UNIVERSIDAD NACIONAL DE ROSARIO
FACULTAD DE CIENCIAS EXACTAS, INGENIERIA Y AGRIMENSURA

I.S.S.N. 03260690

CUADERNOS

DEL

INSTITUTO DE MATEMATICA "BEPPO LEVI"

V SEMINARIO SOBRE PROBLEMAS DE FRONTERA LIBRE Y SUS APLICACIONES

D. A. Tarzia (Editor)

Rosario, 19 al 21 de Diciembre de 1994

26

Rosario - República Argentina
1995

Cuadern. Inst. Mat. Beppo Levi, 26 (1995), 27 – 48.

DETERMINATION OF THERMAL COEFFICIENTS IN SEMI-INFINITE MATERIALS WITH MUSHY ZONE IN PHASE-CHANGE PROCESS

Adriana M. GONZÁLEZ – Domingo A. TARZIA

I. INTRODUCTION

We consider a semi-infinite material with mass densities $\rho > 0$ equal in both solid and liquid phases and we can assume, without loss of generality, that the phase-change temperature is $0^{\underline{o}}$ C.

If the material is initially assumed to be liquid at the constant temperature $E > 0$ and a constant temperature $-D < 0$ is imposed on the fixed face x=0, then three distinct regions can be distinguished (for a mathematical and properties description of this simple model see [Ta4]; for the one-phase model see [SoWiAl1]) :

(H_1) The liquid phase, at temperature $\theta_2 = \theta_2(x,t) > 0$, occupying the region $x > r(t)$, $t > 0$.

(H_2) The solid phase, at temperature $\theta_1 = \theta_1(x,t) < 0$, occupying the region $0 < x < s(t)$, $t > 0$.

(H_3) The mushy zone, at temperature 0, occupying the region $s(t) \leq x \leq r(t)$, $t > 0$. We make two assumptions on its structure :

(a) The material in the mushy zone contains a fixed fraction $\epsilon\lambda$ (with $0 < \epsilon < 1$) of the total latent heat $\lambda > 0$, i.e.,

$$(1) \qquad k_1\, \theta_{1_x}(s(t),t) - k_2\, \theta_{2_x}(r(t),t) = \lambda\, \rho \left(\epsilon\, \dot{s}(t) + (1-\epsilon)\, \dot{r}(t)\right), \quad t > 0.$$

(b) The width of the mushy zone is inversely proportional (with constant $\gamma > 0$) to the temperature gradient at the point $(s(t), t)$, i.e.,

$$\theta_{1_x}(s(t),t)\left(r(t) - s(t)\right) = \gamma \ , \ t > 0. \tag{2}$$

We suppose that the temperature $\theta = \theta(x,t)$ of the material is defined by

$$\theta(x,t) = \begin{cases} \theta_1(x,t) < 0 & \text{if} \quad 0 < x < s(t),\ t > 0 \\ 0 & \text{if} \quad s(t) \le x \le r(t),\ t > 0 \\ \theta_2(x,t) > 0 & \text{if} \quad x > r(t),\ t > 0. \end{cases} \tag{3}$$

The governing differential equations for the solid and liquid phases, the continuity of the temperature, the initial and boundary conditions, and an overspecified heat flux condition [Ca, Ta2] on the fixed face $x=0$ [StTa, Ta1 – Ta4] are given by

$$\alpha_1\, \theta_{1_{xx}}(x,t) = \theta_{1_t}(x,t) \ , \ 0 < x < s(t),\ t > 0 \tag{4}$$

$$\alpha_2\, \theta_{2_{xx}}(x,t) = \theta_{2_t}(x,t) \ , \ x > r(t),\ t > 0 \tag{5}$$

$$\theta_1(s(t),t) = \theta_2(r(t),t) = 0 \ , \ t > 0. \tag{6}$$

$$\theta_1(0,t) = -D < 0 \ , \ t > 0 \ , \tag{7}$$

$$\theta_2(x,0) = \theta_2(+\infty,t) = E > 0 \ , \ x > 0,\ t > 0 \ , \tag{8}$$

$$s(0) = r(0) = 0. \tag{9}$$

$$k_1\, \theta_{1_x}(0,t) = \frac{h_o}{\sqrt{t}} \ , \ t > 0 \ , \ \text{with } h_o > 0. \tag{10}$$

where $c_i > 0$, $k_i > 0$ and $\alpha_i = a_i^2 = k_i / \rho c_i > 0$ are the specific heat, the thermal conductivity and the diffusion coefficient for the phase i ($i=1$: solid phase; $i=2$: liquid phase) respectively.

We shall present some of the results obtained in [GoTa]. If by means of a phase-change experiment we are able to measure certain quantities, then we shall find formulae for the simultaneous determination of the unknown coefficients (ϵ, γ : parameters of the mushy zone; λ, ρ, c_1, c_2, k_1, k_2 : thermal coefficients of the material).

The different problems for determining several unknown coefficients have not always an explicit solution; it does exist iff some complementary conditions for the corresponding data are verified. We generalize the results obtained in [StTa] for the particular case $\epsilon=1$ and $\gamma=0$ (i.e., without mushy region) and those obtained in [Ta3] for the one-phase case. In [Ta2] several references on free-moving boundary problems and determination of physical coefficients are given.

We shall consider the simple mushy zone model for the two-phase Stefan problem for determining one unknown thermal coefficient of a semi-infinite material with an overspecified condition on the fixed face, supposing not knowing the free boundaries $x=s(t)$ and $x=r(t)$. The results obtained for the eight possible cases are considered in Table I which shows both the necessary and sufficient conditions to be verified by the data for the existence and uniqueness of the solution and the expression of the corresponding unknown coefficient.

We shall consider the same model for determining two unknown thermal coefficients of a semi-infinite material with an overspecified condition on the fixed face, supposing known the expression for the moving boundary $x=s(t)$. The results obtained for the twenty-eight possible cases are considered in Table II which shows both the necessary and sufficient conditions to be verified by the data for the existence of the solution and the expression of the corresponding unknown coefficients. There are several cases where the moving boundary problem has a unique solution iff some conditions are verified.

The functions and the restrictions used in the text, Tables I and II are summarized in Appendix I and Appendix II respectively.

II. DETERMINATION OF ONE UNKNOWN THERMAL COEFFICIENT

Taking into account the hypothese $(H_1) - (H_3)$ we can formulate the following

PROBLEM (P_1) : Find the free boundaries $x = s(t)$ and $x = r(t)$, defined for $t > 0$ with $0 < s(t) < r(t)$ and $s(0) = r(0) = 0$, the temperature $\theta = \theta(x,t)$, defined by (I-3) for $x > 0$ and $t > 0$, and one of the eight unknown thermal coefficients ϵ, γ, λ, ρ, c_1, c_2, k_1, k_2 such that they satisfy the conditions (I-1), (I-2), (I-4) – (I-10) where $D > 0$, $E > 0$ and $h_o > 0$ are data and they must be known or determined by an experience of phase-change [ArLaTa].

