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Similarity Solutions for Thawing Processes with a Heat Flux Condition at the Fixed Boundary

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Abstract. Similarity solutions for a mathematical model for thawing in a saturated semi-infinite porous medium is considered when change of phase induces a density jump and a heat flux condition of the type $-q_0 t^{-(1/2)}$ is imposed on the fixed face $x = 0$. Different cases depending on physical parameters are analysed and the explicit solution is obtained if and only if an inequality for the thermal coefficient q_0 is verified. An improvement for the existence of a similarity solution for the same free boundary problem with a constant temperature on the fixed face $x = 0$ is also obtained.

Sommario. Vengono considerate soluzioni di similarità per un modello matematico di disgelo di un mezzo poroso saturo semi-infinito allorquando il cambiamento di fase induce un salto di densità ed una condizione di flusso di calore del tipo $-q_0 t^{-(1/2)}$ viene imposta sulla faccia fissa $x = 0$. Si analizzano differenti casi dipendenti da parametri fisici e la soluzione esplicita viene ottenuta se e solo se risulta verificata una diseguaglianza per il coefficiente termico q_0. Si ottiene altresi un miglioramento della condizione di esistenza di una soluzione di similarità per lo stesso problema al contorno libero con temperatura costante sulla faccia fissa $x = 0$.

Key words: Stefan problem, Free boundary problems, Phase change process, Similarity solution, Density jump, Thawing processes, Freezing, Solidification.

1. Introduction

Phase-change problems appear frequently in industrial processes and other problems of technological interest [2, 4, 9]. A large bibliography on the subject was given in [16]. In this paper, we consider the problem of thawing of a partially frozen porous media, saturated with an incompressible liquid, with the aim of constructing similarity solutions.

We have in mind the following physical assumptions (see [3, 6, 7]):

1. A sharp interface between the frozen part and the unfrozen part of the domain exists (sharp, in the macroscopic sense).
2. The frozen part is at rest with respect to the porous skeleton, which will be considered to be indeformable.
3. Due to density jump between the liquid and solid phase, thawing can induce either desaturation or water movement in the unfrozen region. We will consider the latter situation assuming that liquid is continuously supplied to keep the medium saturated.

Although thawing has received less attention than freezing, our investigation is in the same spirit as [5, 10] (see also [11–13] for further references), with the simplification due to the absence of ice lenses and frozen fringes.

We will study a one-dimensional model of the problem, using the following notation:

$\varepsilon > 0$	:	porosity,
$\rho > 0$	:	density; ρ_w and ρ_I: density of water and ice (g/cm^3),
$c > 0$	:	specific heat at constant density $\left(\frac{\text{cal}}{\text{g°C}}\right)$,
$k > 0$	:	conductivity $\left(\frac{\text{cal}}{\text{s cm°C}}\right)$,
u	:	temperature of unfrozen zone (°C),
v	:	temperature of frozen zone (°C),
$u = v = 0$	:	being the melting point at atmospheric pressure,
$\lambda > 0$	:	latent heat at $u = 0$ (cal/g),
γ	:	coefficient in the Clausius-Clapeyron law (s^{2}cm°C/g),
$\mu > 0$	:	viscosity of liquid (g/cm^3),

and subscripts F, U, I and W refer to the frozen medium, unfrozen medium, pure ice and pure water, respectively, while S refers to the porous skeleton.

The unknowns of the problem are a function $x = s(t)$, representing the free boundary separating $Q_1 = \{(x,t)\colon 0 < x < s(t), t > 0\}$ and $Q_2 = \{(x,t)\colon s(t) < x, t > 0\}$, and the two functions $u(x,t)$ and $v(x,t)$ defined in Q_1 and Q_2, respectively. Besides standard requirements, $s(t)$, $u(x,t)$ and $v(x,t)$ fulfil the following conditions (we refer to [6] for a detailed explanation of the model):

$$u_t = a_1 u_{xx} - b\rho\dot{s}(t)u_x, \quad \text{in } Q_1, \tag{1}$$

$$v_t = a_2 v_{xx}, \quad \text{in } Q_2, \tag{2}$$

$$u(s(t),t) = v(s(t),t) = d\rho s(t)\dot{s}(t), \qquad t > 0, \tag{3}$$

$$k_{\mathrm{F}} v_x(s(t),t) - k_{\mathrm{U}} u_x(s(t),t) = \alpha\dot{s}(t) + \beta\rho s(t)\dot{s}^2(t), \qquad t > 0, \tag{4}$$

