) whose first positive root η_0 satisfies the inequalities

$$\eta_1 < \eta_0 < \xi_0 \tag{25}$$

where ξ_0 is the unique solution of equation (1-3) and η_1 is given by

$$\eta_1 = C\sqrt{\pi} + \frac{Ste}{\xi_0} - \sqrt{(C\sqrt{\pi} + \frac{Ste}{\xi_0})^2 - Ste} > 0. \tag{26}$$

In the particular case where $C > \sqrt{\frac{\text{Ste}}{\pi}}$ we also have

$$\eta_0 < \eta_2 = \pi(C - \sqrt{C^2 - \frac{Ste}{\pi}}). \tag{27}$$

Proof. From elementary computations we get (24). Taking into account that function $H_\star$ is a parabola with properties (24), and F_0 is an exponential type function and satisfies $F_0(0) = 0$, we deduce that there exists at least one solution of the equation (19). The number $\xi_0 > 0$ (unique solution of equation (1-3)) is also the unique solution of the following equation

$$F_0(x) = \frac{\text{Ste}}{\xi_0\sqrt{\pi}}x, \quad x > 0. \tag{28}$$

The equation

$$H_\star(x) = \frac{\text{Ste}}{\xi_0\sqrt{\pi}}x, \quad x > 0, \tag{29}$$

has two positives solutions. The first positive root of (29) is $\eta_1 > 0$, given by (26), which is well defined for all $C > 0$ because $\xi_0^2 = \frac{\text{Ste}}{\sqrt{\pi}}\frac{1}{V(\xi_0)} \le \frac{\text{Ste}}{2}$, after using (1-3) and (2-25). Then we can deduce the inequalities (25). In the particular case where $C > \sqrt{\frac{\text{Ste}}{\pi}}$ the function H_* has two positive roots, whose first positive root is given by η_2, that is (27).

Remark 5. Let A be the positive real constant defined by $A = A(\text{Ste}, \xi_0) = \frac{1}{2} + \xi_0^2(1 + \frac{1}{\text{Ste}}) > 0$, where ξ_0 is the unique solution of equation (I-3). Then we have the following property $\frac{\partial F}{\partial x}(\xi_0, \beta) > 0$, for all continuous real functions β over $\Re^+$ such that $\beta(x) < A(\text{Ste}, \xi_0), x > 0$. This follows from (1-3) and (2-14). Moreover, we get $A(\text{Ste}, \xi_0) \le 1 + \frac{\text{Ste}}{2}$.

Now, let us mention some results concerning to the non-uniqueness of the solution to the free boundary problem (P).

Lemma 9. *Assume that $\beta \ge 0$ over $\Re^+$ and that there are at least two roots $\xi_1 < \xi_2$ of the equation (2-10) for the same function β. Then we have*

$$\begin{aligned} & s_1(t) \le s_2(t), \ \ t \ge 0, \\ & 0 \le \theta_1(x,t) \le \theta_2(x,t), \ \ 0 \le x \le s_1(t), \ t \ge 0, \end{aligned} \tag{30}$$

where $\theta_i = \theta_i(x,t)$ denotes the temperature corresponding to ξ_i $(i = 1, 2)$ and β. Moreover, we have

$$\theta_2(x,t) \ge 0, \ \ 0 \le x \le s_2(t), \ t \ge 0, \tag{31}$$

Proof. We use twice the maximum principle for the functions θ_2 and $\theta_2 - \theta_1$.

To conclude, for a function $\beta \approx 0$ we want to obtain a development of $\xi(\beta)$ in a neighborhood of ξ_0 given by the equation (1-3).

Lemma 10. *Let β_1 be a continuously differentiable real function over $\Re^+$. Let β_λ be a function defined by $\beta_\lambda(x) = \lambda\beta_1(x), x > 0$ and we suppose there exists two positives constants λ_0 and ε_0 such that*

$$\beta_\lambda(x) = \lambda\beta_1(x) < \psi_0(x) - \varepsilon_0, \ \ \textit{for all } x > 0 \ \textit{ and } 0 < \lambda < \lambda_0. \tag{32}$$

Then the unique number $\xi(\beta_\lambda) = \xi_\lambda$, solution of equation (2-10) for function β_λ, can be expressed as

$$\xi_\lambda = \xi(\beta_\lambda) = \xi_0 + \xi_1\lambda + \xi_2\lambda^2 + o(\lambda^2), \tag{33}$$

with

$$\xi_1 = \frac{\xi_0\sqrt{\pi}}{SteA}\int_0^{\xi_0} \exp(r^2)\ erf(r)\beta_1(r)dr, \tag{34}$$

$$\xi_2 = \frac{\xi_1}{A}[\beta_1(\xi_0) - \xi_0\xi_1(1 + \frac{1}{Ste} + A)], \tag{35}$$

where ξ_0 is the unique solution of equation (I-3) and $A = A(Ste, \xi_0)$ is given in Remark 5.

Proof. By the assumption (32) we have a unique number $\xi(\beta_\lambda) = \xi_\lambda > 0$ which is the solution of equation (2-10). Let $I = I(x, \lambda)$ be the real function defined by

$$I(x,\lambda) = F(x,\beta_\lambda) - \frac{\text{Ste}}{\sqrt{\pi}}, \;\; x > 0, \;\; 0 < \lambda < \lambda_0. \tag{36}$$

Taking into account that $I(\xi_0, 0) = 0$, $\frac{\partial I}{\partial x}(\xi_0, 0) = \frac{2\text{Ste}}{\xi_0\sqrt{\pi}}A > 0$, we can apply the Dini implicit function Theorem to deduce the existence of a real function $\xi_\lambda = \xi(\lambda)$, defined in a right neighborhood of $\lambda = 0$ which satisfies $\xi(0) = \xi_0$ and (33).

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