The solution of this problem is given by [CaJa, Ru, SoWiAl2, Ta1, Ta4]

(1) $$\theta_1(x,t) = -D + \frac{D}{f\left(\frac{\sigma}{a_1}\right)} f\left(\frac{x}{2 a_1 \sqrt{t}}\right),$$

(2) $$\theta_2(x,t) = \frac{-E f\left(\frac{\omega}{a_2}\right)}{1 - f\left(\frac{\omega}{a_2}\right)} + \frac{E}{1 - f\left(\frac{\omega}{a_2}\right)} f\left(\frac{x}{2 a_2 \sqrt{t}}\right),$$

(3) $s(t) = 2 \sigma \sqrt{t}, \sigma > 0,$ (4) $r(t) = 2 \omega \sqrt{t}, \omega > \sigma,$

where f is the error function, the coefficient ω is given by

(5) $$\omega = \omega(\sigma) = a_1 W\left(\frac{\sigma}{a_1}\right),$$

and, the coefficient σ and the unknown thermal coefficient are obtained by solving the following system of equations

$$(6)\quad \begin{cases} \text{(a)} \quad \dfrac{h_o}{\lambda\,\rho\,a_1}\exp\left(-\dfrac{\sigma^2}{a_1^{\,2}}\right) - \dfrac{E\,k_2}{\lambda\,\rho\,a_1\,a_2\,\sqrt{\pi}}\,F_1\left(\dfrac{\omega(\sigma)}{a_2}\right) = G\left(\dfrac{\sigma}{a_1}\right)\,, \\ \text{(b)} \quad \dfrac{a_1}{k_1}\,f\left(\dfrac{\sigma}{a_1}\right) = \dfrac{D}{h_o\,\sqrt{\pi}}\,. \end{cases}$$

The eight possible cases for Problem (P_1) are considered in Table I. We remark here that the coefficient ω is always given by the expression (5) as a function of σ and a_1 (a_1 can also be an unknown in some cases, e.g. cases 5, 6 and 8) [GoTa].

III. DETERMINATION OF TWO UNKNOWN THERMAL COEFFICIENTS

Taking into account the hypothese $(H_1) - (H_3)$ we can formulate the following

PROBLEM (P_2) : Find the free boundary $x=r(t)$, defined for $t > 0$ with $r(0)=0$, the temperature $\theta=\theta(x,t)$, defined by (I-3) for $x > 0$ and $t > 0$, and two of the eight unknown thermal coefficients ϵ, γ, λ, ρ, c_1, c_2, k_1, k_2 such that they satisfy the conditions (I-1), (I-2), (I-4) – (I-10) where the moving boundary $x=s(t)$, defined for $t > 0$ with $s(0)=0$, is given by (II-3) with a known coefficient $\sigma > 0$ and D, E, $h_o > 0$ are data and they must be known or determined by an experience of phase-change [ArLaTa].

The solution of that problem is given by (II-1), (II-2) and (II-4) where the coefficient ω and the unknown thermal coefficients are obtained by solving the system of equations (II-6). The twenty-eight cases for Problem (P_2) (cases 9 to 36) are considered in Table II. Now, we shall prove the properties corresponding only for the determination of k_1 and k_2 (case 9).

THEOREM 1 (Case 9). – The necessary and sufficient condition for Problem (P_2), with ω, k_1 and k_2 unknown, to have a unique solution is that data $\sigma > 0$, $D > 0$,

$E > 0$, $h_o > 0$, mushy zone coefficients $0 < \epsilon < 1$ and $\gamma > 0$, and thermal coefficients of the phase-change material λ, ρ, c_1, $c_2 > 0$ do verify the conditions

$$(1) \quad \frac{h_o}{E \rho \sigma c_2} > 1 + \frac{\gamma}{D} + \frac{\lambda}{E c_2}\Big(1 + \frac{(1-\epsilon)\gamma}{D}\Big) \quad , \quad \frac{D \rho \sigma c_1}{h_o \sqrt{\pi}} < H_{20}(x_{23}) \; ,$$

where x_{23} is the unique positive zero of function H_{23} .

In such case, the solution is given by (II-1), (II-2) and (II-4) with

$$(2) \quad \omega = \sigma \; H_{25}(\xi_1) \; , \qquad k_1 = \rho \, \sigma^2 \, c_1 \, \frac{1}{\xi_1^{\;2}} \; , \qquad k_2 = \rho \, \sigma^2 \, c_2 \, \frac{H_{25}^2(\xi_1)}{B^2} \; ,$$

where ξ_1 is the unique solution of the equation

$$(3) \qquad H_{20}(x) = \frac{D \rho \sigma c_1}{h_o \sqrt{\pi}} \; , \; x > 0$$

and B is the only solution of the equation

$$(4) \qquad \frac{1}{H_{16}(x)} = \frac{\lambda}{E c_2} \, \frac{H_{21}(\xi_1)}{W(\xi_1)} \; , \; x > 0.$$

PROOF . – We define

$$(5) \qquad \xi_1 = \frac{\sigma}{a_1} \quad , \quad \text{with} \quad a_1 = \frac{\sqrt{k_1}}{\sqrt{\rho \, c_1}} \, .$$

The coefficients ω and k_1 are obtained using (5) and the element ξ_1 is given from (II-6b) as the solution of (3). From (II-6a) it follows that ξ_1 and k_2 should verify

$$(6) \qquad \frac{E \, k_2}{\lambda \, \rho \, a_1 \, a_2 \sqrt{\pi}} \; F_1\Big(\frac{\sigma}{a_2} \, \frac{W(\xi_1)}{\xi_1}\Big) = H_{21}(\xi_1).$$

If we define

(7) $$B = \frac{\sigma}{a_2} \frac{W(\xi_1)}{\xi_1} = \frac{\sigma}{a_2} H_{25}(\xi_1) \quad , \quad \text{with} \quad a_2 = \sqrt{\frac{k_2}{\rho c_2}} \; ,$$

then equation (6) is equivalent to

(8) $$\frac{F_1(B)}{B} = \frac{\lambda \sqrt{\pi}}{E c_2} \frac{H_{21}(\xi_1)}{W(\xi_1)} \quad , \quad B > 0 \; ,$$

that is, B is the solution of (4). Taking into account the properties of the function H_{16} we can deduce that there exists a unique solution of (4) if and only if

$$H_{23}(\xi_1) > 0 \qquad \text{iff} \qquad H_{23}(0^+) > 0 \text{ and } \xi_1 < x_{23} \; \Big(\text{i.e., (1)}\Big),$$

where x_{23} is the only positive root of H_{23} (because H_{23} is a decreasing function for $x > 0$ and $H_{23}(+\infty) = -\infty$). From (7) we obtain the coefficient k_2 .

REFERENCES

[ArLaTa] J. C. Arderius – M. A. Lara – D. A. Tarzia, "*Experimental-numerical determination of thermal coefficients through a phase-change process*", To appear.

[Ca] J. R. Cannon, "*The one-dimensional heat equation*", Addison-Wesley, Menlo Park, California (1984).

[CaJa] H. S. Carslaw – J. C. Jaeger, "*Conduction of heat in solids*", Oxford University Press, London (1959).

[GoTa] A.M. González – D.A. Tarzia, "*Determination on unknown coefficients of a semi-infinite material through a simple mushy zone model for the two-phase Stefan problem*", Int. J. Engng. Sci., To appear.

[Ru] L. I. Rubinstein, "*The Stefan problem*", Translations of Mathematical Monographs, Vol. 27, Amer. Math. Soc., Providence (1971).

[SoWiAl1] A. D. Solomon – D. G. Wilson – V. Alexiades, "*A mushy zone model with*

an exact solution", Letters Heat Mass Transfer, 9 (1982), 319–324.
[SoWiAl2] A. D. Solomon – D. G. Wilson – V. Alexiades, "*Explicit solutions to phase- change problems*", Quart. Appl. Math., 41 (1983), 237 – 243.
[StTa] M. B. Stampella – D. A. Tarzia, "*Determination of one or two unknown thermal coefficients of a semi-infinite material through a two-phase Stefan problem*", Int. J. Engng. Sci., 27 (1989), 1407–1419.
[Ta1] D. A. Tarzia, "*An inequality for the coefficient σ of the free boundary $s(t)=2\sigma\sqrt{t}$ of the Neumann solution for two-phase Stefan problem*", Quart. Appl. Math., 39 (1981 – 82), 491 – 497.
[Ta2] D. A. Tarzia, "*Simultaneous determination of two unknown thermal coeficients through an inverse one-phase Lamé-Clapeyron (Stefan) problem with an overspecified condition on the fixed face*", Int. J. Heat Mass Transfer, 26 (1983), 1151–1157.
[Ta3] D. A. Tarzia, "*Determination of unknown thermal coefficients of a semi-infinite material for the one-phase Lamé-Clapeyron (Stefan) problem through the Solomon-Wilson-Alexiades' mushy zone model*", Int. Comm. Heat Mass Transfer, 14 (1987), 219–228.
[Ta4] D. A. Tarzia, "*Neumann-like solution for the two-phase Stefan problem with a simple mushy zone model*", Comp. Appl. Math., 9 (1990), 201–211.