$$v(x,0) = v(+\infty,t) = -A < 0, \qquad x,t > 0, \tag{5}$$

$$s(0) = 0, \tag{6}$$

$$k_{\mathrm{U}} u_x(0,t) = -\frac{q_0}{\sqrt{t}}, \qquad t > 0 \tag{7}$$

with

$$a_1 = \alpha_1^2 = \frac{k_{\mathrm{U}}}{\rho_{\mathrm{U}} c_{\mathrm{U}}}, \qquad a_2 = \alpha_2^2 = \frac{k_{\mathrm{F}}}{\rho_{\mathrm{F}} c_{\mathrm{F}}}, \qquad b = \frac{\varepsilon\rho_{\mathrm{W}} c_{\mathrm{W}}}{\rho_{\mathrm{U}} c_{\mathrm{U}}},$$

$$d = \frac{\varepsilon\gamma\mu}{K}, \qquad \rho = \frac{\rho_{\mathrm{W}} - \rho_{\mathrm{I}}}{\rho_{\mathrm{W}}}, \qquad \alpha = \varepsilon\rho_{\mathrm{I}}\lambda,$$

$$\beta = \frac{\varepsilon^2\rho_{\mathrm{I}}(c_{\mathrm{W}} - c_{\mathrm{I}})\gamma\mu}{K} = \varepsilon d\rho_{\mathrm{I}}(c_{\mathrm{W}} - c_{\mathrm{I}}).$$

Problem I consists of equations (1–7), while by Problem II we mean the system (1–6) and (8), where

$$u(0,t) = B > 0, \qquad t > 0. \tag{8}$$

Problem II was previously studied in [7]. In Section 2 we show that the similarity solution of Problem I is given by the expressions (13), where the coefficient ξ, which characterizes the free boundary $s(t)$, must satisfy the equation (10). We also study preliminary properties of some real functions which appear in equation (10). In Section 3 we prove the existence and uniqueness of the similarity solution for different values of the physical parameters ρ, β and d. In Section 4, we discuss the equivalence of the problems I and II, and we extend some existence results for Problem II obtained in [7].

2. Similarity Solutions

We will look for similarity solutions of Problem I in different cases according to the value of parameters ρ, β y d, following the methods introduced in [7, 15].

First of all, we note that the function

$$u(x,t) = \Phi(\eta), \quad \text{with } \eta = \frac{x}{2\alpha_1\sqrt{t}}$$

is a solution of (1) if and only if Φ satisfies the following equation

$$\frac{1}{2}\Phi''(\eta) + \left(\eta - \frac{b\rho}{\alpha_1}\sqrt{t}\dot{s}(t)\right)\Phi'(\eta) = 0.$$

If we assume

$$s(t) = 2\xi\alpha_1\sqrt{t},$$

we obtain that

$$\Phi(\eta) = C_1 + C_2\int_0^\eta \exp(-r^2 + 2b\rho\xi r)\,\mathrm{d}r,$$

where ξ, C_1, y, C_2 are constants to be determined. It is well known that the function

$$v(x,t) = \Psi(\eta)$$

is solution of (2) if and only if

$$\Psi(\eta) = C_3 + C_4\,\mathrm{erf}(\eta),$$

where C_3 and C_4 are constants to be also determined and the error function (erf) and the complementary error function (erfc) are defined by

$$\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-r^2)\,\mathrm{d}r, \qquad \mathrm{erfc}\,(x) = 1 - \mathrm{erf}(x).$$

From conditions (3), (7) and (5) we deduce that

$$C_1 = 2d\rho\alpha_1^2\xi^2 + \frac{2q_0\alpha_1}{k_U}g(2b\rho,\xi),$$
$$C_2 = -\frac{2q_0\alpha_1}{k_U},$$

$$C_3 = \frac{2d\rho\alpha_1^2\xi + A\ \mathrm{erf}\left(\frac{\alpha_1}{\alpha_2}\xi\right)}{\mathrm{erfc}\left(\frac{\alpha_1}{\alpha_2}\xi\right)},$$

$$C_4 = -\frac{2d\rho\alpha_1^2\xi + A}{\mathrm{erfc}\left(\frac{\alpha_1}{\alpha_2}\xi\right)}, \tag{9}$$

where $g = g(p, y)$ is defined by

$$g(p, \xi) = \int_0^y \exp(pyr - r^2)\,\mathrm{d}r.$$

Therefore, the similarity solution is completely determined once the constant ξ is chosen. This is done by imposing the condition (4), which yields that ξ must be a solution of the following equation

$$q_0 \exp\left((p-1)y^2\right) - K_2 F(m, y) = \delta y + \nu y^3, \qquad y > 0, \tag{10}$$

where

$$F(m, y) = (A + my^2)\frac{\exp\left(-\gamma_0^2 y^2\right)}{\mathrm{erfc}\left(\gamma_0 y\right)} \tag{11}$$

and the constants $K_2, m, \delta, \nu, \gamma_0$ are defined as follows:

$$K_2 = \frac{K_F}{\alpha_2\sqrt{\pi}}, \qquad m = 2d\rho\alpha_1^2, \qquad \gamma_0 = \frac{\alpha_1}{\alpha_2} > 0,$$

$$\delta = \alpha\alpha_1 > 0, \qquad \nu = 2\beta\rho\alpha_1^3, \qquad p = 2b\rho. \tag{12}$$

