# A distributed parabolic control with mixed boundary conditions 

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#### Abstract

We study the asymptotic behavior of an optimal distributed control problem where the state is given by the heat equation with mixed boundary conditions. The parameter $\alpha$ intervenes in the Robin boundary condition and it represents the heat transfer coefficient on a portion $\Gamma_{1}$ of the boundary of a given regular $n$-dimensional domain. For each $\alpha$, the distributed parabolic control problem optimizes the internal energy $g$. It is proven that the optimal control $\hat{g}_{\alpha}$ with optimal state $u_{\hat{g}_{\alpha} \alpha}$ and optimal adjoint state $p_{\hat{g}_{\alpha} \alpha}$ are convergent as $\alpha \rightarrow \infty$ (in norm of a suitable Sobolev parabolic space) to $\hat{g}$, $u_{\hat{g}}$ and $p_{\hat{g}}$, respectively, where the limit problem has Dirichlet (instead of Robin) boundary conditions on $\Gamma_{1}$. The main techniques used are derived from the parabolic variational inequality theory.


Keywords: parabolic variational inequalities, distributed evolution optimal control, mixed boundary conditions, adjoint state, optimality condition, asymptotic

## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^{n}$ with a regular boundary $\partial \Omega=\Gamma_{1} \cup \Gamma_{2}$, which is the union of two essentially disjoint (and regular) portions $\Gamma_{1}$ and $\Gamma_{2}$, where $\Gamma_{1}$ has a positive ( $n-1$ )-Hausdorff measure. Also suppose given a time interval $[0, T]$, for some $T>0$. Consider the following two-state evolution heat conduction problems with mixed boundary conditions,

$$
\begin{equation*}
\partial_{t} u-\Delta u=g \quad \text { in } \Omega,\left.\quad u\right|_{\Gamma_{1}}=b, \quad-\left.\partial_{n} u\right|_{\Gamma_{2}}=q, \tag{1.1}
\end{equation*}
$$

and, for a parameter $\alpha>0$,

$$
\begin{equation*}
\partial_{t} u-\Delta u=g \quad \text { in } \Omega, \quad-\left.\partial_{n} u\right|_{\Gamma_{1}}=\alpha(u-b), \quad-\left.\partial_{n} u\right|_{\Gamma_{2}}=q, \tag{1.2}
\end{equation*}
$$

both with an initial condition

$$
\begin{equation*}
u(0)=v_{b}, \tag{1.3}
\end{equation*}
$$

where $g$ is the internal energy in $\Omega, b$ is the temperature (of the external neighborhood) on $\Gamma_{1}$ for (1.1) (for (1.2)), $q$ is the heat flux on $\Gamma_{2}$ and $\alpha$ is the heat transfer coefficient of $\Gamma_{1}$ (Newton's law on $\Gamma_{1}$ ).

All data, $g, q, b, v_{b}$ and the domain $\Omega$ with the boundary $\partial \Omega=\Gamma_{1} \cup \Gamma_{2}$ are assumed to be sufficiently smooth so that the problems (1.1) and (1.2) admit variational solutions in Sobolev spaces.

The data $b, v_{b}$ and $q$ are fixed, sufficiently smooth and satisfy the compatibility condition $v_{b}=b$ on $\Gamma_{1}$, while $g$ is taken as a control variable in $L^{2}\left(0, T ; L^{2}(\Omega)\right)$, and $\alpha$ as a (singular) parameter destined to approaches infinite. Thus, denote by $u_{g}$ and $u_{g \alpha}$ the solution of (1.1) and (1.2), respectively, with the initial condition (1.3) in the following standard variational form

$$
\begin{cases}u_{g}-v_{b} \in L^{2}\left(0, T ; V_{0}\right), \quad u_{g}(0)=v_{b} \quad \text { and } \quad \dot{u}_{g} \in L^{2}\left(0, T ; V_{0}^{\prime}\right)  \tag{1.4}\\ \text { such that }\left\langle\dot{u}_{g}(t), v\right\rangle+a\left(u_{g}(t), v\right)=L_{g}(t, v), \quad \forall v \in V_{0},\end{cases}
$$

and

$$
\left\{\begin{array}{l}
u_{g \alpha} \in L^{2}(0, T ; V), \quad u_{g \alpha}(0)=v_{b} \quad \text { and } \quad \dot{u}_{g \alpha} \in L^{2}\left(0, T ; V^{\prime}\right)  \tag{1.5}\\
\text { such that }\left\langle\dot{u}_{g \alpha}(t), v\right\rangle+a_{\alpha}\left(u_{g \alpha}(t), v\right)=L_{g \alpha}(t, v), \quad \forall v \in V
\end{array}\right.
$$

where

$$
\begin{align*}
& V_{0}:=\left\{v \in H^{1}(\Omega):\left.v\right|_{\Gamma_{1}}=0\right\}, \\
& H:=L^{2}(\Omega), \quad(g, h)_{H}:=\int_{\Omega} g h \mathrm{~d} x, \\
& L_{g}(t, v):=(g(t), v)_{H}-\int_{\Gamma_{2}} q(t) v \mathrm{~d} \gamma, \\
& a(u, v):=\int_{\Omega} \nabla u \cdot \nabla v \mathrm{~d} x,  \tag{1.6}\\
& a_{\alpha}(u, v):=a(u, v)+\alpha \int_{\Gamma_{1}} u v \mathrm{~d} \gamma, \\
& L_{g \alpha}(t, v):=L_{g}(t, v)+\alpha \int_{\Gamma_{1}} b v \mathrm{~d} \gamma,
\end{align*}
$$

and $\langle\cdot, \cdot\rangle$ denotes the duality bracket. Note that the dual space $V_{0}^{\prime}$ (and $V^{\prime}$ ) of $V_{0}$ (and $V$ ) is not an space of distributions, since $\mathcal{D}(\Omega)$ is not dense in $V_{0} \subset V$, due to the non-zero boundary conditions on $\Gamma_{2}$. The norm in $V_{0}$ is given by $v \mapsto\|\nabla v\|_{H}$, while the norm in $V$ is $\left(\|v\|_{H}^{2}+\|\nabla v\|_{H}^{2}\right)^{1 / 2}$. Nevertheless, $v \mapsto L_{g}(t, v)$ and $v \mapsto L_{g \alpha}(t, v)$ are linear continuous functional satisfying

$$
\begin{aligned}
& \left\|L_{g}(t, \cdot)\right\|_{V_{0}^{\prime}} \leqslant\|g(t)\|_{V_{0}^{\prime}}+\|q(t)\|_{H^{-1 / 2}\left(\Gamma_{2}\right)}, \quad \forall v \in V_{0} \\
& \left\|L_{g \alpha}(t, \cdot)\right\|_{V} \leqslant\|g(t)\|_{V^{\prime}}+\|q(t)\|_{H^{-1 / 2}\left(\Gamma_{2}\right)}+\alpha\|b\|_{H^{1 / 2}\left(\Gamma_{1}\right)}, \quad \forall v \in V
\end{aligned}
$$

and $a(\cdot, \cdot)$ and $a_{\alpha}(\cdot, \cdot)$ are bilinear symmetric continuous forms on $V_{0}$ and $V$, respectively. Also, it is clear the compatibility assumption $v_{b}=b$ on $\Gamma_{1}$ and that if $b=0$ then $L_{g}(t, \cdot)=L_{g, \alpha}(t, \cdot)$.

One should remark that an element $u$ of $L^{2}(0, T ; V)$ such that $\dot{u}$ belongs to $L^{2}\left(0, T ; V^{\prime}\right)$ then $u$ can be regarded as a continuous function from $[0, T]$ into $H$. This makes clear the meaning of the initial condition at $t=0$ (and idem with $V_{0}$ replacing $V$ ).