APPENDIX I

The following real functions are defined, for $x > 0$, by

$$f(x)=\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,dt\ , \qquad F_1(x)=\frac{\exp(-x^2)}{1-f(x)}\ , \qquad F_2(x)=\frac{\exp(-x^2)}{f(x)}\ ,$$

$$W(x)=W(x,\gamma)\ =\ x+\frac{\gamma\sqrt{\pi}}{2\,D}\,f(x)\exp(x^2)\ ,\ \ H_3(x)=\exp(-x^2)-\frac{E\,k_2}{h_o\,a_2\sqrt{\pi}}F_1\Big(\frac{a_1}{a_2}W(x)\Big)\ ,$$

$$G(x)=G(x,\gamma)\ =\ x+\frac{(1-\epsilon)\,\gamma\sqrt{\pi}}{2D}\,f(x)\exp(x^2)\ ,\ \ H_2(x)=\frac{h_o}{\lambda\,\rho\,a_1}\exp(-x^2)-G(x)\ ,$$

$$H_4(x)=\frac{h_o}{\lambda\,\rho\,a_1}\exp(-x^2)-x-\frac{E\,k_2}{\lambda\,\rho\,a_1\,a_2\sqrt{\pi}}\,F_1\Big(\frac{a_1}{a_2}W(x)\Big)\ ,\qquad H_1(x)=x\,F_1(x)\ ,$$

$$H_5(x)=\frac{\gamma\sqrt{\pi}}{2\,D}\,f(x)\,\exp(x^2)-H_4(x)\,,\qquad H_6(x)=(1-\epsilon)\,\frac{a_2}{a_1}\,x+\frac{E\,k_2}{\lambda\,\rho\,a_1\,a_2\sqrt{\pi}}\,F_1(x)\,,$$

$$H_7(x)=\frac{h_o}{\lambda\,\rho\,a_1}\exp(-x^2)-x-\frac{E\,k_2}{\lambda\,\rho\,a_1\,a_2\sqrt{\pi}}\,F_1\!\left(\frac{a_1}{a_2}\,x\right),\qquad H_8(x)=\frac{W(x)}{f(x)}\,,$$

$$H_9(x)=\frac{G(x)}{f(x)}\,,\qquad H_{10}(x)=\frac{h_o^{\,2}\sqrt{\pi}}{D\,\lambda\,\rho\,k_1}\exp(-x^2)-\frac{E\,h_o\,k_2}{D\,\lambda\,\rho\,k_1\,a_2}\,F_1\!\left(\frac{D\,k_1}{h_o\,a_2\sqrt{\pi}}\,H_8(x)\right),$$

$$H_{11}(x)=\frac{\beta_2}{\beta_1}\,x+F_1(x)\,,\qquad H_{12}(x)=\beta_1\,\beta_3\,\frac{1}{x}\,,\qquad H_{13}(x)=f(x)\,W(x)\,,$$

$$\beta_1=\frac{D\,k_1}{h_o\,a_2\sqrt{\pi}}\left(\frac{2}{\sqrt{\pi}}+\frac{\gamma\sqrt{\pi}}{2\,D}\right),\quad \beta_2=\frac{D\,\lambda\,\rho\,k_1\,a_2}{E\,h_o\,k_2}\left(\frac{2}{\sqrt{\pi}}+\frac{(1-\epsilon)\,\gamma\sqrt{\pi}}{2\,D}\right),\quad \beta_3=\frac{h_o\,a_2\sqrt{\pi}}{E\,k_2}\,,$$

$$H_{14}(x)=f(x)\,G(x)\,,\qquad H_{15}(x)=\frac{D\,c_1}{\lambda\sqrt{\pi}}\exp(-x^2)-\frac{D\,E\,c_1\,k_2}{\lambda\,\pi\,h_o\,a_2}\,F_1\!\left(\frac{h_o\sqrt{\pi}}{D\,\rho\,c_1\,a_2}\,H_{13}(x)\right),$$

$$H_{16}(x)=\sqrt{\pi}\,x\,\exp(x^2)\Big(1-f(x)\Big)\,,\qquad H_{19}(x)=f(x)\,H_{18}(x)\,,$$

$$H_{17}(x)=\frac{h_o}{E\,\rho\,a_1\,c_2}\exp(-x^2)-\left(1+\frac{(1-\epsilon)\,\lambda}{E\,c_2}\right)\frac{\gamma\sqrt{\pi}}{2\,D}\,f(x)\,\exp(x^2)-\left(1+\frac{\lambda}{E\,c_2}\right)x\,,$$

$$H_{18}(x)=\exp(x^2)\left(G(x)+\frac{E}{\lambda}\sqrt{\frac{c_1\,c_2\,k_2}{\pi\,k_1}}\,F_1\!\left(\sqrt{\frac{k_1\,c_2}{c_1\,k_2}}\,W(x)\right)\right),\qquad H_{20}(x)=x\,f(x)\,,$$

$$H_{21}(x)=H_{21}(x,\gamma)=\frac{h_o}{\lambda\,\rho\,\sigma}\,x\exp(-x^2)-G(x,\gamma)\,,\qquad H_{23}(x)=\frac{H_{22}(x)}{x}\,,$$

$$H_{22}(x)=\frac{h_o}{E\,\rho\,\sigma\,c_2}\,x\,\exp(-x^2)-\left(1+\frac{(1-\epsilon)\,\lambda}{E\,c_2}\right)\frac{\gamma\sqrt{\pi}}{2\,D}\,f(x)\,\exp(x^2)-\left(1+\frac{\lambda}{E\,c_2}\right)x\,,$$

$$H_{24}(x)=\frac{f(x)}{x}\,,\qquad H_{25}(x)=\frac{W(x)}{x}\,,\qquad H_{27}(x)=\frac{H_{26}(x)}{x}\,,$$

$$H_{26}(x)=x\,\exp(-x^2)-\frac{E\,k_2}{h_o\,a_2\sqrt{\pi}}\,x\,F_1\!\left(\frac{\sigma}{a_2}\,H_{25}(x)\right)\,,\qquad H_{28}(x)=\frac{H_{21}(x)}{x}\,,$$

$$H_{29}(x)=\frac{h_o}{\lambda\,\rho\,\sigma}\exp(-x^2)-1-\frac{E\,k_2}{\lambda\,\rho\,\sigma\,a_2\sqrt{\pi}}\,F_1\!\left(\frac{\sigma}{a_2}\,H_{25}(x)\right)\,,$$

$$H_{30}(x)=\frac{\gamma\sqrt{\pi}}{2\,D}\,\frac{1}{x\,F_2(x)}-H_{29}(x)\,,\qquad H_{31}(x)=(1-\epsilon)\,\frac{a_2}{\sigma}\,x+\frac{E\,k_2}{\lambda\,\rho\,\sigma\,a_2\sqrt{\pi}}\,F_1(x)\,,$$

$$H_{32}(x)=\frac{h_o}{\lambda\,\rho\,\sigma}\exp(-x^2)-1-\frac{E\,k_2}{\lambda\,\rho\,\sigma\,a_2\sqrt{\pi}}F_1\!\left(\frac{\sigma}{a_2}\right), \qquad H_{34}(x)=\frac{E\,c_2\sqrt{\pi}}{D\,c_1}\,\frac{W(x)}{F_2(x)},$$

$$H_{33}(x)=\frac{\sigma\,h_o\,c_1}{\lambda\,k_1}\exp(-x^2)-x\,G(x)-\frac{E\,c_2}{\lambda}x\,W(x), \qquad H_{35}(x)=1-H_{34}(x),$$