THEOREM 1. *The free boundary Problem I has the similarity solutions*

$$s(t) = 2\xi\alpha_1\sqrt{t},$$

$$u(x, t) = m\xi^2 + \frac{2q_0\alpha_1}{K_U}g(p, \xi) - \frac{2q_0\alpha_1}{K_U}\int_0^{x/2\alpha_1\sqrt{t}} \exp\left(pyr - r^2\right)\mathrm{d}r,$$

$$v(x, t) = \frac{m\xi^2 + A\ \mathrm{erf}\left(\gamma_0\xi\right)}{\mathrm{erfc}\left(\gamma_0\xi\right)} - \frac{m\xi^2}{\mathrm{erfc}\left(\gamma_0\xi\right)}\ \mathrm{erf}\left(\frac{x}{2\alpha_1\sqrt{t}}\right), \tag{13}$$

if and only if the coefficient ξ satisfies the equation (10).

In order to analyze (10) we need some preliminary results. The following result is proved in [7].

PROPOSITION 2. *If $m > 0$, then F grows from A to $+\infty$, when y grows from* 0 *to* $+\infty$. *If $m < 0$, then F has a unique positive maximum, from which it decreases to $-\infty$. In both cases, $F(m, y) \sim \sqrt{\pi}\gamma_0 m y^3$ when $y \to +\infty$.*

PROPOSITION 3. *For all $p > 0$, we have*

1. $g(p, y) \geqslant \frac{1}{py}\left(\exp\left((p-1)y^2\right)\right) - \exp\left(-y^2\right), \qquad y > 0$
2. $g_y(p, y) \geqslant \exp\left((p-1)y^2\right) + \frac{p}{2}\left(1 - \exp\left(-y^2\right)\right) > 0, \qquad y > 0$

3. $g(p,0)=0, \qquad g_y(p,0)=1, \qquad g(p,+\infty)=+\infty.$
4. $\lim_{y\to+\infty}\frac{g(p,y)}{y^2}=0$ if $p\leqslant 0$, and $\lim_{y\to+\infty}\frac{g(p,y)}{y^2}=+\infty$ if $p>0$.
5. $\lim_{y\to+\infty}\frac{y\,g(p,y)}{\exp\left((p-1)y^2\right)}=\begin{cases}\frac{1}{p-2}\ if\ p>2,\\ +\infty\ if\ p\leqslant 2.\end{cases}$

Proof. The assertions 1, 2, 3 were proved in [7], and 4 and 5 easily follow from the identity (see [1])

$$\frac{2}{\sqrt{\pi}}g(p,y)=\exp\left(\frac{p^2y^2}{4}\right)\left(\operatorname{erf}\left(\frac{py}{2}\right)+\operatorname{erf}\left(\frac{(2-p)y}{2}\right)\right).$$

For example, if $p>2$ we have

$$\begin{aligned}
\lim_{y\to+\infty}\frac{yg(p,y)}{\exp\left((p-1)y^2\right)} &= \lim_{y\to+\infty}\frac{\sqrt{\pi}}{2}\frac{y\left(\operatorname{erf}\left(\frac{py}{2}\right)+\ \operatorname{erf}\left(\frac{(2-p)y}{2}\right)\right)}{\exp\left(-\frac{(p-2)^2y^2}{4}\right)}\\
&= \lim_{y\to+\infty}\left[\frac{\sqrt{\pi}}{2}\frac{\left(\operatorname{erf}\left(\frac{py}{2}\right)+\ \operatorname{erf}\left(\frac{(2-p)y}{2}\right)\right)}{-\frac{(p-2)^2}{2}\,y\exp\left(-\frac{(p-2)^2y^2}{4}\right)}+\right.\\
&\qquad\left.+\frac{\left(\frac{p}{2}\exp\left(-\frac{p^2y^2}{4}\right)+\frac{2-p}{2}\exp\left(-\frac{(2-p)^2y^2}{4}\right)\right)}{-\frac{(p-2)^2}{2}\exp\left(-\frac{(p-2)^2y^2}{4}\right)}\right]\\
&= \frac{1}{p-2}-\lim_{y\to+\infty}\frac{p}{(p-2)^2}\exp\left((1-p)y^2\right)
\end{aligned} \tag{14}$$

and therefore

$$\lim_{y\to+\infty}\frac{yg(p,y)}{\exp\left((p-1)y^2\right)}=\frac{1}{p-2}\quad \text{if } p>2. \tag{15}$$