On the space $\mathcal{H}:=L^{2}(\Omega \times] 0, T[)$ with norm $\|\cdot\|_{\mathcal{H}}$ and inner product $(\cdot, \cdot)_{\mathcal{H}}$, i.e.,

$$
(u, v)_{\mathcal{H}}=\int_{0}^{T}(u(t), v(t))_{H} \mathrm{~d} t, \quad \forall u, v \in \mathcal{H}
$$

consider the nonnegative functional costs $J$ and $J_{\alpha}$, defined by the expressions

$$
\begin{equation*}
J(g):=\frac{1}{2}\left\|u_{g}-z_{d}\right\|_{\mathcal{H}}^{2}+\frac{m}{2}\|g\|_{\mathcal{H}}^{2} \tag{1.7}
\end{equation*}
$$

and

$$
\begin{equation*}
J_{\alpha}(g):=\frac{1}{2}\left\|u_{g \alpha}-z_{d}\right\|_{\mathcal{H}}^{2}+\frac{m}{2}\|g\|_{\mathcal{H}}^{2}, \tag{1.8}
\end{equation*}
$$

where $z_{d}$ is a given element in $\mathcal{H}=L^{2}(\Omega \times] 0, T[)$ and $m$ is a strictly positive constant.
Our interest is on the distributed parabolic (or evolution) optimal control problems
Find $\quad \hat{g} \quad$ such that $\quad J(\hat{g}) \leqslant J(g), \quad \forall g \in \mathcal{H}$
and

Find $\quad \hat{g}_{\alpha} \quad$ such that $\quad J_{\alpha}\left(\hat{g}_{\alpha}\right) \leqslant J_{\alpha}(g), \quad \forall g \in \mathcal{H}$,
as well as the asymptotic behavior as the parameter $\alpha$ approaches infinite.
This type of optimal distributed control problems have been extensively studied, e.g., see the book Lions [10] among others. As point out early, our interest is the convergence as $\alpha \rightarrow \infty$, a parabolic version of Gariboldi and Tarzia [8], which is related to Ben Belgacem et al. [4] and Tabacman and Tarzia [11].

## 2. Parabolic equations with mixed conditions

Note that if via Riesz' representation $H=H^{\prime}$ then one has $V \subset H \subset V^{\prime}$ and $V_{0} \subset H \subset V_{0}^{\prime}$ with a continuous and dense inclusion.

As mentioned early the control parameter $g$ belongs to $\mathcal{H}$, and the data for the optimal control problems are $z_{d}$ and $m$ satisfying

$$
\begin{equation*}
z_{d} \in \mathcal{H}=L^{2}\left(0, T ; L^{2}(\Omega)\right) \quad \text { and } \quad m>0 \tag{2.1}
\end{equation*}
$$

The regularity of the domain $\Omega$, the boundary $\Gamma_{1} \cup \Gamma_{2}$ and the regularity of the boundary data $v_{b}, b$ and $q$ are summarized on the assumption

$$
\begin{align*}
& \text { there exists } \quad \psi \in L^{2}\left(0, T ; H^{2}(\Omega)\right) \quad \text { with } \dot{\psi} \in L^{2}\left(0, T ; L^{2}(\Omega)\right)  \tag{2.2}\\
& \text { such that } \quad \psi(0)=v_{b},\left.\quad \psi\right|_{\Gamma_{1}}=b,\left.\quad \partial_{n} \psi\right|_{\Gamma_{1}}=0, \quad-\left.\partial_{n} \psi\right|_{\Gamma_{2}}=q,
\end{align*}
$$

with the standard notation of Sobolev and Lebesgue spaces and the compatibility assumption $v_{b}=b$ on $\Gamma_{1}$. Note the over conditioning for $\psi$ on $\Gamma_{1}$, which is not necessary but convenient in some way (e.g., the adjoint state has a very similar equation with homogeneous boundary conditions).

Thus, the change of unknown function $u$ into $u-\psi$ reduces to analysis the case where the boundary data $v_{b}, b$ and $q$ are all zero, and $g$ is replaced by $g-\left(\partial_{t}-\Delta\right) \psi$. However, for $\alpha>0$ a new term appears, namely,

$$
\begin{equation*}
\left\langle g_{\psi}(t), v\right\rangle=(g(t), v)_{H}+\int_{\Gamma_{1}} v \partial_{n} \psi(t) \mathrm{d} \gamma, \quad \forall v \in V \tag{2.3}
\end{equation*}
$$

i.e., the new Robin boundary condition is non-homogeneous and

$$
\left\|g_{\psi}(t)\right\|_{V^{\prime}}=\sup _{\|v\|_{V} \leqslant 1}\left|\left\langle g_{\psi}(t), v\right\rangle\right| \leqslant\|g(t)\|_{L^{2}(\Omega)}+\left\|\partial_{n} \psi(t)\right\|_{H^{-1 / 2}\left(\Gamma_{1}\right)}
$$

Thus, because of the over conditioning on $\Gamma_{1}$ one has $g_{\psi}=g$. Anyway, both problems, (1.4) and (1.5) become

$$
\left\{\begin{array}{l}
u_{g} \in L^{2}\left(0, T ; V_{0}\right), \quad \text { with } u_{g}(0)=0 \text { and } \dot{u}_{g} \in L^{2}\left(0, T ; V_{0}^{\prime}\right)  \tag{2.4}\\
\text { such that } \quad\left\langle\dot{u}_{g}(t), v\right\rangle+a\left(u_{g}(t), v\right)=(g(t), v)_{H}, \quad \forall v \in V_{0}
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
u_{g \alpha} \in L^{2}(0, T ; V), \quad \text { with } u_{g \alpha}(0)=0 \text { and } \dot{u}_{g \alpha} \in L^{2}\left(0, T ; V^{\prime}\right)  \tag{2.5}\\
\text { such that }\left\langle\dot{u}_{g \alpha}(t), v\right\rangle+a_{\alpha}\left(u_{g \alpha}(t), v\right)=(g(t), v)_{H}, \quad \forall v \in V
\end{array}\right.
$$

where $(\cdot, \cdot)_{H}, a(\cdot, \cdot)$ and $a_{\alpha}(\cdot, \cdot)$ are as in (1.6). Again $V_{0} \subset V$ with inclusion continuous but not dense, so that $V^{\prime}$ is not identifiable with a subset of $V_{0}^{\prime}$. However, by Hahn-Banach Theorem, any element in $V_{0}^{\prime}$ can be extended to an element in $V^{\prime}$ preserving its norm.

Recall that for any element $u$ in $L^{2}(0, T ; V)$ with $\dot{u}$ in $L^{2}\left(0, T ; V^{\prime}\right)$ such that the distribution $\left(\partial_{t}-\Delta\right) u$ belongs to $L^{2}(\Omega \times] 0, T[)$ one can integrate by parts to interpret $\partial_{n} u$ as an element in $L^{2}\left(0, T ; H^{-1 / 2}(\partial \Omega)\right)$, where $H^{-1 / 2}(\partial \Omega)$ is the dual space of $H^{1 / 2}(\partial \Omega)=\gamma\left(H^{1}(\Omega)\right)$ and $\gamma$ is the trace operator from $H^{1}(\Omega)$ onto $H^{1 / 2}(\partial \Omega)$. Again, to simplify the arguments, one may assume that $\partial \Omega=\Gamma_{1} \cup \Gamma_{2}$ such that for any $v_{i}$ in $H^{1 / 2}\left(\Gamma_{i}\right)$ there exists $v$ in $H^{1}(\Omega)$ satisfying $v=v_{i}$ on $\Gamma_{i}$, for $i=1,2$, e.g., the two pieces of the boundary are strictly disjoint, $\Gamma_{1} \cap \Gamma_{2}=\emptyset$ (i.e., $\Gamma_{i}=\partial \Omega_{i}$ and $\bar{\Omega}_{1} \subset \Omega_{2}$ ). Therefore, the parabolic equations (2.4) and (2.5) mean the following:

- space of the solution: $u_{g}$ in $L^{2}\left(0, T ; V_{0}\right)$ with $\dot{u}_{g}$ in $L^{2}\left(0, T ; V_{0}^{\prime}\right)$, and $u_{g \alpha}$ in $L^{2}(0, T ; V)$ with $\dot{u}_{g \alpha}$ in $L^{2}\left(0, T ; V^{\prime}\right)$,
- initial condition: for either $u=u_{g}$ or $u=u_{g \alpha}$ the solution $u$ belongs to $C^{0}\left(0, T ; L^{2}(\Omega)\right)$ and so $u(0)=0$ in $L^{2}(\Omega)$,
- equation in $\Omega \times] 0, T\left[\right.$ : for either $u=u_{g}$ or $u=u_{g \alpha}$ the solution $u$ is considered as a distribution so that $\left(\partial_{t}-\Delta\right) u=g$ in $\mathcal{D}^{\prime}(\Omega \times] 0, T[)$,
- boundary condition on $\Gamma_{2}$ : for either $u=u_{g}$ or $u=u_{g \alpha}$ the trace of the solution $u$ is defined and $\partial_{n} u=0$ in $L^{2}\left(0, T ; H^{-1 / 2}\left(\Gamma_{2}\right)\right)$,
- boundary condition on $\Gamma_{1}: u_{g}=0$ in $L^{2}\left(0, T ; H^{1 / 2}\left(\Gamma_{1}\right)\right)$ and $\partial_{n} u_{g \alpha}+\alpha u_{g \alpha}=0$ in $L^{2}\left(0, T ; H^{-1 / 2}\left(\Gamma_{1}\right)\right)$.
Firstly, note that $\left.u_{g \alpha}\right|_{\Gamma_{1}}$ belongs to $L^{2}\left(0, T ; H^{1 / 2}\left(\Gamma_{1}\right)\right)$ and

$$
L^{2}\left(0, T ; H^{1 / 2}\left(\Gamma_{1}\right)\right) \subset L^{2}\left(0, T ; L^{2}\left(\Gamma_{1}\right)\right) \subset L^{2}\left(0, T ; H^{-1 / 2}\left(\Gamma_{1}\right)\right)
$$

with continuous and dense inclusion. Secondly, when comparing the solutions $u_{g}$ and $u_{g \alpha}$ one has both in the larger space $L^{2}(0, T ; V)$. However, the continuous inclusion $V_{0} \subset V$ is not dense, and so the inclusion $V^{\prime} \subset V_{0}^{\prime}$ is not injective, one has $\dot{u}_{g}$ and $\dot{u}_{g \alpha}$ elements in $L^{2}\left(0, T ; V_{0}^{\prime}\right)$, which are not identifiable as distributions.