$$H_{36}(x)=\frac{D\,c_1}{\lambda\sqrt{\pi}}F_2(x)-x-\frac{E\,c_2}{\lambda}W(x), \qquad H_{37}(x)=H_{36}(x)-\frac{\gamma\sqrt{\pi}}{2\,D}\,\frac{1}{F_2(x)},$$

$$H_{38}(x)=x+\frac{E\,c_2}{\lambda\sqrt{\pi}}F_1(x), \qquad H_{39}(x)=\sqrt{H_{20}(x)}\left(1+\frac{\gamma\sqrt{\pi}}{2\,D}\,\frac{1}{x\,F_2(x)}\right),$$

$$H_{40}(x)=G(x)+\frac{E}{\lambda}\,\frac{1}{\sqrt{H_{24}(x)}}\sqrt{\frac{D\,c_1\,c_2\,k_2}{\sigma\,h_o\sqrt{\pi^3}}}\,F_1\!\left(\sqrt{\frac{\sigma\,h_o\,c_2\sqrt{\pi}}{D\,c_1\,k_2}}\,H_{39}(x)\right), \quad H_{42}(x)=\frac{G(x)}{x},$$

$$H_{41}(x)=\frac{D\,c_1}{\lambda\sqrt{\pi}}F_2(x), \qquad H_{43}(x)=\frac{h_o}{\lambda\,\rho\,\sigma}\exp(-x^2)-\left(1+\frac{E\,c_2}{\lambda}\right),$$

$$H_{44}(x)=\frac{E\,c_2}{\lambda\sqrt{\pi}}W(\xi_1)\,H_1(x)+G(\xi_1)\,x^2\quad(\xi_1>0), \qquad H_{46}(x)=\frac{D\,c_1}{\lambda\sqrt{\pi}}F_2(x)-W(x),$$

$$H_{45}(x)=\frac{\sigma\,h_o\,c_1}{\lambda\,k_1}\exp(-x^2)-x\,G(x), \qquad H_{47}(x)=H_{46}(x)+\frac{\gamma\sqrt{\pi}}{2\,D}\,\frac{1}{F_2(x)},$$

$$H_{48}(x)=\frac{h_o}{\lambda\,\rho\,\sigma}\exp(-x^2)-1, \quad H_{49}(x)=\frac{\exp(-x^2)}{x}-\frac{E}{\sigma\,h_o}\sqrt{\frac{k_1\,c_2\,k_2}{\pi\,c_1}}\,F_1\!\left(\sqrt{\frac{k_1\,c_2}{c_1\,k_2}}\,W(x)\right),$$

$$H_{50}(x)=\frac{D\,c_1}{\lambda\sqrt{\pi}}F_2(x)-\left(1+\frac{E\,c_2}{\lambda}\right)x, \qquad H_{52}(x)=H_{51}(x)-\frac{\gamma\sqrt{\pi}}{2\,D}\,\frac{x}{F_2(x)},$$

$$H_{51}(x)=\frac{\sigma\,h_o\,c_1}{\lambda\,k_1}\exp(-x^2)-x^2-\frac{E}{\lambda}\sqrt{\frac{c_1\,c_2\,k_2}{\pi\,k_1}}\,x\,F_1\!\left(\sqrt{\frac{k_1\,c_2}{c_1\,k_2}}\,W(x)\right),$$

$$H_{53}(x)=(1-\epsilon)\sqrt{\frac{c_1\,k_2}{k_1\,c_2}}\,x+\frac{E}{\lambda}\sqrt{\frac{c_1\,c_2\,k_2}{\pi\,k_1}}\,F_1(x), \qquad H_{55}(x)=x\,f(x)\exp(x^2)=\frac{x}{F_2(x)},$$

$$H_{54}(x)=\frac{\sigma\,h_o\,c_1}{\lambda\,k_1}\exp(-x^2)-\left(\epsilon+(1-\epsilon)\sqrt{\frac{c_1\,k_2}{k_1\,c_2}}\right)x^2-\frac{E}{\lambda}\sqrt{\frac{c_1\,c_2\,k_2}{\pi\,k_1}}\,x\,F_1\!\left(\sqrt{\frac{k_1\,c_2}{c_1\,k_2}}\,x\right),$$

$$H_{56}(x)=\frac{1}{H_{55}(x)}-\frac{E\,k_2}{D\,\rho\,\sigma\,c_1\,a_2}F_1\!\left(\frac{\sigma}{a_2}H_{25}(x)\right), \qquad H_{58}(x)=x\,F_2(x),$$

$$H_{57}(x)=H_{57}(x,\gamma)=\frac{D\,c_1}{\lambda\sqrt{\pi}}F_2(x)-G(x,\gamma).$$

APPENDIX II

The restrictions used in the text are the following

(R1) $$h_o > \frac{E\,k_2}{a_2\sqrt{\pi}}$$

(R2) $$\frac{D\,k_1}{h_o\,a_1\sqrt{\pi}} < f(x_2)\ ,\ x_2 :\ \text{the unique positive zero of } H_2$$

(R3) $$\frac{D\,k_1}{h_o\,a_1\sqrt{\pi}} < f(x_3)\ ,\ x_3 :\ \text{the unique positive zero of } H_3$$

(R4) $$\frac{D\,k_1}{h_o\,a_1\sqrt{\pi}} < f(x_4)\ ,\ x_4 :\ \text{the unique positive zero of } H_4$$

(R5) $$\frac{D\,k_1}{h_o\,a_1\sqrt{\pi}} > f(x_5)\ ,\ x_5 :\ \text{the unique positive zero of } H_5$$

(R6) $$\frac{D\,k_1}{h_o\,a_1\sqrt{\pi}} < f(x_7)\ ,\ x_7 :\ \text{the unique positive zero of } H_7$$

(R7) $$h_o > \frac{D\,k_1}{a_2\sqrt{\pi}}\left(\frac{2}{\sqrt{\pi}} + \frac{\gamma\sqrt{\pi}}{2\,D}\right)\frac{1}{\eta}\ ,$$

η : the unique positive solution of the equation $H_{11}(x) = H_{12}(x)\ ,\ x > 0$

(R8) $$\frac{D\,k_1}{h_o\,a_1\sqrt{\pi}} < f(x_{17})\ ,\ x_{17} :\ \text{the unique positive zero of } H_{17}$$

(R9) $$\frac{h_o}{E\,\rho\,\sigma\,c_2} > 1 + \frac{\gamma}{D} + \frac{\lambda}{E\,c_2}\left(1 + \frac{(1-\epsilon)\,\gamma}{D}\right)$$

(R10) $$h_o > \frac{D\,k_1}{2\,\sigma}$$

(R11) $$\frac{D\,\rho\,\sigma\,c_1}{h_o\sqrt{\pi}} < H_{20}(x_{23})\ ,\ x_{23} :\ \text{the unique positive zero of } H_{23}$$

(R12) $$\frac{D\,k_1}{\sigma\,h_o\sqrt{\pi}} > H_{24}(x_{23})\ ,\ x_{23} :\ \text{the unique positive zero of } H_{23}$$

(R13) $$h_o > \frac{E\, k_2}{a_2 \sqrt{\pi}} F_1\Big(\frac{\sigma}{a_2}(1 + \frac{\gamma}{D})\Big)$$

(R14) $$\frac{h_o}{\lambda \rho \sigma} > 1 + \frac{(1-\epsilon)\gamma}{D}$$

(R15) $$\frac{h_o}{\lambda \rho \sigma} > 1 + \frac{E\, k_2}{\lambda \rho \sigma a_2 \sqrt{\pi}} F_1\Big(\frac{\sigma}{a_2}(1 + \frac{\gamma}{D})\Big)$$

(R16) $$\frac{h_o}{\lambda \rho \sigma} > 1 + \frac{\gamma}{D} + \frac{E\, k_2}{\lambda \rho \sigma a_2 \sqrt{\pi}} F_1\Big(\frac{\sigma}{a_2}(1 + \frac{\gamma}{D})\Big)$$