Moreover, by using the first equality in (14) it follows that

$$\lim_{y\to+\infty}\frac{yg(p,y)}{\exp\left((p-1)y^2\right)}=+\infty,\quad \text{if } 0<p\leqslant 2. \tag{16}$$

Finally

$$\lim_{y\to+\infty}\frac{g(p,y)}{\exp\left((p-1)y^2\right)}=\begin{cases}+\infty & \text{if } p<0,\\ 0 & \text{if } p>2,\end{cases}$$

from which follows immediately that

$$\lim_{y\to+\infty}\frac{yg(p,y)}{\exp\left((p-1)y^2\right)}=+\infty\quad \text{if } p<0. \qquad \square$$

In order to study function

$$H(y)=F(m,y)+\frac{\nu}{K_2}y^3, \tag{17}$$

we prove the following propositions.

LEMMA 4. *Let f and h be two real functions defined in* $[0,+\infty)$, *with* $h>0$, $h'<0$, $\frac{f'}{h'}$ *strictly increasing, such that* $\lim_{x\to+\infty}\frac{f'(x)}{h'(x)}=\lim_{x\to+\infty}\frac{f(x)}{h(x)}$. *Then*

1. $\frac{f(0)}{h(0)} > \frac{f'(0)}{h'(0)}$.
2. $\frac{f}{h}$ is a strictly increasing function.

Proof. We follow the same arguments as in ([8], p. 106). □

PROPOSITION 5. *For all* $\alpha\geqslant 2$

$$\Phi_\alpha(y)=\frac{1}{\sqrt{\pi}}\frac{y^{\alpha-1}\exp(-y^2)}{\operatorname{erfc}(y)}-y^\alpha,\qquad y>0$$

is an increasing function of y.

Proof. Having in mind that

$$\Phi_\alpha(y)=y^{\alpha-2}\Phi_2(y),$$

it is sufficient to prove that Φ_2 is increasing function. We note that

$$\Phi_2(y)=\frac{\frac{1}{\sqrt{\pi}}y\exp(-y^2)-y^2\operatorname{erfc}(y)}{\operatorname{erfc}(y)}=\frac{f(y)}{h(y)},$$

where functions f and h are defined for $y>0$ by

$$f(y)=\frac{1}{\sqrt{\pi}}y\exp(-y^2)-y^2\operatorname{erfc}(y),\qquad h(y)=\operatorname{erfc}(y).$$

Then, h is positive and decreasing. Also we have

$$\frac{f'(y)}{h'(y)}=-\frac{1}{2}+\sqrt{\pi}\,y\exp(y^2)\operatorname{erfc}(y),$$

from which, $\frac{f'}{h'}$ is increasing, since the function

$$Q(y)=\sqrt{\pi}\,y\exp(y^2)\operatorname{erfc}(y)$$

is strictly increasing and verifies $Q(0)=0$ and $Q(+\infty)=1$. We have

$$\frac{f(0)}{g(0)}=0,\qquad \frac{f'(0)}{h'(0)}=-\frac{1}{2}$$

and

$$\lim_{y\to+\infty}\frac{f(y)}{h(y)}=\lim_{y\to+\infty}\frac{f'(y)}{h'(y)}=\frac{1}{2}.$$

Then, by Lemma 4, it follows that Φ_2 is increasing. □

PROPOSITION 6. *If* $m>0$ *and* $\nu\geqslant -m\sqrt{\pi}\gamma_0K_2$ *then the function* H, *defined in* (17), *is strictly increasing.*

Proof. We can write

$$H(y) = A\frac{\exp(-\gamma_0^2 y^2)}{\operatorname{erfc}(\gamma_0 y)} + \frac{m\sqrt{\pi}}{\gamma_0} y\Phi_2(\gamma_0 y) + \left(\sqrt{\pi} m\gamma_0 + \frac{\nu}{K_2}\right) y^3. \tag{18}$$

The first and third terms on right-hand side of (18) are strictly increasing functions of y, and the second one is also increasing by Proposition 5. □

3. Existence and Uniqueness of Similarity Solutions

Now, we are in position to discuss the solvability of the equation (10). We introduce the following function

$$Q_0(y) = \frac{\delta y + \nu y^3 + K_2 F(m, y)}{\exp((p-1)y^2)} \tag{19}$$

defined for $y > 0$ which verifies $Q_0(0) = K_2 A > 0$.