## 3. State and adjoint state equations

To study the optimal control problem (1.9), denote by $u_{0}$ the solution $u_{g}$ of the parabolic variational equality either (1.4) or equivalently (2.4) corresponding to $g=0$, and define the (linear) operator $C: \mathcal{H} \rightarrow L^{2}\left(0, T ; V_{0}\right)$, given by $C(g):=u_{g}-u_{0}$. We have

Proposition 3.1. With the previous notation and assumptions, the functional (1.7) can be expressed as

$$
J(g)=\frac{1}{2} \pi(g, g)-\ell(g)+\frac{1}{2}\left\|z_{d}-u_{0}\right\|_{\mathcal{H}}^{2}, \quad \forall g \in \mathcal{H}
$$

where $\pi(g, h):=(C(g), C(h))_{\mathcal{H}}+m(g, h)_{\mathcal{H}}$ is a symmetric, continuous and coercive bilinear form on $\mathcal{H}$ and $\ell(g):=\left(C(g), z_{d}-u_{0}\right)_{\mathcal{H}}$ is a linear continuous functional on $\mathcal{H}$. Moreover, $J$ is strictly convex and its Gateaux derivative is given by $\left\langle J^{\prime}(g), h\right\rangle=\left(u_{g}-z_{d}, C(g)\right)_{\mathcal{H}}+m(g, h)_{\mathcal{H}}$. Furthermore, as a consequence, the optimal control problem (1.9) has a unique minimizer $\hat{g}$ in $\mathcal{H}$, i.e., $J(\hat{g}) \leqslant J(g)$, for every $g$ in $\mathcal{H}$, any solution $\bar{g}$ of the equation $J^{\prime}(\bar{g})=0$ is indeed a minimizer. Also, if $p_{g}$ is the adjoint state defined by the parabolic variational equality with a terminal condition

$$
\left\{\begin{array}{l}
p_{g} \in L^{2}\left(0, T ; V_{0}\right), \quad \text { with } p_{g}(T)=0 \text { and } \dot{p}_{g} \in L^{2}\left(0, T ; V_{0}^{\prime}\right)  \tag{3.1}\\
\text { such that }-\left\langle\dot{p}_{g}(t), v\right\rangle+a\left(p_{g}(t), v\right)=\left(u_{g}-z_{d}, v\right)_{H}, \quad \forall v \in V_{0}
\end{array}\right.
$$

then $J^{\prime}(g)=m g+p_{g}$ for every $g$ in $\mathcal{H}$ and $J^{\prime}(\hat{g})=m \hat{g}+p_{\hat{g}}=0$.
Proof. Note the boundary conditions for the adjoint state $p_{g}$ are

$$
p_{g}(t)=0 \quad \text { on } \Gamma_{1} \quad \text { and } \quad \partial_{n} p_{g}(t)=0 \quad \text { on } \Gamma_{2}
$$

for almost every $t$ in $] 0, T[$.

First, we check the expression of $J$, if $z_{d}^{\prime}:=z_{d}-u_{0}$ then

$$
\begin{aligned}
J(g) & =\frac{1}{2}\left\|C(g)-z_{d}^{\prime}\right\|_{\mathcal{H}}^{2}+\frac{m}{2}\|g\|_{\mathcal{H}}^{2} \\
& =\frac{1}{2}\left[\|C(g)\|_{\mathcal{H}}^{2}+\left\|z_{d}^{\prime}\right\|_{\mathcal{H}}^{2}-2\left(C(g), z_{d}^{\prime}\right)_{\mathcal{H}}\right]+\frac{m}{2}\|g\|_{\mathcal{H}}^{2} \\
& =\frac{1}{2} \pi(g, g)-L(g)+\frac{1}{2}\left\|z_{d}-u_{0}\right\|_{\mathcal{H}}^{2} .
\end{aligned}
$$

To verify that $g \mapsto C(g)$ is a linear application, one checks that the function $r_{1} u_{g_{1}}+r_{2} u_{g_{2}}+(1-$ $\left.r_{1}-r_{2}\right) u_{0}$ is a solution of the parabolic variational equality (1.4) with $g=r_{1} g_{1}+r_{2} g_{2}$, for every real numbers $r_{1}, r_{2}$; and by uniqueness one has

$$
\begin{equation*}
u_{r_{1} g_{1}+r_{2} g_{2}}=r_{1} u_{g_{1}}+r_{2} u_{g_{2}}+\left(1-r_{1}-r_{2}\right) u_{0} \tag{3.2}
\end{equation*}
$$

for every $r_{i}, r_{2}$ in $\mathbb{R}$ and $g_{1}, g_{2}$ in $\mathcal{H}$. Hence,

$$
\begin{aligned}
C\left(r_{1} g_{1}+r_{2} g_{2}\right) & =u_{r_{1} g_{1}+r_{2} g_{2}}-u_{0}=r_{1} u_{g_{1}}+r_{2} u_{g_{2}}+\left(1-r_{1}-r_{2}\right) u_{0}-u_{0} \\
& =r_{1}\left(u_{g_{1}}-u_{0}\right)+r_{2}\left(u_{g_{2}}-u_{0}\right)=r_{1} C\left(g_{1}\right)+r_{2} C\left(g_{2}\right),
\end{aligned}
$$

i.e., the operator $C$ is linear.

Now to check the continuity of $C$, we note that since $\Gamma_{1}$ has positive measure, Poincaré inequality implies that the bilinear form $a(\cdot, \cdot)$ is coercive on $V_{0}$, i.e., there exists $\lambda_{0}>0$ such that

$$
\begin{equation*}
a(v, v) \geqslant \lambda_{0}\|\nabla v\|_{H}^{2}, \quad \forall v \in V_{0} \tag{3.3}
\end{equation*}
$$

We have

$$
\left(\dot{u}_{g}(t)-\dot{u}_{0}(t), v\right)_{H}+a\left(u_{g}(t)-u_{0}(t), v\right)=(g(t), v)_{H}, \quad \forall v \in V_{0}
$$

and, in particular, for $v=u_{g}(t)-u_{0}(t)$,

$$
\begin{aligned}
& \frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\left(\left\|u_{g}(t)-u_{0}(t)\right\|_{H}^{2}\right)+\lambda_{0}\left\|\nabla\left(u_{g}(t)-u_{0}(t)\right)\right\|_{H}^{2} \\
& \quad \leqslant\left(g(t), u_{g}(t)-u_{0}(t)\right)_{H} \leqslant \frac{1}{2 \lambda_{0}}\|g(t)\|_{V_{0}^{\prime}}^{2}+\frac{\lambda_{0}}{2}\left\|\nabla\left(u_{g}(t)-u_{0}(t)\right)\right\|_{H}^{2}
\end{aligned}
$$

where the dual norm is given by

$$
\|v\|_{V_{0}^{\prime}}^{2}=\sup \left\{(v, \varphi)_{H}: \varphi \in V_{0},\|\varphi\|_{V_{0}} \leqslant 1\right\} .
$$

This yields

$$
\begin{aligned}
& \|\nabla C(g)\|_{\mathcal{H}} \leqslant \frac{1}{\lambda_{0}}\left[\int_{0}^{T}\|g(t)\|_{V_{0}^{\prime}}^{2} \mathrm{~d} t\right]^{1 / 2}, \\
& \sup _{0 \leqslant t \leqslant T}\|C(g)(t)\|_{H} \leqslant \frac{1}{\sqrt{\lambda_{0}}}\left[\int_{0}^{T}\|g(t)\|_{V_{0}^{\prime}}^{2} \mathrm{~d} t\right]^{1 / 2},
\end{aligned}
$$

and going back to the equation, we get

$$
\left[\int_{0}^{T}\left\|\frac{\mathrm{~d}}{\mathrm{~d} t}(C(g)(t))\right\|_{V_{0}^{\prime}}^{2} \mathrm{~d} t\right]^{1 / 2} \leqslant \frac{2}{\lambda_{0}}\left[\int_{0}^{T}\|g(t)\|_{V_{0}^{\prime}}^{2} \mathrm{~d} t\right]^{1 / 2}
$$