(R17) $$\frac{h_o}{\lambda \rho \sigma} > 1 + \frac{E\, k_2}{\lambda \rho \sigma a_2 \sqrt{\pi}} F_1\Big(\frac{\sigma}{a_2}\Big)$$

(R18) $$\frac{D \rho \sigma c_1}{h_o \sqrt{\pi}} < H_{20}(x_{27}) \;,\; x_{27}: \text{ the unique positive zero of } H_{27}$$

(R19) $$\frac{D \rho \sigma c_1}{h_o \sqrt{\pi}} < H_{20}(x_{28}) \;,\; x_{28}: \text{ the unique positive zero of } H_{28}$$

(R20) $$H_{20}(x_{29}) > \frac{D \rho \sigma c_1}{h_o \sqrt{\pi}} > H_{20}(x_{30}) \;,$$

x_{29} : the unique positive zero of H_{29} , x_{30} : the unique positive zero of H_{30}

(R21) $$\frac{D \rho \sigma c_1}{h_o \sqrt{\pi}} < H_{20}(x_{32}) \;,\; x_{32}: \text{ the unique positive zero of } H_{32}$$

(R22) $$\frac{D\, k_1}{\sigma h_o \sqrt{\pi}} > H_{24}(x_{33}) \;,\; x_{33}: \text{ the unique positive zero of } H_{33}$$

(R23) $$h_o = \frac{D\, k_1}{a_1 f\Big(\frac{\sigma}{a_1}\Big) \sqrt{\pi}}$$

(R24) $$H_{35}\Big(\frac{\sigma}{a_1}\Big) > 0 \text{ or } \frac{\sigma}{a_1} < x_{35} \;,\; x_{35}: \text{ the unique positive zero of } H_{35}$$

(R25) $$H_{36}\Big(\frac{\sigma}{a_1}\Big) > 0 \text{ or } \frac{\sigma}{a_1} < x_{36} \;,\; x_{36}: \text{ the unique positive zero of } H_{36}$$

(R26) $$\frac{D\,k_1}{E\,\rho\,a_1\,a_2\,c_2}\,F_2\!\left(\frac{\sigma}{a_1}\right) \;\le\; 1$$

(R27) $$\frac{\lambda\,\rho\,\sigma\,a_2\,\sqrt{\pi}}{E\,k_2}\left(\frac{D\,k_1}{\lambda\,\rho\,\sigma\,a_1\,\sqrt{\pi}}\,F_2\!\left(\frac{\sigma}{a_1}\right)-1\right) \;>\; 1$$

(R28) $$F_1\!\left(\frac{\sigma}{a_2}\right) \;<\; \frac{\lambda\,\rho\,\sigma\,a_2\,\sqrt{\pi}}{E\,k_2}\left(\frac{D\,k_1}{\lambda\,\rho\,\sigma\,a_1\,\sqrt{\pi}}\,F_2\!\left(\frac{\sigma}{a_1}\right)-1\right)$$

(R29) $H_{57}\!\left(\frac{\sigma}{a_1}\right) > 0$ or $\frac{\sigma}{a_1} < x_{57}$, x_{57} : the unique positive zero of H_{57}

(R30) $$\frac{h_o}{\lambda\,\rho\,\sigma} \;>\; 1+\frac{(1-\epsilon)\,\gamma}{D}+\frac{E\,k_2}{\lambda\,\rho\,\sigma\,a_2\,\sqrt{\pi}}\,F_1\!\left(\frac{\sigma}{a_2}\left(1+\frac{\gamma}{D}\right)\right)$$

(R31) $\frac{D\,k_1}{\sigma\,h_o\,\sqrt{\pi}} > H_{24}(x_{27})$, x_{27} : the unique positive zero of H_{27}

(R32) $h_o \;>\; \rho\,\sigma\,(\lambda + E\,c_2)$

(R33) $H_{43}(\frac{\sigma}{a_1}) > 0$ or $\frac{\sigma}{a_1} < x_{43}$,

x_{43} : the unique positive zero of H_{43} when (**R32**) is verified

(R34) $\frac{D\,k_1}{\sigma\,h_o\,\sqrt{\pi}} > H_{24}(x_{28})$, x_{28} : the unique positive zero of H_{28}

(R35) $\frac{D\,k_1}{\sigma\,h_o\,\sqrt{\pi}} > H_{24}(x_{29})$, x_{29} : the unique positive zero of H_{29}

(R36) $\frac{D\,k_1}{\sigma\,h_o\,\sqrt{\pi}} < H_{24}(x_{30})$, x_{30} : the unique positive zero of H_{30}

(R37) $\frac{D\,k_1}{\sigma\,h_o\,\sqrt{\pi}} > H_{24}(x_{32})$, x_{32} : the unique positive zero of H_{32}

(R38) $\frac{D\,k_1}{\sigma\,h_o\,\sqrt{\pi}} > H_{24}(x_{45})$, x_{45} : the unique positive zero of H_{45}

(R39) $H_{47}\left(\frac{\sigma}{a_1}\right) > 0$ or $\frac{\sigma}{a_1} < x_{47}$, x_{47} : the unique positive zero of H_{47}

(R40) $H_{55}\left(\frac{\sigma}{a_1}\right) < \frac{D\, c_1}{\lambda \sqrt{\pi}}$

(R41) $\frac{D\, k_1}{\sigma\, h_o \sqrt{\pi}} > H_{24}(x_{54})$, x_{54} : the unique positive zero of H_{54}

(R42) $\frac{D\, k_1}{\sigma\, h_o \sqrt{\pi}} > H_{24}(x_{49})$, x_{49} : the unique positive zero of H_{49}

(R43) $H_{50}\left(\frac{\sigma}{a_1}\right) > 0$ or $\frac{\sigma}{a_1} < x_{50}$, x_{50} : the unique positive zero of H_{50}

(R44) $\frac{D\, k_1\, a_2}{E\, a_1\, k_2} F_2\left(\frac{\sigma}{a_1}\right) > 1$

(R45) $F_1\left(\frac{\sigma}{a_2}\right) < \frac{D\, k_1\, a_2}{E\, a_1\, k_2} F_2\left(\frac{\sigma}{a_1}\right)$

(R46) $\frac{D\, k_1}{\sigma\, h_o \sqrt{\pi}} > H_{24}(x_{51})$, x_{51} : the unique positive zero of H_{51}

(R47) $\frac{D\, k_1}{\sigma\, h_o \sqrt{\pi}} < H_{24}(x_{52})$, x_{52} : the unique positive zero of H_{52}

(R48) $H_{56}\left(\frac{\sigma}{a_1}\right) > 0$ or $\frac{\sigma}{a_1} < x_{56}$, x_{56} : the unique positive zero of H_{56} .

Table I

CASE	*UNKNOWN COEFFICIENTS*	*RESTRICTIONS*	*SOLUTION*
1	c_2 , σ , ω	(R2)	$\sigma = a_1\, \xi_1 \quad , \quad c_2 = \frac{c_1\, k_2}{k_1} \frac{B^2}{W^2(\xi_1)} \quad , \quad \omega = a_1\, W(\xi_1)$ where ξ_1 is the unique positive solution of the equation $f(x) = \frac{D\, k_1}{h_o\, a_1 \sqrt{\pi}} \;,\; x > 0$ and B is the only positive solution of the equation $H_1(x) = \frac{\lambda\, k_1 \sqrt{\pi}}{E\, c_1\, k_2} W(\xi_1)\, H_2(\xi_1) \;,\; x > 0.$
2	λ , σ , ω	(R1) , (R3)	$\sigma = a_1\, \xi_1 \quad , \quad \lambda = \frac{h_o}{\rho\, a_1} \frac{H_3(\xi_1)}{G(\xi_1)} \quad , \quad \omega = a_1\, W(\xi_1)$ where ξ_1 is given as in case 1.
3	ϵ , σ , ω	(R1) , (R4) (R5)	$\sigma = a_1\, \xi_1 \quad , \quad \epsilon = \frac{2\, D}{\gamma \sqrt{\pi}} F_2(\xi_1)\, H_5(\xi_1) \quad , \quad \omega = a_1\, W(\xi_1)$ where ξ_1 is given as in case 1.
4	γ , σ , ω	(R1) , (R6)	$\sigma = a_1\, \xi_1 \;,\; \gamma = \frac{2\, D}{\sqrt{\pi}} \left(\frac{a_2}{a_1} B - \xi_1 \right) F_2(\xi_1) \;,\; \omega = a_1\, W(\xi_1)$ where ξ_1 is given as in case 1 and B is the only positive solution of the equation $H_6(x) = \frac{h_o}{\lambda\, \rho\, a_1} \exp(-\xi_1{}^2) - \epsilon\, \xi_1 \;,\; x > \frac{a_1}{a_2} \xi_1 \,.$
5	c_1 , σ , ω	(R7)	$\sigma = \frac{D\, k_1}{h_o \sqrt{\pi}} \frac{1}{H_{21}(\xi_1)} \,,\, c_1 = \frac{\pi\, h_o{}^2}{D^2\, \rho\, k_1} f^2(\xi_1) \,,\, \omega = \frac{D\, k_1}{h_o \sqrt{\pi}} H_8(\xi_1)$ where ξ_1 is the unique positive solution of the equation $H_9(x) = H_{10}(x) \;,\; x > 0.$