THEOREM 7. *Let m be a positive real number. We define the following sets in the plane* ν, p*:*

$$\begin{aligned} R_1 &= \{(\nu, p) \in \mathbb{R}^2 : -\sqrt{\pi} K_2 \gamma_0 m \leqslant \nu,\ p \leqslant 1\}, \\ R_2 &= \mathbb{R}^2 - R_1. \end{aligned}$$

We have

If $(\nu, p) \in R_1$ *then the Problem I has a unique similarity solution if and only if*

$$q_0 > \frac{K_F}{\alpha_2\sqrt{\pi}} A.$$

If $(\nu, p) \in R_2$ *then the Problem I has a similarity solution if and only if*

$$0 < q_0 \leqslant \max_{y \geqslant 0} Q_0(y).$$

Proof. We split the proof in four cases. To prove the existence (and uniqueness) of similarity solution to Problem I, it is necessary and sufficient to verify that the equation (10) has a (unique) solution. The equation (10) has a solution ξ if and only if $q_0 = Q_0(\xi)$.

For the case $m > 0$, $\nu \geqslant 0$ and $p \leqslant 1$, by Proposition 2, $F(m, y)$ is increasing, and then $Q_0(y)$ is in this case strictly increasing, since $Q_0(0) = K_2 A$, from which the assertion follows.

For the case $m > 0$, $\nu \geqslant 0$ and $p > 1$, by Proposition 2 we have

$$\delta y + \nu y^3 + K_2 F(m, y) = \mathrm{O}(y^3) \quad \text{when} \quad y \to +\infty$$

and then

$$\lim_{y\to+\infty} Q_0(y) = 0$$

and besides $Q_0(0) = K_2 A$. Then, the function Q_0 have a finite positive maximum.

If $\nu < -\sqrt{\pi} K_2 \gamma_0 m$, then

$$\lim_{y\to+\infty} (\delta y + \nu y^3 + F(m, y)) = -\infty$$

and $Q_0(0)=K_2A$. Then $Q_0(y)<0$ if y is sufficiently large. If $\nu \geqslant -\sqrt{\pi}K_2\gamma_0 m$ and $p>1$, then $Q_0(y)\geqslant 0$ for all $y>0$ and $\lim_{y\to+\infty} Q_0(y)=0$. Therefore, in both cases Q_0 has a finite positive maximum in R^+ and then the thesis holds.

Finally, for $m>0$, $-\sqrt{\pi}K_2\gamma_0 m \leqslant \nu < 0$ and $p\leqslant 1$, we have that by Proposition 6, $Q_0(y)$ is strictly increasing and $Q_0(0) = K_2A$, and then the thesis holds. □

Remark 8. For $m=0$, that is, $d\rho=0$, there exist a unique solution of equation (10) if and only if the inequality $q_0 > K_2A$ is verified. This result has already been found in [14].

Remark 9. We note that in the case $p>1$, if $\max_{y\geqslant 0} Q_0(y)>q_0>K_2A$ there exist at least two solutions. On the other hand, if q_0 is sufficiently small, then there exists a unique solution. The situation is a bit different in the Problem II, studied in [7], where it was proved the existence and uniqueness of similarity solutions in the case $m>0$, $\nu\geqslant 0$, $p\leqslant 2$.

Similarly, we can obtain the following result.

THEOREM 10. *Let $m<0$. We define the sets*

$$R_3 = \{(\nu, p)\in\mathbb{R}^2 : \nu > -\sqrt{\pi}K_2\gamma_0 m,\ \mathrm{p}\leqslant 1\},$$
$$R_4 = \mathbb{R}^2 - R_3.$$

Then

1. *If $(\nu, p)\in R_3$, there exists a solution when*
$$q_0 > K_2A.$$
2. *If $(\nu, p)\in R_4$, there exists a solution when*
$$0 < q_0 \leqslant \max_{y>0} Q_0(y).$$

Proof. By Proposition 2 we have $F(m, y) \sim \sqrt{\pi}\gamma_0 m y^3$, and then

$$\delta y + \nu y^3 + K_2F(m, y) \sim \nu + K_2\sqrt{\pi}\gamma_0 m y^3$$

and it follows that if $(\nu, p)\in R_3$ then

$$\lim_{y\to+\infty} Q_0(y) = +\infty$$

from which we have $[K_2A, +\infty) \subset \text{Range}\,(Q_0)$. This proves part 1.

If $(\nu, p)\in R_4$, it is easy to see that the function Q_0 has a positive finite maximum, and then part 2 is proved. □

Remark 11. When $c_\mathrm{W} \neq c_\mathrm{I}$ the temperature

$$u^* = \frac{\lambda}{c_\mathrm{I} - c_\mathrm{W}}$$

is the intersection of the two lines representing the energy of the solid phase and the liquid phase, as a function of the temperature.