Hence the operator

$$
C: L^{2}\left(0, T ; V_{0}^{\prime}\right) \rightarrow\left\{v \in L^{2}\left(0, T ; V_{0}\right) \cap L^{\infty}(0, T ; H): \dot{v} \in L^{2}\left(0, T ; V_{0}^{\prime}\right)\right\}
$$

is actually continuous. As a consequence, the bilinear form $\pi(\cdot, \cdot)$ is symmetric, continuous and coercive on $\mathcal{H} \times \mathcal{H}$, since $\mathcal{H} \subset L^{2}\left(0, T ; V_{0}^{\prime}\right)$.
To complete the argument, we choose $v=C(h)$ in (3.1) and $v=p_{g}$ in (1.4) with $g=0$ and $g=h$ to obtain, after integrating in $t$, the equalities

$$
-\left(\dot{p}_{g}, C(h)\right)_{\mathcal{H}}+\int_{0}^{T} a\left(p_{g}(t), C(h)(t)\right) \mathrm{d} t=\left(u_{g}-z_{d}, C(h)\right)_{\mathcal{H}}
$$

and

$$
\left(\dot{u}_{h}-\dot{u}_{0}, p_{g}\right)_{\mathcal{H}}+\int_{0}^{T} a\left(u_{h}(t)-u_{0}(t), p_{g}(t)\right) \mathrm{d} t=\left(h, p_{g}\right)_{\mathcal{H}} .
$$

Thus

$$
-\int_{0}^{T} \frac{\mathrm{~d}}{\mathrm{~d} t}\left(p_{g}(t), C(h)(t)\right)_{H} \mathrm{~d} t+\left(h, p_{g}\right)_{\mathcal{H}}=\left(u_{g}-z_{d}, C(h)\right)_{\mathcal{H}},
$$

and because $p_{g}(T)=0$ and $C(h)(0)=0$, we deduce $J^{\prime}(g)=m g+p_{g}$.
To show that $g \mapsto J(g)$ is strictly convex, one makes use of (1.7) and (3.2) to check that

$$
(1-\theta) J\left(g_{2}\right)+\theta J\left(g_{1}\right)-J\left((1-\theta) g_{1}+\theta g_{2}\right)=\frac{1}{2} \theta(1-\theta)\left[\left\|u_{g_{1}}-u_{g_{2}}\right\|_{\mathcal{H}}^{2}+m\left\|g_{1}-g_{2}\right\|_{\mathcal{H}}^{2}\right]
$$

for every $\theta$ in $[0,1]$ and any $g_{1}, g_{2}$ in $\mathcal{H}$.
Similarly, to study the optimal control problem (1.10), denote by $u_{0 \alpha}$ the solution $u_{g \alpha}$ of the parabolic variational equality either (1.5) or equivalently (2.5) corresponding to $g=0$, and define the (linear) operator $C_{\alpha}: \mathcal{H} \rightarrow L^{2}(0, T ; V)$, given by $C_{\alpha}(g):=u_{g \alpha}-u_{0 \alpha}$. We have

Proposition 3.2. With the previous notation and assumptions, the functional (1.8) can be expressed as

$$
J_{\alpha}(g)=\frac{1}{2} \pi_{\alpha}(g, g)-\ell_{\alpha}(g)+\frac{1}{2}\left\|z_{d}-u_{0 \alpha}\right\|_{\mathcal{H}}^{2}, \quad g \in \mathcal{H}
$$

where $\pi_{\alpha}(g, h):=\left(C_{\alpha}(g), C_{\alpha}(h)\right)_{\mathcal{H}}+m(g, h)_{\mathcal{H}}$ is a symmetric, continuous and coercive bilinear form on $\mathcal{H}$ and $\ell_{\alpha}(g):=\left(C_{\alpha}(g), z_{d}-u_{0 \alpha}\right)_{\mathcal{H}}$ is a linear continuous functional on $\mathcal{H}$. Moreover, $J_{\alpha}$ is strictly
convex and its Gateaux derivative of $J_{\alpha}$ is given by $\left\langle J_{\alpha}^{\prime}(g), h\right\rangle=\left(u_{g}-z_{d}, C_{\alpha}(g)\right)_{\mathcal{H}}+m(g, h)_{\mathcal{H}}$. Furthermore, as a consequence, the optimal control problem (1.10) has a unique minimizer $\hat{g}_{\alpha}$ in $\mathcal{H}$, i.e., $J_{\alpha}\left(\hat{g}_{\alpha}\right) \leqslant J_{\alpha}(g)$, for every $g$ in $\mathcal{H}$, and any solution $\bar{g}_{\alpha}$ of the equation $J^{\prime}\left(\bar{g}_{\alpha}\right)=0$ is indeed a minimizer. Also if $p_{g \alpha}$ is the adjoint state defined by the parabolic variational equality with a terminal condition

$$
\left\{\begin{array}{l}
p_{g \alpha} \in L^{2}(0, T ; V), \quad \text { with } p_{g \alpha}(T)=0 \text { and } \dot{p}_{g \alpha} \in L^{2}\left(0, T ; V^{\prime}\right)  \tag{3.4}\\
\text { such that } \quad-\left\langle\dot{p}_{g \alpha}(t), v\right\rangle+a_{\alpha}\left(p_{g \alpha}(t), v\right)=\left(u_{g \alpha}-z_{d}, v\right)_{H}, \quad \forall v \in V
\end{array}\right.
$$

then $J_{\alpha}^{\prime}(g)=m g_{\alpha}+p_{g_{\alpha}}$ for every $g$ in $\mathcal{H}$ and $J_{\alpha}^{\prime}\left(\hat{g}_{\alpha}\right)=m \hat{g}_{\alpha}+p_{\hat{g}_{\alpha}}=0$.
Proof. The calculations are similar to the previous proposition. We remark that the boundary conditions for the adjoint state $p_{g \alpha}$ are

$$
-\partial_{n} p_{g \alpha}(t)=\alpha p_{g \alpha} \quad \text { on } \Gamma_{1} \quad \text { and } \quad \partial_{n} p_{g \alpha}(t)=0 \quad \text { on } \Gamma_{2}
$$

for almost every $t$ in $] 0, T[$. Moreover, we assume $\alpha>0$ so that the coerciveness (3.3) becomes

$$
\begin{equation*}
a_{\alpha}(v, v) \geqslant \lambda_{1} \min \{1, \alpha\}\left[\|\nabla v\|_{H}^{2}+\|v\|_{H}^{2}\right], \quad \forall v \in V \tag{3.5}
\end{equation*}
$$

Indeed, by contradiction one can show that $a_{1}(v, v) \geqslant c_{1}\|v\|_{H}^{2}$ for every $v$ in $V$, which implies (3.5). The continuity of $a(\cdot, \cdot)$ in $V$ uses the continuity of the trace in $H^{1}(\Omega)$, namely, for some $\Lambda_{1}>0$ one has

$$
\begin{equation*}
a_{\alpha}(u, v) \leqslant \Lambda_{1} \max \{1, \alpha\}\|u\|_{V}\|v\|_{V}, \quad \forall v \in V \tag{3.6}
\end{equation*}
$$

which depends on $\alpha>0$.
The operator $C_{\alpha}$ actually maps the space $L^{2}\left(0, T ; V^{\prime}\right)$ into the space

$$
\left\{v \in L^{2}(0, T ; V) \cap L^{\infty}(0, T ; H): \dot{v} \in L^{2}\left(0, T ; V^{\prime}\right)\right\}
$$

and the estimates

$$
\begin{aligned}
& \left\|\nabla C_{\alpha}(g)\right\|_{\mathcal{H}} \leqslant \frac{1}{\lambda_{1}}\left[\int_{0}^{T}\|g(t)\|_{V^{\prime}}^{2} \mathrm{~d} t\right]^{1 / 2} \\
& \sup _{0 \leqslant t \leqslant T}\left\|C_{\alpha}(g)(t)\right\|_{H} \leqslant \frac{1}{\sqrt{\lambda_{1}}}\left[\int_{0}^{T}\|g(t)\|_{V^{\prime}}^{2} \mathrm{~d} t\right]^{1 / 2}, \\
& {\left[\int_{0}^{T}\left\|\frac{\mathrm{~d}}{\mathrm{~d} t}\left(C_{\alpha}(g)(t)\right)\right\|_{V_{0}^{\prime}}^{2} \mathrm{~d} t\right]^{1 / 2} \leqslant \frac{2}{\lambda_{1}}\left[\int_{0}^{T}\|g(t)\|_{V^{\prime}}^{2} \mathrm{~d} t\right]^{1 / 2}}
\end{aligned}
$$

are independent of $\alpha>1$, but

$$
\left[\int_{0}^{T}\left\|\frac{\mathrm{~d}}{\mathrm{~d} t}\left(C_{\alpha}(g)(t)\right)\right\|_{V^{\prime}}^{2} \mathrm{~d} t\right]^{1 / 2} \leqslant \frac{1+\alpha}{\lambda_{1}}\left[\int_{0}^{T}\|g(t)\|_{V^{\prime}}^{2} \mathrm{~d} t\right]^{1 / 2}
$$

is depends on $\alpha$. Certainly, also one deduces

$$
\alpha \int_{0}^{T}\left|C_{\alpha}(g)(t)\right|_{L^{2}\left(\Gamma_{1}\right)}^{2} \mathrm{~d} t \leqslant\|g\|_{L^{2}\left(0, T ; V^{\prime}\right)}\left\|C_{\alpha}(g)\right\|_{L^{2}(0, T ; V)}
$$

which is uniformly bounded in $\alpha>1$. On the other hand, note that the functions $b$ and $q($ or $\psi)$ intervene to estimate $u_{0 \alpha}$ and $\dot{u}_{0 \alpha}$.