6	k_1 , σ , ω	(R1)	$\sigma = \frac{h_o \sqrt{\pi}}{D \rho c_1} H_{20}(\xi_1) , k_1 = \frac{\pi h_o^2}{D^2 \rho c_1} f^2(\xi_1) , \omega = \frac{h_o \sqrt{\pi}}{D \rho c_1} H_{13}(\xi_1)$ where ξ_1 is the unique positive solution of the equation $H_{14}(x) = H_{15}(x) \;,\; x > 0.$
7	k_2 , σ , ω	(R8)	$\sigma = a_1 \xi_1 \;,\; k_2 = \frac{k_1 c_2}{c_1} \frac{W^2(\xi_1)}{B^2} \;,\; \omega = a_1 W(\xi_1)$ where ξ_1 is given as in case 1 and B is the only positive solution of the equation $\frac{1}{H_{16}(x)} = \frac{\lambda}{E c_2} \frac{H_2(\xi_1)}{W(\xi_1)} \;,\; x > 0.$
8	ρ , σ , ω	——	$\sigma = \frac{\lambda k_1}{h_o c_1} \xi_1 H_{18}(\xi_1) \;,\; \rho = \frac{h_o^2 c_1}{\lambda^2 k_1} \frac{1}{H_{18}^2(\xi_1)} \;,$ $\omega = \frac{\lambda k_1}{h_o c_1} W(\xi_1) H_{18}(\xi_1)$ where ξ_1 is the unique positive solution of the equation $H_{19}(x) = \frac{D c_1}{\lambda \sqrt{\pi}} \;,\; x > 0.$

Table II

CASE	UNKNOWN COEFFICIENTS	RESTRICTIONS	SOLUTION
9	ω , k_1 , k_2	(R9) , (R11)	$\omega = \sigma H_{25}(\xi_1)$, $k_1 = \rho \sigma^2 c_1 \frac{1}{\xi_1^2}$, $k_2 = \rho \sigma^2 c_2 \frac{H_{25}^2(\xi_1)}{B^2}$ where ξ_1 is the unique solution of the equation $H_{20}(x) = \frac{D \rho \sigma c_1}{h_o \sqrt{\pi}}$, $x > 0$ and B is the only solution of the equation $\frac{1}{H_{16}(x)} = \frac{\lambda}{E c_2} \frac{H_{21}(\xi_1)}{W(\xi_1)}$, $x > 0$.
10	ω , c_1 , k_2	(R9) , (R10) (R12)	$\omega = \sigma H_{25}(\xi_1)$, $c_1 = \frac{k_1}{\rho \sigma^2} \xi_1^2$, $k_2 = \rho \sigma^2 c_2 \frac{H_{25}^2(\xi_1)}{B^2}$ where ξ_1 is the unique solution of the equation $H_{24}(x) = \frac{D k_1}{\sigma h_o \sqrt{\pi}}$, $x > 0$ and B is given as in case 9.
11	ω , λ , k_1	(R13) , (R18)	$\omega = \sigma H_{25}(\xi_1)$, $\lambda = \frac{h_o}{\rho \sigma} \frac{H_{26}(\xi_1)}{G(\xi_1)}$, $k_1 = \rho \sigma^2 c_1 \frac{1}{\xi_1^2}$ where ξ_1 is given as in case 9.
12	ω , k_1 , c_2	(R14) , (R19)	$\omega = \sigma H_{25}(\xi_1)$, $k_1 = \rho \sigma^2 c_1 \frac{1}{\xi_1^2}$, $c_2 = \frac{k_2}{\rho \sigma^2} \frac{B^2}{H_{25}^2(\xi_1)}$ where ξ_1 is given as in case 9 and B is the only solution of the equation $H_1(x) = \frac{\lambda \rho \sigma^2 \sqrt{\pi}}{E k_2} H_{25}(\xi_1) H_{28}(\xi_1)$, $x > 0$.
13	ω , ϵ , k_1	(R16) , (R20)	$\omega = \sigma H_{25}(\xi_1)$, $\epsilon = 1 - \frac{2 D}{\gamma \sqrt{\pi}} H_{29}(\xi_1) H_{58}(\xi_1)$, $k_1 = \rho \sigma^2 c_1 \frac{1}{\xi_1^2}$ where ξ_1 is given as in case 9.

14	ω , γ , k_1	(R17) , (R21)	$\omega = B a_2$, $\gamma = \frac{2D}{\sqrt{\pi}}\left(\frac{a_2}{\sigma} B - 1\right) H_{58}(\xi_1)$, $k_1 = \rho \sigma^2 c_1 \frac{1}{\xi_1^2}$ where ξ_1 is given as in case 9 and B is the only solution of the equation $H_{31}(x) = \frac{h_0}{\lambda \rho \sigma} \exp(-\xi_1^2) - \epsilon \; , \; x > \frac{\sigma}{a_2}$.
15	ω , ρ , k_2	(R10) , (R22)	$\omega = \sigma H_{25}(\xi_1)$, $\rho = \frac{k_1}{\sigma^2 c_1} \xi_1^2$, $k_2 = \frac{k_1 c_2}{c_1} \frac{W^2(\xi_1)}{B^2}$ where ξ_1 is given as in case 10 and B is the only solution of the equation $\frac{1}{H_{16}(x)} = \frac{\lambda}{E c_2} \frac{H_{45}(\xi_1)}{\xi_1 W(\xi_1)} \; , \; x > 0.$
16	ω , λ , k_2	(R23) , (R24)	$\omega = a_1 W\left(\frac{\sigma}{a_1}\right)$, $\lambda = \frac{1}{G\left(\frac{\sigma}{a_1}\right)} \left(\frac{D c_1}{\sqrt{\pi}} F_2\left(\frac{\sigma}{a_1}\right) - E c_2 \frac{W\left(\frac{\sigma}{a_1}\right)}{H_{16}(B)}\right)$, $k_2 = \frac{k_1 c_2}{c_1} \frac{W^2\left(\frac{\sigma}{a_1}\right)}{B^2}$ for any $B > H_{16}^{-1}\left(\frac{E c_2 \sqrt{\pi}}{D c_1} \frac{W\left(\frac{\sigma}{a_1}\right)}{F_2\left(\frac{\sigma}{a_1}\right)}\right)$.
17	ω , ϵ , k_2	(R23) , (R25)	$\omega = a_1 W\left(\frac{\sigma}{a_1}\right)$, $k_2 = \frac{k_1 c_2}{c_1} \frac{W^2\left(\frac{\sigma}{a_1}\right)}{B^2}$, $\epsilon = \frac{2D}{\gamma \sqrt{\pi}} W\left(\frac{\sigma}{a_1}\right) F_2\left(\frac{\sigma}{a_1}\right) -$ $- \frac{2D}{\lambda \gamma \sqrt{\pi}} F_2\left(\frac{\sigma}{a_1}\right) \left(\frac{D c_1}{\sqrt{\pi}} F_2\left(\frac{\sigma}{a_1}\right) - E c_2 \frac{W\left(\frac{\sigma}{a_1}\right)}{H_{16}(B)}\right)$ for any $H_{16}^{-1}\left(\frac{1}{A}\right) < B < H_{16}^{-1}\left(\frac{1}{C}\right)$ if $\frac{\sigma}{a_1} < x_{37}$ or for any $B > H_{16}^{-1}\left(\frac{1}{A}\right)$ if $x_{37} \le \frac{\sigma}{a_1} < x_{36}$ where $A = \frac{\lambda}{E c_2 W\left(\frac{\sigma}{a_1}\right)} \left(H_{41}\left(\frac{\sigma}{a_1}\right) - \frac{\sigma}{a_1}\right)$ and $C = \frac{\lambda}{E c_2 W\left(\frac{\sigma}{a_1}\right)} H_{46}\left(\frac{\sigma}{a_1}\right)$.