Note that the similarity solution is such that the interface temperature is constant for all time, being

$$u(s(t), t) = v(s(t), t) = m\xi^2.$$

Then it seems appropriate to say that the similarity solution is physically acceptable if [7]

$$m\xi^2 < \frac{\lambda}{c_{\mathrm{I}} - c_{\mathrm{W}}} \quad \text{when } c_{\mathrm{I}} > c_{\mathrm{W}},$$
$$m\xi^2 > \frac{\lambda}{c_I - c_W} \quad \text{when } c_{\mathrm{I}} < c_{\mathrm{W}},$$

that is, if

$$2d\rho\alpha_1^2(c_{\mathrm{I}} - c_{\mathrm{W}})\xi^2 < \lambda. \tag{20}$$

Then, it is easy to verify that

1. If $d\rho > 0$ and $c_{\mathrm{I}} < c_{\mathrm{W}}$ (i.e. $m > 0$ and $\nu > 0$) then (20) is always satisfied, and therefore all solutions provided by Theorem 7 are physically acceptable.
2. If $d\rho > 0$ and $-\frac{\sqrt{\pi}K_2}{\varepsilon\rho_{\mathrm{I}}\alpha_2} \leqslant c_{\mathrm{W}} - c_{\mathrm{I}} < 0$ (i.e. $m > 0$ and $-\sqrt{\pi}K_2\gamma_0 m < \nu < 0$) and $p \leqslant 1$, then, if q_0 is sufficiently small the solution provided by Theorem 7 is physically acceptable, while if q_0 is too large, the similarity solution is not physically acceptable. More specifically, the similarity solution is physically acceptable if and only if

   $$K_2 A < q_0 < Q_0\left(\sqrt{\frac{\lambda}{2d\rho(c_{\mathrm{I}} - c_{\mathrm{W}})\alpha_1^2}}\right).$$

   In the same conditions, but with $p > 1$, there exists a physically acceptable solution if and only if

   $$q_0 \in Q_0\left(\left[0, \sqrt{\frac{\lambda}{2d\rho(c_{\mathrm{I}} - c_{\mathrm{W}})\alpha_1^2}}\right]\right).$$
3. If $d\rho < 0$ and $c_{\mathrm{I}} > c_W$ (i.e. $m < 0$ and $\nu > 0$) then (20) is always true and then the solutions are physically acceptable.

4. Relationship between Problems I and II

Let (s, u, v) be given by (13), for some constant $\xi > 0$. Then $u(0, t)$ is a constant given by

$$u(0, t) = m\xi^2 + \frac{2q_0}{K_{\mathrm{U}}}g(p, \xi) > 0. \tag{21}$$

Then, we can consider the Problem II, by imposing this new temperature as $u(0, t)$ at the fixed face $x = 0$.

THEOREM 12. *Let $m > 0$, $\nu > 0$, $p \leqslant 1$ and $q_0 > K_2 A$. If (s, u, v) is the unique similarity solution of Problem I, then (s, u, v) is the unique similarity solution of Problem II, provided the constant B in the condition* (8) *is given by*

$$B = m\xi^2 + \frac{2q_0\alpha_1}{K_U}g(p, \xi), \tag{22}$$

where ξ is the unique solution of equation (10).

Proof. We know that (s, u, v) is given by (13) where ξ is the unique solution of (10), which can be written as

$$Q_0(y) = q_0, \qquad y > 0. \tag{23}$$

By the results obtained in [7], there exists one unique solution to Problem II, with the parameter B defined in (22), given by

$$\bar{s}(t) = 2\bar{\xi}\alpha_1\sqrt{t},$$

$$\bar{u}(x,t) = B - \frac{m\bar{\xi} - B}{g(p,\bar{\xi})}\int_0^{x/2\alpha_1\sqrt{t}} \exp\,(p\bar{\xi}r - r^2)\,\mathrm{d}r,$$
$$v(x,t) = \frac{m\bar{\xi}^2\mathrm{erfc}\left(\frac{x}{2\alpha_2\sqrt{t}}\right) + A\left(\mathrm{erf}(\gamma_0\bar{\xi}) - \mathrm{erf}\left(\frac{x}{2\alpha_2\sqrt{t}}\right)\right)}{\mathrm{erfc}\,(\gamma_0\bar{\xi})}, \tag{24}$$

where $\bar{\xi}$ is the unique solution of

$$\frac{\sqrt{\pi}}{2}K_1(B - my^2)\frac{\exp\,((p-1)y^2)}{g(p,y)} - K_2F(m,y) = \delta y + \nu y^3, \qquad y > 0 \tag{25}$$

and $K_1 = \frac{K_{\mathrm{U}}}{\alpha_1\pi}$. It is easy to see that the solutions given by (13) and (24) are coincident if and only if $\xi = \bar{\xi}$. Then, it is sufficient to see that ξ is a solution of (25). In fact, we have

