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ON SOME THERMIC FLUX OPTIMIZATION PROBLEMS IN A DOMAIN WITH FOURIER BOUNDARY CONDITION AND STATE RESTRICTIONS

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I INTRODUCTION

We consider a regular bounded domain Ω of R^n (n=1, 2, 3 for the applications) with a sufficiently regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ ($|\Gamma_1|=meas(\Gamma_1)>0$ and $|\Gamma_2|=meas(\Gamma_2)>0$) and it is assumed that the phase change temperature is 0°C . We denote with $|\Gamma|$ the (n−1)-dimensional Lebesque measure of Γ . On portion Γ_1 of the boundary we have a Fourier boundary condition (a Newton law with transfer coefficient $\alpha>0$ with an exterior temperature $b>0$), and on portion Γ_2 of the remaining boundary a heat flux $q>0$ is imposed. We consider in Ω a steady-state heat conduction problem and we are interested in studying under which condition on data we have a steady-state phase change problem, i.e. the temperature is of non-constant sign in Ω.

Following [Ta1] we study the temperature $\Theta=\Theta(x)$, defined for $x \in \Omega$. If we define the function u in Ω as follows:

(1.1) $$u=k_2\,\theta^+ - k_1\,\theta^- \quad \text{in } \Omega ,$$

where Θ^+ and θ^- represent the positive and the negative parts of the function Θ respectively, k_i=const. >0 is the thermal conductivity of the phase i (i=1: solid phase, i=2: liquid phase), then the variables $u=u(x)$, $q=q(x)$ on Γ_2, α=const.>0, $B=B(x)=k_2\,b(x)>0$ on Γ_1 are related in the following way

(1.2) $$\Delta u = 0 \ \text{ in } \Omega , \quad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q \ , \quad -\frac{\partial u}{\partial n}\Big|_{\Gamma_1} = \alpha\,(u-B) ,$$

whose variational formulation is given by

(1.3) $$a_\alpha(u,v) = L_{\alpha q B}(v)\ ,\ \forall v\in V\ ,\ u\in V\ ,$$

where

(1.4)
$$V=H^1(\Omega)\ ,\quad a(u,v)=\int_\Omega \nabla u.\nabla v\,dx\ ,\quad L_q(v)= -\int_{\Gamma_2} q\,v\,d\gamma\ ,$$
$$a_\alpha(u,v)= a(u,v)+\alpha\int_{\Gamma_1} u\,v\,d\gamma\ ,\qquad L_{\alpha q B}(v)=L_q(v)+\alpha\int_{\Gamma_1} B\,v\,d\gamma\ .$$

The bilinear form a_α is coercive on V for each $\alpha>0$ because there exists $M_\alpha>0$ such that [KiSt,Ta1]

(1.5) $$a_\alpha(v,v) \geq M_\alpha \|v\|^2\quad ,\forall\ v\in V\ ,$$

where $\|\ \|$ represents the classic norm of the Sobolev space V.

In [TaTa], a sufficient condition for the existence of a phase change in Ω was obtained (that is, there exist in Ω the liquid and solid phases, i.e. function u (or equivalently Θ) is a solution of non-constant sign of (1.2) or (1.3)) and it is given in the following way : There exists a steady-state two-phase Stefan problem in Ω (i.e. u is of non-constant sign in Ω) for

(1.6) $$q_m(\alpha,B) < q < q_M(\alpha,B)\quad ,\ \alpha>0\ ,$$

for each B=const.>0, where the function q_m and q_M are given by

(1.7) $$q_m(\alpha,B)= \frac{B\,|\Gamma_2|}{\Lambda(\alpha)}\ ,\ q_M(\alpha,B)= \frac{B\,|\Gamma_1|\,\alpha}{|\Gamma_2|}\ ,$$

and $\Lambda=\Lambda(\alpha)$ has an adequate expression. Moreover, $q_m= q_m(\alpha,B)$ is an increasing monotone function of $\alpha>0$, which satisfies

(1.8) $$q_m(0^+,B) = q_M(0^+,B) = 0\ ,\ q_m(+\infty,B) = q_o(B) = \frac{B\,|\Gamma_2|}{C}\ ,$$

where $C = C(\Omega, \Gamma_1, \Gamma_2) > 0$ is an adequate positive constant [Ta3,TaTa].

In §II., a necessary and sufficient condition to obtain a steady-state two-phase Stefan problem in Ω is given. Moreover; we obtain that for each α=const. >0 and B=const. >0 , there exists one and only one interval ($q_1(\alpha,B)$, $q_2(\alpha,B)$) for q in which the solution u of (1.2) or (1.3) is of non-constant sign in Ω. We also characterize the expression of q_1 and q_2 as a function of $\alpha>0$ and B>0.

In §III, for the general case q=q(x) on Γ_2 and B=B(x)>0 on Γ_1 , we can state the following optimization problem :

$$\operatorname{Sup}_{q} \int_{\Gamma_2} q \, d\gamma \quad \text{such that } u \geq 0 \text{ in } \bar{\Omega} ,$$

In §IV we give three examples in which the solution of the different problems presented is explicitly known[Ta2].