To show that $g \mapsto J_{\alpha}(g)$ is strictly convex, one show that

$$
\begin{aligned}
& (1-\theta) J_{\alpha}\left(g_{2}\right)+\theta J_{\alpha}\left(g_{1}\right)-J_{\alpha}\left((1-\theta) g_{1}+\theta g_{2}\right) \\
& \quad=\frac{1}{2} \theta(1-\theta)\left[\left\|u_{g_{1} \alpha}-u_{g_{2} \alpha}\right\|_{\mathcal{H}}^{2}+m\left\|g_{1}-g_{2}\right\|_{\mathcal{H}}^{2}\right],
\end{aligned}
$$

for every $\theta$ in $[0,1]$ and any $g_{1}, g_{2}$ in $\mathcal{H}$.
Remark that one has nice estimates for the affine application $g \mapsto u_{g \alpha}$, namely

$$
\begin{aligned}
& \left\|\nabla u_{g_{1} \alpha}-\nabla u_{g_{2} \alpha}\right\|_{\mathcal{H}} \leqslant \frac{1}{\lambda_{1}}\left\|g_{1}-g_{2}\right\|_{L^{2}\left(0, T ; V^{\prime}\right)}, \\
& \sup _{0 \leqslant t \leqslant T}\left\|u_{g_{1} \alpha}(t)-u_{g_{2} \alpha}(t)\right\|_{H} \leqslant \frac{1}{\sqrt{\lambda_{1}}}\left\|g_{1}-g_{2}\right\|_{L^{2}\left(0, T ; V^{\prime}\right)} \\
& \left\|\dot{u}_{g_{1} \alpha}-\dot{u}_{g_{2} \alpha}\right\|_{L^{2}\left(0, T ; V_{0}^{\prime}\right)} \leqslant \frac{2}{\lambda_{1}}\left\|g_{1}-g_{2}\right\|_{L^{2}\left(0, T ; V^{\prime}\right)}, \\
& \left\|\dot{u}_{g_{1} \alpha}-\dot{u}_{g_{2} \alpha}\right\|_{L^{2}\left(0, T ; V^{\prime}\right)} \leqslant \frac{1+\alpha}{\lambda_{1}}\left\|g_{1}-g_{2}\right\|_{L^{2}\left(0, T ; V^{\prime}\right)}, \\
& \left\|u_{g_{1} \alpha}-u_{g_{2} \alpha}\right\|_{L^{2}\left(0, T ; L^{2}\left(\Gamma_{1}\right)\right.} \leqslant \frac{1}{\sqrt{\lambda_{1} \alpha}}\left\|g_{1}-g_{2}\right\|_{L^{2}\left(0, T ; V^{\prime}\right)}
\end{aligned}
$$

and similarly, for the adjoint state mapping $g \mapsto p_{g \alpha}$, one obtain estimates as above replacing $u_{g_{i} \alpha}$ with $p_{g_{i} \alpha}$.

On the other hand, $u_{g_{1} \alpha}-u_{g_{2} \alpha}$ is the unique solution of a parabolic variational equality (1.5) with $q=0, b=0$ and $g=g_{1}-g_{2}$, i.e., $\left(\partial_{t}-\Delta\right)\left(u_{g_{1} \alpha}-u_{g_{2} \alpha}\right)=g$ in $L^{2}(\Omega \times] 0, T[)$ with homogeneous mixed (Robin on $\Gamma_{1}$ and Neumann on $\Gamma_{2}$ ) boundary conditions. Hence, regularity results implies that $u_{g_{1} \alpha}-u_{g_{2} \alpha}$ belongs to $L^{2}\left(0, T ; H^{2}(\Omega)\right) \cap H^{1}\left(0, T ; L^{2}(\Omega)\right)$. Similar arguments apply to $u_{g_{1}}-u_{g_{2}}$, i.e., $\left(\partial_{t}-\Delta\right)\left(u_{g_{1}}-u_{g_{2}}\right)=g$ in $L^{2}(\Omega \times] 0, T[)$ with homogeneous mixed (Dirichlet on $\Gamma_{1}$ and Neumann on $\Gamma_{2}$ ) boundary conditions. Note that some difficulties due to the mixed boundary conditions do arrives, e.g., see Grisvard [9], but our interest is on the asymptotic behavior as $\alpha$ becomes infinite.

## 4. Asymptotic estimates

First one needs to obtain estimates on $u_{g \alpha}$ and $p_{g \alpha}$ uniformly in $\alpha>1$ and any given $g$.

Proposition 4.1. Under the previous assumptions one has the estimate

$$
\begin{equation*}
\left\|u_{g \alpha}\right\|_{L^{\infty}(0, T ; H)}+\left\|u_{g \alpha}\right\|_{L^{2}(0, T ; V)}+\sqrt{(\alpha-1)}\left\|u_{g \alpha}-b\right\|_{L^{2}\left(\Gamma_{1} \times\right] 0, T \mathrm{D}} \leqslant C\left(1+\left\|g_{\psi}\right\|_{L^{2}\left(0, T ; V^{\prime}\right)}\right) \tag{4.1}
\end{equation*}
$$

for every $\alpha>1$ and any $g$ in $\mathcal{H}$, where the constant $C$ depends only on the norms $\left\|\dot{u}_{g}\right\|_{L^{2}\left(0, T ; V^{\prime}\right)}$, $\left\|\nabla u_{g}\right\|_{L^{2}(0, T ; H)}$, and the coerciveness constant $\lambda_{1}$ in (3.5). Moreover, as $\alpha \rightarrow \infty$ one has $u_{g \alpha} \rightarrow u_{g}$ strongly in $L^{2}(0, T ; V) \cap L^{\infty}(0, T ; H)$ and $\dot{u}_{g \alpha} \rightarrow \dot{u}_{g}$ in norm $L^{2}\left(0, T ; V_{0}^{\prime}\right)$.

Proof. First note that $V_{0} \subset V$ is a continuous (nondense) inclusion and the norms $\|v\|_{V_{0}}=\|\nabla v\|_{H}$ is equivalently to $\|v\|_{V}=\sqrt{\|v\|_{V_{0}}+\|v\|_{H}}$ on $V_{0}$.

Let $\varphi$ be a function in $L^{2}(0, T ; V)$ such that $\dot{\varphi}$ belongs to $L^{2}\left(0, T ; V^{\prime}\right), \varphi(0)=v_{b}$ and $\varphi=b$ on $\Gamma_{1}$, e.g., an extension of $b$ and $v_{b}$ such as $\psi$ in (2.2). Now, on the equality (1.5) defining $u_{g \alpha}$ take $v=u_{g \alpha}(t)-\varphi(t):=z_{g \alpha}(t)$ to get

$$
\begin{aligned}
& \left\langle\dot{u}_{g \alpha}(t), z_{g \alpha}(t)\right\rangle+\left(\nabla u_{g \alpha}(t), \nabla z_{g \alpha}(t)\right)_{H}+\alpha\left\langle u_{g \alpha}(t), z_{g \alpha}(t)\right\rangle_{\Gamma_{1}} \\
& \quad=\left(g(t), z_{g \alpha}(t)\right)_{H}-\left\langle q(t), z_{g \alpha}(t)\right\rangle_{\Gamma_{2}}+\alpha\left\langle b, z_{g \alpha}(t)\right\rangle_{\Gamma_{1}},
\end{aligned}
$$

and because $\varphi=b$ on $\Gamma_{1}$ one deduces

$$
\begin{align*}
& \frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\left\|z_{g \alpha}(t)\right\|_{H}^{2}+\left\|\nabla z_{g \alpha}(t)\right\|_{H}^{2}+\alpha\left\|z_{g \alpha}(t)\right\|_{L^{2}\left(\Gamma_{1}\right)}^{2}  \tag{4.2}\\
& \quad=\left(g(t), z_{g \alpha}(t)\right)_{H}-\left\langle q(t), z_{g \alpha}(t)\right\rangle_{L^{2}\left(\Gamma_{2}\right)}-\left\langle\dot{\varphi}(t), z_{g \alpha}(t)\right\rangle-\left(\nabla \varphi, \nabla z_{g \alpha}\right)_{H}
\end{align*}
$$

which together with coerciveness (3.5) and the condition $z_{g \alpha}(0)=0$ yield the bound (4.1). By means of estimate (4.1), there exists a sequence $\alpha_{n} \rightarrow \infty$ and $z$ in $L^{2}(0, T ; V) \cap L^{\infty}(0, T ; H)$ such that $z_{g \alpha_{n}} \rightarrow z$ weakly in $L^{2}(0, T ; V)$ and weakly* in $L^{\infty}(0, T ; H)$, and $z=0$ on $\Gamma_{1}$, i.e., $z$ belongs to $L^{2}\left(0, T ; V_{0}\right)$.