$$\begin{aligned}
&\frac{\sqrt{\pi}}{2}K_1(B - m\xi^2)\frac{\exp\,((p-1)\xi^2)}{g(p,\xi)} - K_2F(m,\xi) - \delta\xi + \nu\xi^3\\
&\quad = \frac{\sqrt{\pi}}{2}K_1\left(m\xi^2 + \frac{2q_0\alpha_1}{K_{\mathrm{U}}}g(p,\xi) - m\xi^2\right)\times\\
&\qquad \times\frac{\exp\,((p-1)\xi^2)}{g(p,\xi)} - K_2F(m,\xi) - \delta\xi + \nu\xi^3\\
&\quad = q_0(p-1)\xi^2 - K_2F(m,\xi) - \delta\xi + \nu\xi^3 = 0
\end{aligned}$$

using the equations (25) and (22). □

Suppose that (s, u, v) is a solution to Problem I, with the boundary condition (7). By the results of Section 2, we know that s, u, v are given by (13), where ξ must satisfy the equation (10). For this solution, the temperature in the fixed boundary is constant and equal to $B = u(0,t) = T_0(q_0, \xi)$, where T_0 is the real function defined by

$$T_0(q,y) = my^2 + \frac{2}{\sqrt{\pi}}\frac{q}{K_1}g(p,y), \qquad q > 0, \quad y > 0. \tag{26}$$

Assuming that $q > 0$, we will describe some properties of function T_0. First of all, we note that $T_0(q,0) = 0$. Besides, it follows from the Proposition 3 that if $m > 0$ and $p > 0$, then $T_0(q,y)$ is an increasing function in both of its arguments, with $T_0(q,+\infty) = +\infty$. If $m < 0$ and $p > 0$, then $T_0(q,+\infty) = +\infty$, and if $m < 0$ and $p \leqslant 0$, then $T_0(q,+\infty) = -\infty$. Finally, if $m > 0$ and $p < 0$ then $T_0(q,+\infty) = +\infty$.

Suppose that $m > 0$. For each $\bar{\xi} > 0$ let

$$\bar{q}_0 = Q_0(\bar{\xi}), \tag{27}$$

where Q_0 is the function defined by (19). Let

$$\bar{B} = T_0(\bar{q}_0, \bar{\xi}) = m\bar{\xi}^2 + \frac{2}{\sqrt{\pi}} \frac{\bar{q}_0}{K_1} g(p, \bar{\xi}),$$

then a solution to Problem I with $q_0 = \bar{q}_0$, which is given by (13) with $\xi = \bar{\xi}$ (because of (27)), corresponds to a solution to Problem II with $B = \bar{B}$. Then, given $B > 0$, we can show the existence of solution for the Problem II, by proving that B belongs to the image set of the function

$$J(\cdot) = T_0(Q_0(\cdot), \cdot).$$

Being $m > 0$, we know that if $\nu \geqslant -\sqrt{\pi} K_2 m \gamma_0$ then

$$Q_0(0) = K_2 A, \qquad Q_0(y) > 0 \,\forall\, y > 0.$$

For $q_0 > 0$, we have

$$T_0(q_0, \xi) > T_0(0, \xi) = m\xi^2 \to +\infty \quad \text{if } \xi \to +\infty,$$

where the convergence is independent of parameter p and is uniform with respect to q_0. It follows that

$$J(0) = 0, \qquad J(\xi) > 0 \,\forall\, \xi > 0, \qquad J(+\infty) = +\infty.$$

Therefore we have proved the existence of solution to Problem II under the conditions $m > 0$ and $\nu > -\sqrt{\pi} K_2 m \gamma_0$.

If we also have $0 \leqslant p \leqslant 1$, then $Q_0(y)$ is increasing for $y > 0$, and taking into account the properties of T_0 described above, it follows

$$J' = \frac{\partial T_0}{\partial q} Q_0' + \frac{\partial T_0}{\partial \xi} > 0,$$

that is, J is strictly increasing. Therefore, we have that Range $(J) = \Re_0^+$, and then the solution to Problem II is unique in the class of similarity solutions.

If $\nu < -\sqrt{\pi} K_2 m \gamma_0$, then a sufficient condition to have $\bar{q}_0 > 0$, is that

$$\bar{\xi} < \min\{\text{Zeroes}\,(\delta y + \nu y^3 + K_2 F(m, y), y)\}.$$

In particular

$$\bar{\xi} < \sqrt{\frac{\delta}{|\nu|}} \quad \Rightarrow \quad \bar{q}_0 = Q_0(\bar{\xi}) > 0.$$

Then, a sufficient condition in order to have the existence of similarity solutions to Problem II, if $m > 0$ and $\nu < -\sqrt{\pi} K_2 m \gamma_0$, is that

$$B < m\frac{\delta}{|\nu|} + \frac{2}{\sqrt{\pi}} \frac{M}{K_1} g\left(p, \sqrt{\frac{\delta}{|\nu|}}\right), \tag{28}$$

where

$$M = \min_{0 \leqslant y \leqslant \sqrt{\frac{\delta}{|\nu|}}} Q_0(y).$$

Indeed, this is the case, because $J(\xi)$ is a continuous function and $Q_0 > 0$ in $\left[0, \sqrt{\frac{\delta}{|\nu|}}\right]$. Then, we have proved the following.