This paper was motivated by [GoTa,TaTa]. For a general introduction for studying a mixed boundary value problem for the Laplace equation with the finality of deciding when it exhibits a solution of non-constant sign, see [Ta3].

Remark 1. The boundary portions Γ_1 and Γ_2 may be separated by a boundary portion Γ_3 (disjoint from Γ_1 and Γ_2) that will behave like a heat-isolating wall, i.e. with a null heat flux over it. This new variant does not introduce any essential modification in the analysis of the problems to be formulated.

II. NECESSARY AND SUFFICIENT CONDITIONS FOR A SOLUTION OF (1.3) OF NON-CONSTANT SIGN

We generalise Theorem 18 of [TaTa] for problem (1.2.) or (1.3) with α, q, B=const.>0.

Theorem 1. Problem (1.2) or (1.3) represents a steady-state two-phase Stefan problem (or equivalently, the solution of (1.2) or (1.3) is of non-constant sign) if and only if the heat flux q verifies the following inequalities

(2.1) $$q_1(\alpha,B) < q < q_2(\alpha,B) \quad , \alpha > 0 \ , \ B > 0 ,$$

where $q_1 = q_1(\alpha,B)$ and $q_2 = q_2(\alpha,B)$, are given by (2.7) and (2.8) respectively.

Proof. Function $u = u_{\alpha q B}$, solution of (1.2) or (1.3), can be expressed by

(2.2) $$u_{\alpha q B} = B - q \, U_\alpha \quad \text{in } \Omega ,$$

where $U_\alpha = U_\alpha(x)$ is defined by

(2.3) $$\Delta U_\alpha = 0 \text{ in } \Omega , \quad - \frac{\partial U_\alpha}{\partial n}\Big|_{\Gamma_1} = \alpha \, U_\alpha , \quad \frac{\partial U_\alpha}{\partial n}\Big|_{\Gamma_2} = 1 ,$$

whose variational formulation is given by [KiSt]

(2.4) $$a_\alpha(U_\alpha, v) = \int_{\Gamma_2} v \, d\gamma , \ \forall v \in V , \quad U_\alpha \in V .$$

If we choose $v = U_\alpha^-$ in (2.4), we obtain

$$M_\alpha \| U_\alpha^- \|^2 \le a_\alpha (U_\alpha^-, U_\alpha^-) = - \int_{\Gamma_2} U_\alpha^- \, d\gamma \le 0 ,$$

that is $U_\alpha^- = 0$ in $\bar{\Omega}$, i.e. $U_\alpha \ge 0$ in $\bar{\Omega}$. Moreover, if we choose $v = (U_\alpha - \operatorname{Inf}_{\Gamma_1} U_\alpha)^- \in V$ in (2.4), we also obtain that $U_\alpha \ge \operatorname{Inf}_{\Gamma_1} U_\alpha$ in $\bar{\Omega}$. Therefore, we can deduce that [KiSt,PrWe]:

(2.5) $$U_\alpha > 0 \text{ in } \bar{\Omega} .$$

By using the following results for the function $u_{\alpha q B}$ [TaTa]

(2.6) $$\operatorname{Min}_{\Gamma_2} u_{\alpha q B} = \operatorname{Min}_{\bar{\Omega}} u_{\alpha q B} \le u_{\alpha q B} \le \operatorname{Max}_{\bar{\Omega}} u_{\alpha q B} = \operatorname{Max}_{\Gamma_1} u_{\alpha q B} \text{ in } \bar{\Omega} ,$$

we can obtain the thesis by virtue of the following equivalences (a) and (b), given by

(a) $$u_{\alpha q B} \ge 0 \text{ in } \bar{\Omega} \Leftrightarrow u_{\alpha q B} \ge 0 \text{ on } \Gamma_2 \Leftrightarrow q \le q_1(\alpha,B) ,$$

where

(2.7) $$q_1(\alpha,B) = \operatorname{Min}_{\Gamma_2} \left(\frac{B}{U_\alpha}\right) = \frac{B}{\operatorname{Max}_{\Gamma_2} (U_\alpha)} ,$$

and

(b) $$u_{\alpha q B} \le 0 \text{ in } \bar{\Omega} \Leftrightarrow u_{\alpha q B} \le 0 \text{ on } \Gamma_1 \Leftrightarrow q \ge q_2(\alpha,B) ,$$

where

(2.8) $$q_2(\alpha,B) = \operatorname{Max}_{\Gamma_1} \left(\frac{B}{U_\alpha}\right) = \frac{B}{\operatorname{Min}_{\Gamma_1} (U_\alpha)} .$$

<u>Remark 2</u>. We can generalize the above results for a given $B = B(x) > 0$ on Γ_1 by considering