18	$\omega , \epsilon , \gamma$	(R23) , (R26) (R27) , (R28)	$\omega = \omega(\gamma) = \sigma \left(1 + \frac{\gamma \sqrt{\pi}}{2 D} \frac{1}{H_{58}\left(\frac{\sigma}{a_1}\right)}\right)$, $\epsilon = \epsilon(\gamma) = 1 - \frac{\sigma}{\omega(\gamma) - \sigma} \left(\frac{D k_1}{\lambda \rho \sigma \sqrt{\pi}} H_{58}\left(\frac{\sigma}{a_1}\right) - 1\right) - \frac{1}{\omega(\gamma) - \sigma} \frac{E k_2}{\lambda \rho a_2 \sqrt{\pi}} F_1\left(\frac{\omega(\gamma)}{a_2}\right)$ for any $0 < \gamma < \frac{2 D a_2}{a_1 \sqrt{\pi}} F_2\left(\frac{\sigma}{a_1}\right)\left(F_1^{-1}(B) - \frac{\sigma}{a_2}\right)$ where $B = \frac{\lambda \rho \sigma a_2 \sqrt{\pi}}{E k_2} \left(\frac{D k_1}{\lambda \rho \sigma a_1 \sqrt{\pi}} F_2\left(\frac{\sigma}{a_1}\right) - 1\right).$
19	ω , c_2 , k_2	(R23) , (R29)	$\omega = a_1 W\left(\frac{\sigma}{a_1}\right)$, $c_2 = \frac{\lambda}{E} \frac{H_{16}(B) H_{57}\left(\frac{\sigma}{a_1}\right)}{W\left(\frac{\sigma}{a_1}\right)}$, $k_2 = \frac{\lambda k_1 \sqrt{\pi}}{E c_1} \frac{W\left(\frac{\sigma}{a_1}\right) H_{57}\left(\frac{\sigma}{a_1}\right)}{H_1(B)}$ for any $B > 0$.
20	ω , ρ , k_1	————	$\omega = \sigma H_{25}(\xi_1)$, $\rho = \frac{h_o \sqrt{\pi}}{D \sigma c_1} H_{20}(\xi_1)$, $k_1 = \frac{\sigma h_o \sqrt{\pi}}{D} H_{24}(\xi_1)$ where ξ_1 is the unique solution of the equation $H_{40}(x) = H_{41}(x)$, $x > 0$.
21	ω , c_1 , k_1	(R30)	$\omega = \sigma H_{25}(\xi_1)$, $c_1 = \frac{h_o \sqrt{\pi}}{D \rho \sigma} H_{20}(\xi_1)$, $k_1 = \frac{\sigma h_o \sqrt{\pi}}{D} H_{24}(\xi_1)$ where ξ_1 is the unique solution of the equation $\frac{h_o}{\lambda \rho \sigma} H_{27}(x) = H_{42}(x)$, $x > 0$.
22	ω , λ , c_1	(R10) , (R13) (R31)	$\omega = \sigma H_{25}(\xi_1)$, $\lambda = \frac{h_o}{\rho \sigma} \frac{H_{26}(\xi_1)}{G(\xi_1)}$, $c_1 = \frac{k_1}{\rho \sigma^2} \xi_1^2$ where ξ_1 is given as in case 10.

23	ω , γ , k_2	(R23) , (R43)	$\omega = a_1 \, W\left(\frac{\sigma}{a_1},\gamma\right) \quad , \quad k_2 = \frac{k_1 \, c_2}{c_1} \, \frac{W^2\left(\frac{\sigma}{a_1},\gamma\right)}{B^2}$ for any $0 < \gamma < \frac{2 \, D \, F_2\left(\frac{\sigma}{a_1}\right)}{\left(1 - \epsilon + \frac{E \, c_2}{\lambda}\right) \sqrt{\pi}} \, H_{50}\left(\frac{\sigma}{a_1}\right)$ where $B = B(\gamma)$ is the unique solution of the equation $\frac{1}{H_{16}(x)} = \frac{\lambda}{E \, c_2} \, \frac{H_{57}\left(\frac{\sigma}{a_1},\gamma\right)}{W\left(\frac{\sigma}{a_1},\gamma\right)} \quad , \quad x > 0.$
24	ω , ϵ , c_1	(R16) , (R35)	$\omega = \sigma \, H_{25}(\xi_1) \, , \, \epsilon = 1 - \frac{2 \, D}{\gamma \, \sqrt{\pi}} \, H_{29}(\xi_1) \, H_{58}(\xi_1) \, , \, c_1 = \frac{k_1}{\rho \, \sigma^2} \, \xi_1^2$ where ξ_1 is given as in case 10.
25	ω , c_1 , c_2	(R10) , (R14) (R34)	$\omega = \sigma \, H_{25}(\xi_1) \quad , \quad c_1 = \frac{k_1}{\rho \, \sigma^2} \, \xi_1^2 \quad , \quad c_2 = \frac{k_2}{\rho \, \sigma^2} \, \frac{B^2}{H_{25}^2(\xi_1)}$ where ξ_1 is given as in case 10 and B is given as in case 12.
26	ω , ρ , c_1	(R10)	$\omega = \sigma \, H_{25}(\xi_1) \quad , \quad \rho = \frac{k_2}{\sigma^2 \, c_2} \, \frac{B^2}{H_{25}^2(\xi_1)} \quad , \quad c_1 = \frac{k_1 \, c_2}{k_2} \, \frac{W^2(\xi_1)}{B^2}$ where ξ_1 is given as in case 10 and B is the only solution of the equation $H_{44}(x) = \frac{\sigma \, h_0 \, c_2}{\lambda \, k_2} \, \frac{W^2(\xi_1)}{\xi_1 \, \exp(\xi_1^2)} \quad , \quad x > 0.$
27	ω , γ , c_1	(R10) , (R17) (R37)	$\omega = B \, a_2 \quad , \quad \gamma = \frac{2 \, D}{\sqrt{\pi}} \left(\frac{a_2}{\sigma} \, B - 1\right) H_{58}(\xi_1) \quad , \quad c_1 = \frac{k_1}{\rho \, \sigma^2} \, \xi_1^2$ where ξ_1 is given as in case 10 and B is given as in case 14.
28	ω , λ , c_2	(R23)	$\omega = a_1 \, W\left(\frac{\sigma}{a_1}\right) , \, \lambda = \frac{1}{G\left(\frac{\sigma}{a_1}\right)} \left(\frac{D \, c_1}{\sqrt{\pi}} \, F_2\left(\frac{\sigma}{a_1}\right) - \frac{E \, c_1 \, k_2}{k_1 \, \sqrt{\pi}} \, \frac{H_1(B)}{W\left(\frac{\sigma}{a_1}\right)}\right) ,$ $c_2 = \frac{c_1 \, k_2}{k_1} \, \frac{B^2}{W^2\left(\frac{\sigma}{a_1}\right)}$ for any $0 < B < H_1^{-1}\left(\frac{D \, k_1}{E \, k_2} \, W\left(\frac{\sigma}{a_1}\right) F_2\left(\frac{\sigma}{a_1}\right)\right)$