Hence, note that $a_{\alpha}(u, v)=a(u, v)$ and $L_{g \alpha}(t, v)=L_{g}(t, v)$ if $u$ belongs to $V$ and $v$ belongs to $V_{0}$, and take $v$ in $V_{0}$ in Eqs (1.4) and (1.5) defining $u_{g}$ and $u_{g \alpha}$ to obtain $\left\langle\dot{z}_{g \alpha}, v\right\rangle+a\left(z_{g \alpha}, v\right)=0$, for every $v \in V_{0}$. Therefore, $\dot{z}_{g \alpha_{n}} \rightarrow \dot{z}$ weakly in $L^{2}\left(0, T ; V_{0}^{\prime}\right)$ and because $z_{g \alpha}(0)=0$ and $z=0$ on $\Gamma_{1}$, one deduces $z=0$ in $L^{2}(0, T ; V)$.
Thus, as $\alpha \rightarrow \infty$ one has $z_{g \alpha} \rightarrow 0$ weakly in $L^{2}(0, T ; V)$ and weakly* in $L^{\infty}(0, T ; H)$. It is clear that the inclusion $V_{0} \subset V$ is continuous and because the norm of $V$ restricted to $V_{0}$ is equivalent to the norm of $V_{0}$, Hahn-Banach Theorem implies that any element $\vartheta$ of $V_{0}^{\prime}$ can be extended to an element in $V^{\prime}$ preserving its norm, in particular $\dot{u}_{g}$ can be extended to be an element in $L^{2}\left(0, T ; V^{\prime}\right)$. Then, take $\varphi=u_{g}$ in the equality (4.2) and considering $\dot{u}_{g}$ an element in $L^{2}\left(0, T ; V^{\prime}\right)$, one deduces that the convergence of $u_{g \alpha}$ toward $u_{g}$ is indeed strongly in $L^{2}(0, T ; V) \cap L^{\infty}(0, T ; H)$. Moreover, $z_{g \alpha} \rightarrow 0$ in norm $L^{2}(\Gamma \times] 0, T[)$ and $\dot{z}_{g \alpha} \rightarrow 0$ in norm $L^{2}\left(0, T ; V_{0}^{\prime}\right)$.

Proposition 4.2. Under the previous assumptions one has the estimate

$$
\begin{equation*}
\left\|p_{g \alpha}\right\|_{L^{\infty}(0, T ; H)}+\left\|p_{g \alpha}\right\|_{L^{2}(0, T ; V)}+\sqrt{(\alpha-1)}\left\|p_{g \alpha}\right\|_{L^{2}\left(\Gamma_{1} \times\right] 0, T \mathrm{D}} \leqslant C\left(1+\left\|u_{g \alpha}\right\|_{L^{2}\left(0, T ; V^{\prime}\right)}\right) \tag{4.3}
\end{equation*}
$$

for every $\alpha>1$ and any $g$ in $\mathcal{H}$, where the constant $C$ depends only on the norms $\left\|z_{d}\right\|_{\mathcal{H}},\left\|\dot{p}_{g}\right\|_{L^{2}\left(0, T ; V^{\prime}\right)}$, $\left\|\nabla p_{g}\right\|_{L^{2}(0, T ; H)}$, and the coerciveness constant $\lambda_{1}$ in (3.5). Moreover, as $\alpha \rightarrow \infty$ one has $p_{g \alpha} \rightarrow p_{g}$ strongly in $L^{2}(0, T ; V) \cap L^{\infty}(0, T ; H)$ and $\dot{p}_{g \alpha} \rightarrow \dot{p}_{g}$ in norm $L^{2}\left(0, T ; V_{0}^{\prime}\right)$.

Proof. Note that even when $b \neq 0$ the (Robin) boundary condition of $p_{g}$ and $p_{g \alpha}$ on $\Gamma_{1}$ does not involve $b$ directly. Certainly, the norm $\left\|u_{g \alpha}\right\|_{L^{2}\left(0, T ; V^{\prime}\right)}$ is bounded by $\left\|u_{g \alpha}\right\|_{L^{2}(0, T ; H)}$, which is uniformly bounded in $\alpha$.

The technique used in Proposition 4.1 applies for the adjoint states $p_{g \alpha}$ and $p_{g}$. Perhaps the only point to remark is the convergence as $\alpha \rightarrow \infty$. Indeed, one needs to make use of the weak (and later strong) convergence $u_{g \alpha} \rightarrow u_{g}$ in $L^{2}\left(0, T ; V^{\prime}\right)$, which is deduced for the convergence in $L^{2}(0, T ; H)$.

## 5. Optimal control problems

We are now ready to consider the distributed control problems (1.9) and (1.10). Our purpose is to establish

Theorem 5.1. Let assumptions (2.1) and (2.2) be hold, and $\hat{g}$ and $\hat{g}_{\alpha}$ be the minimizers in $\mathcal{H}$ of problems (1.9) and (1.10), respectively. Then, as the parameter $\alpha \rightarrow \infty$, the minimizers $\hat{g}_{\alpha} \rightarrow \hat{g}$ strongly in $\mathcal{H}$. Moreover the corresponding optimal state and adjoint state satisfy $\left(u_{\hat{g}_{\alpha} \alpha}, \dot{u}_{\hat{g}_{\alpha} \alpha}\right) \rightarrow\left(u_{\hat{g}}, \dot{u}_{\hat{g}}\right)$ and $\left(p_{\hat{g}_{\alpha} \alpha}, \dot{p}_{\hat{g}_{\alpha} \alpha}\right) \rightarrow\left(p_{\hat{g}}, \dot{p}_{\hat{g}}\right)$ strongly in $L^{2}(0, T ; V) \times L^{2}\left(0, T ; V_{0}^{\prime}\right)$.

Proof. We make several steps. First, be means of the estimate (4.1) in Proposition 4.1 one has

$$
\left\|u_{0 \alpha}\right\|_{\mathcal{H}} \leqslant C, \quad \forall \alpha>1
$$

for some constant $C$. Now, from the inequality $J\left(\hat{g}_{\alpha}\right) \leqslant J(0)$ we deduce

$$
\left\|\hat{g}_{\alpha}\right\|_{\mathcal{H}}+\left\|u_{\hat{g}_{\alpha} \alpha}\right\|_{\mathcal{H}} \leqslant C, \quad \forall \alpha>1
$$

for some constant independent of $\alpha>1$.
Again, estimate (4.1) in Proposition 4.1 and estimate (4.2) in Proposition 4.2 yield

$$
\left\|u_{\hat{g}_{\alpha} \alpha}\right\|_{L^{2}(0, T ; V)}+\left\|\dot{u}_{\hat{g}_{\alpha} \alpha}\right\|_{L^{2}\left(0, T ; V_{0}^{\prime}\right)}+\sqrt{(\alpha-1)}\left\|u_{\hat{g}_{\alpha} \alpha}-b\right\|_{L^{2}\left(0, T ; L^{2}\left(\Gamma_{1}\right)\right)} \leqslant C, \quad \forall \alpha>1
$$

and

$$
\left\|p_{\hat{g}_{\alpha} \alpha}\right\|_{L^{2}(0, T ; V)}+\left\|\dot{p}_{\hat{g}_{\alpha} \alpha}\right\|_{L^{2}\left(0, T ; V_{0}^{\prime}\right)}+\sqrt{(\alpha-1)}\left\|p_{\hat{g}_{\alpha} \alpha}\right\|_{L^{2}\left(0, T ; L^{2}\left(\Gamma_{1}\right)\right)} \leqslant C, \quad \forall \alpha>1 .
$$

Hence, there exist $\bar{g}$ in $\mathcal{H}, \hat{u}$ and $\hat{p}$ in $L^{2}\left(0, T ; V_{0}\right)$ with $\dot{\hat{u}}$ and $\dot{\hat{p}}$ in $L^{2}\left(0, T ; V_{0}^{\prime}\right)$ such that, for a convenient subsequence as $\alpha \rightarrow \infty$ we has $\hat{g}_{\alpha} \rightharpoonup \bar{g}$ weakly in $\mathcal{H}, u_{\hat{g}_{\alpha} \alpha} \rightharpoonup \hat{u}$ weakly in $L^{2}(0, T ; V), \dot{u}_{\hat{g}_{\alpha} \alpha} \rightharpoonup \dot{\hat{u}}$ weakly in $L^{2}\left(0, T ; V_{0}^{\prime}\right)$, $p_{\hat{g}_{\alpha} \alpha} \rightharpoonup \hat{p}$ weakly in $L^{2}(0, T ; V), \dot{p}_{\hat{g}_{\alpha} \alpha} \rightharpoonup \dot{\hat{p}}$ weakly in $L^{2}\left(0, T ; V_{0}^{\prime}\right)$.