THEOREM 13. *Let $m > 0$. If $\nu \geqslant -\sqrt{\pi} K_2 m \gamma_0$, then there exists a similarity solution to Problem II. If, in addition, $0 \leqslant p \leqslant 1$, then the similarity solution is unique. For the case $\nu < -\sqrt{\pi} K_2 m \gamma_0$, the inequality* (28) *for B is a sufficient condition in order to have the existence of a solution to Problem II.*

Remark 14. We note that $M > K_2 A$ for $m > 0$, $\nu < -\sqrt{\pi} K_2 m \gamma_0$ and $0 \leqslant p \leqslant 1$, and therefore we can replace M by $K_2 A$ in (28) in order to obtain a weaker conditon, but that depends only on the parameters of the model.

Remark 15. The last Theorem, extends a result of [7], where it was proved that if $m > 0$, $\nu < 0$, then there exists a solution to Problem II when

$$B < \frac{m\delta}{|\nu|}.$$

Now we consider $m < 0$ and $\nu > \sqrt{\pi} K_2 |m| \gamma_0$. We know that

$$\lim_{y \to +\infty} Q_0(y) = \begin{cases} +\infty & \text{if } p \leqslant 1, \\ 0+ & \text{if } p > 1, \end{cases}$$

and then for both cases, there exist $y_0 > 0$ such that

$$Q_0(y) > 0 \quad \text{if } y > y_0.$$

On the other hand, we have $J(0) = 0$ and taking into account (15), (16) and the equality

$$\lim_{y \to +\infty} \frac{\delta y + \nu y^3 + K_2 F(m, y)}{y^3} = \nu + \sqrt{\pi} K_2 m \gamma_0 > 0,$$

it follows that

$$\begin{aligned} \lim_{y \to +\infty} J(y) &= \lim_{y \to +\infty} y^2 \left(m + \frac{2}{\sqrt{\pi} K_1} (\delta y + \nu y^3 + F(m, y)) \frac{g(p, y)}{y^2 \exp((p-1)y)} \right) \\ &= \lim_{y \to +\infty} y^2 \left(m + \frac{2}{\sqrt{\pi} K_1} \frac{\delta y + \nu y^3 + F(m, y)}{y^3} \frac{y\, g(p, y)}{\exp((p-1)y)} \right) \\ &= +\infty, \end{aligned}$$

when either

$$p > 2 \quad \text{and} \quad m + \frac{2(\nu + \sqrt{\pi} K_2 m \gamma_0)}{\sqrt{\pi} K_1 (p-2)} > 0 \tag{29}$$

or $p \leqslant 2$. Then, in these cases we have obtained

$$\text{Range}(J) = \Re_0^+$$

and the following proposition holds.

PROPOSITION 16. *Suppose $m < 0$ and $\nu > \sqrt{\pi} K_2 |m| \gamma_0$, then:*

1. *If $p \leqslant 2$ the Problem II has a similarity solution if*

$$B > \max(0, J(y_0)). \tag{30}$$

2. *If $p > 2$ the Problem II has a similarity solution if* (29) *and* (30) *are verified.*

In the conditions of the Proposition 16 we note that

$$\begin{aligned} p &\leqslant 1 \quad \Rightarrow \quad \lim_{y \to +\infty} Q_0(y) = +\infty, \\ p &> 1 \quad \Rightarrow \quad \lim_{y \to +\infty} Q_0(y) = 0, \end{aligned} \tag{31}$$

and then if ξ_B verifies $J(\xi_B) = B$, we can deduce that if $p \leqslant 1$, then

$$B \to +\infty \quad \Rightarrow \quad q_0 = Q_0(\xi_B) \to +\infty,$$

while if $p > 1$

$$B \to +\infty \quad \Rightarrow \quad q_0 = Q_0(\xi_B) \to 0.$$

5. Conclusions

A mathematical model for thawing in a saturated semi-infinite porous medium with a density jump is considered. A flux condition of the type (7) on the fixed face $x = 0$ is imposed. Similarity solutions for different cases depending on three physical parameters are analysed (see Theorem 1). This explicit solution is obtained if and only if an inequality for the thermal coefficient q_0 is verified (see Theorems 7 and 10). Moreover, by studing the application T_0 defined by (26), some results of [7] concerning the similar case with a constant temperature condition on the fixed face $x = 0$ of the type (8), are improved (see Theorem 13). Taking into account the relation (22) the two free boundary problems are equivalent (see Theorem 12).

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