29 ω, ρ, c_2 (R10), (R38)

$$\omega = \sigma\, H_{25}(\xi_1) \quad , \quad \rho = \frac{k_1}{\sigma^2 c_1}\, \xi_1^{\,2} \quad , \quad c_2 = \frac{c_1 k_2}{k_1}\, \frac{B^2}{W^2(\xi_1)}$$

where ξ_1 is given as in case 10 and B is the only solution of the equation

$$H_1(x) = \frac{\lambda\, k_1 \sqrt{\pi}}{E\, c_1\, k_2}\, H_{25}(\xi_1)\, H_{45}(\xi_1) \quad , \quad x > 0.$$

---

30 ω, ϵ, c_2 (R23), (R39)

$$\omega = a_1\, W\!\left(\frac{\sigma}{a_1}\right) \quad , \quad c_2 = \frac{c_1 k_2}{k_1}\, \frac{B^2}{W^2\!\left(\frac{\sigma}{a_1}\right)} \quad ,$$

$$\epsilon = \frac{2\,D}{\gamma \sqrt{\pi}}\, W\!\left(\frac{\sigma}{a_1}\right) F_2\!\left(\frac{\sigma}{a_1}\right) - \frac{2\,D}{\lambda \gamma \sqrt{\pi}}\, F_2\!\left(\frac{\sigma}{a_1}\right)\left(\frac{D\, c_1}{\sqrt{\pi}}\, F_2\!\left(\frac{\sigma}{a_1}\right) - \frac{E\, c_1\, k_2}{k_1 \sqrt{\pi}}\, \frac{H_1(B)}{W\!\left(\frac{\sigma}{a_1}\right)}\right)$$

for any $H_1^{-1}(A) < B < H_1^{-1}(C)$ if $\frac{\sigma}{a_1} < x_{46}$

or for any $0 < B < H_1^{-1}(C)$ if $x_{46} \le \frac{\sigma}{a_1} < x_{47}$

where $A = \frac{\lambda\, k_1 \sqrt{\pi}}{E\, c_1\, k_2}\, W\!\left(\frac{\sigma}{a_1}\right) H_{46}\!\left(\frac{\sigma}{a_1}\right)$ and

$$C = \frac{\lambda\, k_1 \sqrt{\pi}}{E\, c_1\, k_2}\, W\!\left(\frac{\sigma}{a_1}\right) H_{47}\!\left(\frac{\sigma}{a_1}\right).$$

---

31 ω, γ, c_2 (R23), (R40)

$$\omega = a_1\, W\!\left(\frac{\sigma}{a_1}, \gamma\right) \quad , \quad c_2 = \frac{k_2}{\rho\, a_1^{\,2}}\, \frac{B^2}{W^2\!\left(\frac{\sigma}{a_1}, \gamma\right)} \quad ,$$

for any $0 < \gamma < \frac{2\,D\, H_{58}\left(\frac{\sigma}{a_1}\right)}{(1-\epsilon)\sqrt{\pi}}\left(\frac{D\, c_1}{\lambda \sqrt{\pi}}\, \frac{1}{H_{55}\left(\frac{\sigma}{a_1}\right)} - 1\right)$

where $B = B(\gamma)$ is the unique solution of the equation

$$H_1(x) = \frac{\lambda\, k_1 \sqrt{\pi}}{E\, c_1\, k_2}\, W\!\left(\frac{\sigma}{a_1}, \gamma\right) H_{57}\!\left(\frac{\sigma}{a_1}, \gamma\right) \quad , \quad x > 0.$$

---

32 ω, λ, ρ (R10), (R42)

$$\omega = \sigma\, H_{25}(\xi_1) \quad , \quad \lambda = \frac{\sigma\, h_0\, c_1}{k_1}\, \frac{H_{49}(\xi_1)}{G(\xi_1)} \quad , \quad \rho = \frac{k_1}{\sigma^2 c_1}\, \xi_1^{\,2}$$

where ξ_1 is given as in case 10.

33	$\omega , \lambda , \epsilon$	(R23) , (R48)	$\omega = a_1 W\left(\frac{\sigma}{a_1}\right), \epsilon = 1 - \frac{2D}{\gamma\sqrt{\pi}}\left(\frac{Dc_1}{\lambda\sqrt{\pi}} H_{56}\left(\frac{\sigma}{a_1}\right) - 1\right) H_{58}\left(\frac{\sigma}{a_1}\right)$ for any $\dfrac{\frac{Dc_1}{\sqrt{\pi}} H_{56}\left(\frac{\sigma}{a_1}\right)}{1 + \frac{\gamma\sqrt{\pi}}{2D}\,\frac{1}{H_{58}\left(\frac{\sigma}{a_1}\right)}} < \lambda < \frac{Dc_1}{\sqrt{\pi}} H_{56}\left(\frac{\sigma}{a_1}\right).$
34	$\omega , \lambda , \gamma$	(R23) , (R44) (R45)	$\omega = a_1 W\left(\frac{\sigma}{a_1},\gamma\right)$, $\lambda = \dfrac{Dc_1}{G\left(\frac{\sigma}{a_1},\gamma\right)\sqrt{\pi}}\left(F_2\left(\frac{\sigma}{a_1}\right) - \frac{E}{D}\sqrt{\frac{c_2 k_2}{c_1 k_1}}\, F_1\left(\frac{\sigma}{a_2} H_{25}\left(\frac{\sigma}{a_1}\right)\right)\right)$ for any $0 < \gamma < \dfrac{2D\, H_{58}\left(\frac{\sigma}{a_1}\right)}{(1-\epsilon)\sqrt{\pi}}\left(\frac{a_2}{\sigma} F_1^{-1}\left(\frac{D k_1 a_2}{E a_1 k_2} F_2\left(\frac{\sigma}{a_1}\right)\right) - 1\right).$
35	ω , ϵ , ρ	(R46) , (R47)	$\omega = \sigma H_{25}(\xi_1)$, $\epsilon = 1 - \frac{2D}{\gamma\sqrt{\pi}}\,\frac{H_{51}(\xi_1)}{H_{55}(\xi_1)}$, $\rho = \frac{k_1}{\sigma^2 c_1}\,\xi_1^{\,2}$ where ξ_1 is given as in case 10.
36	ω, γ , ρ	(R10) , (R41)	$\omega = \sigma\sqrt{\frac{c_1 k_2}{k_1 c_2}}\,\frac{B}{\xi_1}$, $\gamma = \frac{2D}{\sqrt{\pi}}\left(\sqrt{\frac{c_1 k_2}{k_1 c_2}}\,B - \xi_1\right) F_2(\xi_1)$, $\rho = \frac{k_1}{\sigma^2 c_1}\,\xi_1^{\,2}$ where ξ_1 is given as in case 10 and B is the only solution of the equation $H_{53}(x) = \frac{\sigma h_0 c_1}{\lambda k_1}\,\frac{\exp(-\xi_1^{\,2})}{\xi_1} - \epsilon\,\xi_1$, $x > \sqrt{\frac{k_1 c_2}{c_1 k_2}}\,\xi_1$.

Adriana M. GONZÁLEZ
Depto. de Matemática,
Facultad de Cs. Exactas, Fco-Qcas y Naturales,
Univ. Nac. de Río Cuarto,
Campus Universitario, Ruta 36, km 601,
(5800) Río Cuarto, Argentina.

Domingo A. TARZIA
Depto. de Matemática
FCE, Univ. Austral,
Paraguay 1950,
(2000) Rosario, Argentina.
and
PROMAR (CONICET – UNR),
Instituto de Matemática "B. Levi",
Av. Pellegrini 250, (2000) Rosario, Argentina.