By taking $v$ in $V_{0}$ in the parabolic variational equality (2.5) and letting $\alpha \rightarrow \infty$ we deduce that $\hat{u}$ solves parabolic variational equality (2.4), and by uniqueness $\hat{u}=u_{\hat{g}}$. In particular $u_{\hat{g}_{\alpha} \alpha} \rightharpoonup u_{\hat{g}}$ weakly in $L^{2}\left(0, T ; V_{0}^{\prime}\right)$. Thus, by taking $v$ in $V_{0}$ in the parabolic variational equality defining the adjoint state $p_{\hat{g}_{\alpha} \alpha}$ in Proposition 3.2 and letting $\alpha \rightarrow \infty$ we deduce that $\hat{p}=p_{\bar{g}}$. On the other hand, taking limit in the equality $m \hat{g}_{\alpha}+p_{\hat{g}_{\alpha} \alpha}=0$ we deduce $m \bar{g}+p_{\bar{g}}=0$. Thus, by using Proposition 3.1, this proves that $\bar{g}$ is a minimizer for the control problem (1.9), and by uniqueness $\hat{g}=\bar{g}$.

At this point, we have

$$
\left(\hat{g}_{\alpha}, u_{\hat{g}_{\alpha} \alpha}, \dot{u}_{\hat{g}_{\alpha} \alpha}, p_{\hat{g}_{\alpha} \alpha}, \dot{p}_{\hat{g}_{\alpha} \alpha}\right) \rightharpoonup\left(\hat{g}, u_{\hat{g}}, \dot{u}_{\hat{g}}, p_{\hat{g}}, \dot{p}_{\hat{g}}\right)
$$

weakly in the corresponding spaces, initially for a convenient subsequence as $\alpha \rightarrow \infty$, but in view of the uniqueness of the limit, the weak convergence whole as $\alpha \rightarrow \infty$.

To prove the strong convergence we use the weak semicontinuity of the norm and the optimality of $\hat{g}, \hat{g}_{\alpha}$, namely,

$$
\begin{aligned}
J(\hat{g}) & =\frac{1}{2}\left\|u_{\hat{g}}-z_{d}\right\|_{\mathcal{H}}^{2}+\frac{m}{2}\|\hat{g}\|_{\mathcal{H}}^{2} \leqslant \liminf _{\alpha \rightarrow \infty}\left[\frac{1}{2}\left\|u_{\hat{g}_{\alpha} \alpha}-z_{d}\right\|_{\mathcal{H}}^{2}+\frac{m}{2}\left\|\hat{g}_{\alpha}\right\|_{\mathcal{H}}^{2}\right] \\
& \leqslant \limsup _{\alpha \rightarrow \infty}\left[\frac{1}{2}\left\|u_{\hat{g}_{\alpha} \alpha}-z_{d}\right\|_{\mathcal{H}}^{2}+\frac{m}{2}\left\|\hat{g}_{\alpha}\right\|_{\mathcal{H}}^{2}\right] \leqslant \limsup _{\alpha \rightarrow \infty} J_{\alpha}(g),
\end{aligned}
$$

for any $g$ in $\mathcal{H}$. In view of Proposition 4.1, $u_{g \alpha} \rightarrow u_{g}$ strongly in $L^{2}(0, T ; V)$ as $\alpha \rightarrow \infty$, which implies that

$$
\limsup _{\alpha \rightarrow \infty} J_{\alpha}(g)=\lim _{\alpha \rightarrow \infty}\left[\frac{1}{2}\left\|u_{g \alpha}-z_{d}\right\|_{\mathcal{H}}^{2}+\frac{m}{2}\|g\|_{\mathcal{H}}^{2}\right]=J(g) .
$$

By taking infimum on $g$, all the above inequalities become equalities and therefore

$$
\frac{1}{2}\left\|u_{\hat{g}_{\alpha} \alpha}-z_{d}\right\|_{\mathcal{H}}^{2}+\frac{m}{2}\left\|\hat{g}_{\alpha}\right\|_{\mathcal{H}}^{2} \rightarrow \frac{1}{2}\left\|u_{\hat{g}}-z_{d}\right\|_{\mathcal{H}}^{2}+\frac{m}{2}\|\hat{g}\|_{\mathcal{H}}^{2}
$$

This and the weak convergence imply that $\left(\hat{g}_{\alpha}, u_{\hat{g}_{\alpha} \alpha}\right) \rightarrow\left(\hat{g}, u_{\hat{g}}\right)$ strongly in $\mathcal{H} \times \mathcal{H}$, as $\alpha \rightarrow \infty$.
Finally, if $z_{\alpha}=u_{\hat{g}_{\alpha} \alpha}-u_{\hat{g}}$ then we deduce

$$
\begin{aligned}
& \int_{0}^{T}\left[\left\langle\dot{z}_{\alpha}(t), z_{\alpha}(t)\right\rangle+a_{1}\left(z_{\alpha}(t), z_{\alpha}(t)\right)+(\alpha-1) \int_{\Gamma_{1}}\left|z_{\alpha}(x, t)\right|^{2} \mathrm{~d} x\right] \mathrm{d} t \\
& \quad \leqslant \int_{0}^{T}\left[\left\langle\hat{g}_{\alpha}-\dot{u}_{\hat{g}}, z_{\alpha}\right\rangle-a\left(u_{\hat{g}}, z_{\alpha}\right)-\int_{\Gamma_{2}} q(x, t) z_{\alpha}(x, t) \mathrm{d} x\right] \mathrm{d} t .
\end{aligned}
$$

Since $z_{\alpha} \rightarrow 0$ weakly in $L^{2}(0, T ; V)$ and $\hat{g}_{\alpha} \rightarrow \hat{g}$ strongly in $\mathcal{H}$, we obtain $u_{\hat{g}_{\alpha} \alpha} \rightarrow u_{\hat{g}}$ strongly in $L^{2}(0, T ; V)$, as $\alpha \rightarrow \infty$. Now, going back to the equation one has

$$
\left\langle\dot{z}_{\alpha}(t), v\right\rangle+a\left(z_{\alpha}(t), v\right)=\left\langle\hat{g}_{\alpha}-\hat{g}, v\right\rangle .
$$

Now, taking sup for $v$ in $V_{0}$ with $\left\|v_{0}\right\|_{V_{0}} \leqslant 1$ and integrating in $] 0, T$ [ one obtains the strong convergence of the time derivative. Similarly, $\left(p_{\hat{g}_{\alpha} \alpha}, \dot{p}_{\hat{g}_{\alpha} \alpha}\right) \rightarrow\left(p_{\hat{g}}, \dot{p}_{\hat{g}}\right)$ strongly in $L^{2}(0, T ; V) \times L^{2}\left(0, T ; V_{0}^{\prime}\right)$, as $\alpha \rightarrow \infty$. This completes the proof.

Also we have

Proposition 5.2. If $\alpha_{2} \geqslant \alpha_{1} \geqslant \alpha_{0}>0$ then there exists a constant $C=C_{\alpha_{0}}$ such that for every $g$ in $\mathcal{H}$ one has

$$
\begin{equation*}
\left\|u_{g \alpha_{1}}-u_{g \alpha_{2}}\right\|_{L^{2}(0, T ; V)} \leqslant C_{\alpha_{0}}\left(\alpha_{2}-\alpha_{1}\right)\left\|b-u_{g \alpha_{2}}\right\|_{L^{2}\left(0, T ; H^{-1 / 2}\left(\Gamma_{1}\right)\right)}, \tag{5.1}
\end{equation*}
$$

and

$$
\begin{align*}
\left\|p_{g \alpha_{1}}-p_{g \alpha_{2}}\right\|_{L^{2}(0, T ; V)} \leqslant & C_{\alpha_{0}}\left(\alpha_{2}-\alpha_{1}\right)\left(\left\|p_{g \alpha_{2}}\right\|_{L^{2}\left(0, T ; H^{-1 / 2}\left(\Gamma_{1}\right)\right)}\right. \\
& \left.+\left\|b-u_{g \alpha_{2}}\right\|_{L^{2}\left(0, T ; H^{-1 / 2}\left(\Gamma_{1}\right)\right)}\right) \tag{5.2}
\end{align*}
$$

i.e., the dependency in $\alpha$ is Lipschitz continuous.

Proof. For a fixed $g$ and $\alpha_{2} \geqslant \alpha_{1} \geqslant \alpha_{0}>0$ set $z=u_{g \alpha_{2}}-u_{g \alpha_{1}}$ to obtain from Eq. (1.5) with $\alpha_{i}$ the identity

$$
\langle\dot{z}(t), v\rangle+a_{\alpha_{1}}(z(t), v)=\left(\alpha_{2}-\alpha_{1}\right) \int_{\Gamma_{1}}\left(b-u_{g \alpha_{2}}\right) v \mathrm{~d} \gamma, \quad \forall v \in V .
$$

By taking $v=z(t)$ and by means of the inequalities

$$
\left|\int_{0}^{T} \mathrm{~d} t \int_{\Gamma_{1}}\left(b-u_{g \alpha_{2}}\right) z \mathrm{~d} \gamma\right| \leqslant C_{0}\left\|b-u_{g \alpha_{2}}\right\|_{L^{2}\left(0, T ; H^{-1 / 2}\left(\Gamma_{1}\right)\right)}\|z\|_{L^{2}(0, T ; V)}
$$

and

$$
a_{\alpha}(v, v) \geqslant \lambda\left(\alpha_{0}\right)\|v\|_{V}^{2}, \quad \forall v \in V, \alpha \geqslant \alpha_{0}
$$

we deduce the desired estimate with $C_{\alpha_{0}}=C_{0} / \lambda\left(\alpha_{0}\right)$.
Similarly, for a fixed $g$ and $\alpha_{2} \geqslant \alpha_{1} \geqslant \alpha_{0}>0$ set $w=p_{g \alpha_{2}}-p_{g \alpha_{1}}$ to obtain from Eq. (3.5) with $\alpha_{i}$ the identity

$$
\langle\dot{w}(t), v\rangle+a_{\alpha_{1}}(w(t), v)=\left(\alpha_{1}-\alpha_{2}\right) \int_{\Gamma_{1}} p_{g \alpha_{2}} v \mathrm{~d} \gamma+\left(u_{g \alpha_{2}}-u_{g \alpha_{1}}, v\right)_{H}
$$

for every $v$ in $V$. By taking $v=w(t)$ and in view of the estimate (5.1), we conclude.
Under some more restrict assumption we have monotonicity on $\alpha$
Proposition 5.3. Let us assume the data $b$ constant on $\Gamma_{1}, v_{b} \leqslant b$ on $\Omega, g \leqslant 0$ in $\left.\Omega \times\right] 0, T$ [ and $q \geqslant 0$ on $\left.\Gamma_{2} \times\right] 0, T\left[\right.$. Then $u_{g \alpha} \leqslant u_{g} \leqslant b$ for every $\alpha>0$. Moreover, if $0<\alpha_{1} \leqslant \alpha_{2}$ then $u_{g \alpha_{1}} \leqslant u_{g \alpha_{2}} \leqslant u_{g} \leqslant b$ in $\Omega \times] 0, T\left[\right.$. Furthermore, if $b \leqslant z_{d}$ in $\left.\Omega \times\right] 0, T\left[\right.$ then $p_{g \alpha_{1}} \leqslant p_{g \alpha_{2}} \leqslant p_{g} \leqslant 0$ in $\left.\Omega \times\right] 0, T[$, for every $\alpha_{2} \geqslant \alpha_{1}>0$.

Proof. First, the maximum principle implies that $u_{g \alpha} \leqslant b$. Indeed, if $z=\left(u_{g \alpha}-b\right)$ then we have

$$
\left\langle\dot{z}(t), z^{+}(t)\right\rangle+a\left(z(t), z^{+}(t)\right)+\alpha \int_{\Gamma_{1}} z(t) z^{+}(t) \mathrm{d} \gamma=\left(g(t), z^{+}(t)\right)-\int_{\Gamma_{2}} q(t) z^{+}(t) \mathrm{d} \gamma
$$

after using the fact that $b$ is constant, which implies $z^{+}=0$.

Similarly, if $w=u_{g \alpha_{2}}-u_{g \alpha_{1}}$ with $\alpha_{2}>\alpha_{1}$ then we get

$$
\left\langle\dot{w}(t), w^{+}(t)\right\rangle+a_{\alpha_{1}}\left(w(t), w^{+}(t)\right)+\left(\alpha_{2}-\alpha_{1}\right) \int_{\Gamma_{1}}\left(b-u_{g \alpha_{2}}(t)\right) z^{+}(t) \mathrm{d} \gamma=0
$$

which yields $w \leqslant 0$, i.e., $u_{g \alpha_{2}} \leqslant u_{g \alpha_{1}}$.
Finally, if $y=u_{g \alpha}-u_{g}$ then we obtain

$$
\left\langle\dot{y}(t), y^{+}(t)\right\rangle+a\left(y(t), y^{+}(t)\right)+\alpha \int_{\Gamma_{1}}\left(b-u_{g \alpha}(t)\right) y^{+}(t) \mathrm{d} \gamma=0
$$

which yields $y \leqslant 0$, i.e., $u_{g \alpha} \leqslant u_{g}$.
The estimate on the adjoint state follows from a comparison with the solution $r$ of the parabolic variational equality with terminal condition

$$
\left\{\begin{array}{l}
r \in L^{2}(0, T ; V), \quad r(T)=0 \quad \text { and } \quad \dot{r} \in L^{2}\left(0, T ; V^{\prime}\right)  \tag{5.3}\\
\text { such that }-\langle\dot{r}(t), v\rangle+a(r(t), v)=\left(b-z_{d}, v\right)_{H}, \quad \forall v \in V
\end{array}\right.
$$

Indeed, if $b \leqslant z_{d}$ in $\left.\Omega \times\right] 0, T$ [ then the maximum principle (as above) yields $p_{g} \leqslant r \leqslant 0$. Next, similarly to the state $u$ with $b=0$, one deduces that $p_{g \alpha_{1}} \leqslant p_{g \alpha_{2}} \leqslant p_{g} \leqslant r \leqslant 0$ in $\left.\Omega \times\right] 0, T$, for every $\alpha_{2} \geqslant \alpha_{1}>0$.

Certainly, the maximum principle yields $u_{g_{1}} \leqslant u_{g_{2}}$ and $u_{g_{1} \alpha} \leqslant u_{g_{2} \alpha}$ if $g_{1} \leqslant g_{2}$, but a priori, it is not clear when the minimizers satisfy $\hat{g} \geqslant \hat{g}_{\alpha}$ to deduce the monotonicity $u_{g_{\alpha_{1}} \alpha_{1}} \leqslant u_{g_{\alpha_{2}} \alpha_{2}} \leqslant u_{g_{\alpha}} \leqslant b$.

## 6. Final comments

Variational inequalities was popular in the 70's, most of the main techniques for parabolic variational inequalities can be found in various classic books, e.g., Bensoussan and Lions [5], among other.

It is well known that the regularity of the mixed problem is problematic when both portions of the boundary $\Gamma_{1}$ and $\Gamma_{2}$ have a nonempty intersection, e.g. see the book Grisvard [9]. Recently, sufficient conditions (on the data) to obtain a $H^{2}$ regularity for a (elliptic) mixed boundary conditions are given in Bacuta et al. [3], see also Azzam and Kreyszig [1], among others.

Numerical analysis of a parabolic PDE with mixed boundary conditions (Dirichlet and Neumann) is studied in Babuska and Ohnimus [2], while a parabolic control problem with Robin boundary conditions is considered in Chrysafinos et al. [7] and Bergounioux and Troltzsch [6].

The state equation, i.e., a parabolic PDE with mixed boundary conditions (Robin and Neumann) has been discussed in Ben Belgacem et al. [4] and Tarzia [12].

Certainly, there are several possible extensions, e.g., a state equation of the form

$$
\left.\partial_{t} u-\operatorname{div}(A(x, t) \nabla u)+b(t, x) u=f \quad \text { in } \Omega \times\right] 0, T[,
$$

with mixed boundary conditions. A carefully analysis is necessary, but the main techniques used to let $\alpha \rightarrow \infty$ in the parabolic variational inequality seems to be very well adaptable to more general situations.

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