(2.9) $$q_1(\alpha,B) = \operatorname{Min}_{\Gamma_2} \left(\frac{u_{\alpha B}}{U_\alpha}\right) , \quad q_2(\alpha,B) = \operatorname{Max}_{\Gamma_1} \left(\frac{u_{\alpha B}}{U_\alpha}\right) ,$$

where function $u_{\alpha B} = u_{\alpha B}(x)$ is defined by

(2.10) $$\Delta u_{\alpha B} = 0 \text{ in } \Omega , \quad -\frac{\partial u_{\alpha B}}{\partial n}\Big|_{\Gamma_1} = \alpha (u_{\alpha B} - B) , \quad \frac{\partial u_{\alpha B}}{\partial n}\Big|_{\Gamma_2} = 0 ,$$

whose variational formulation is given by [KiSt]

(2.11) $$a_\alpha(u_{\alpha B}, v) = \alpha \int_{\Gamma_1} B v \, d\gamma , \ \forall v \in V , \quad u_{\alpha B} \in V ,$$

for each $\alpha > 0$. Moreover, we have

(2.2 bis) $$u_{\alpha q B} = u_{\alpha B} - q \, U_\alpha .$$

Now, we can obtain a relationship among functions q_m and q_M, defined by (1.7) [TaTa], and functions q_1 and q_2, defined by (2.7) and (2.8) respectively.

Theorem 2. (i) Function U_α verifies the following properties ($\alpha>0$):

(2.12) $$\int_{\Gamma_1} U_\alpha \, d\gamma = \frac{|\Gamma_2|}{\alpha} \, ,$$ (2.13) $$\int_{\Gamma_2} U_\alpha \, d\gamma = A(\alpha) \, ,$$

(2.14) $$a(U_\alpha,U_\alpha) = \frac{d[\alpha \, A(\alpha)]}{d\alpha} \, ,$$

where function $A=A(\alpha)>0$ is defined in [TaTa].

(ii) We have the following inequalities:

(2.15) $$q_1(\alpha,B) \le q_m(\alpha,B) < q_M(\alpha,B) \le q_2(\alpha,B) \, , \quad \forall \, \alpha,B>0 \, .$$

Moreover, we have that (for all $B>0$):

(2.16) $$q_1(\alpha,B) = q_m(\alpha,B) \iff U_\alpha|_{\Gamma_2} = \text{Const.} \left(= \frac{A(\alpha)}{|\Gamma_2|}\right),$$

(2.17) $$q_2(\alpha,B) = q_M(\alpha,B) \iff U_\alpha|_{\Gamma_1} = \text{Const.} \left(= \frac{|\Gamma_2|}{\alpha|\Gamma_1|}\right) .$$

(iii) The particular case, defined in §V of [TaTa] is characterized by

(2.18) $$a(U_\alpha,U_\alpha) = \text{Const.} = C>0 \, , \ \forall \, \alpha>0 \, ,$$

where $C>0$ is a positive constant defined in [Ta3].

Proof. (i) By choosing $v=1 \in V$ in (2.4) we obtain (2.12). By using (2.2) and formula (IV-26) of [TaTa] we deduce for $A(\alpha)$ the expression (2.13). Moreover, we have (2.14) by using formula (IV.40) of [TaTa] and the fact that

(2.19) $$a(u_{\alpha qB},u_{\alpha qB}) = q^2 \, a(U_\alpha,U_\alpha) \, , \ \forall \, \alpha,q,B>0 \, .$$

Therefore we also obtain (iii).

(ii) By using the above expression (2.12) and (2.13) and the definitions of q_m, q_M, q_1 and q_2 we deduce after elementary manipulations the following inequalities

(2.20) $$q_1(\alpha,B) \le q_m(\alpha,B) \ \text{ and } \ q_M(\alpha,B) \le q_2(\alpha,B) \, .$$

The remaining inequality $q_m < q_M$ for $\alpha,B>0$ was proved in [TaTa].

Remark 3. We remark here that function $A(\alpha)$ is explicitely known for the particular case, defined in [TaTa]. In this case, we have that

(2.21) $$A(\alpha) = C + \frac{1}{\alpha}\frac{|\Gamma_2|^2}{|\Gamma_1|}$$

Moreover, constant C can be also obtained by the following expression

(2.22) $$C = a(U_\alpha, U_\alpha) \ , \ \forall \ \alpha > 0 \ .$$

III. SOME OPTIMIZATION PROBLEM WITH STATE RESTRICTIONS

We consider the general case with $q \in L^2(\Gamma_2)$ and b or $B \in H^{\frac{1}{2}}(\Gamma_1)$ and α=const.>0. Let $T : Q \to S$ be the application defined by

$$T(q) = u_{\alpha q B}$$

where

(3.1) $$S = \{ v \in V / \ \Delta v = 0 \text{ in } \Omega \, , \ -\frac{\partial v}{\partial n}|_{\Gamma_1} = \alpha \ (v - B) \} \, ,$$

$$S_o = \{ v \in V / \ \Delta v = 0 \text{ in } \Omega \, , \ -\frac{\partial v}{\partial n}|_{\Gamma_1} = \alpha \ v \} \, , \quad Q = L^2(\Gamma_2) \ ,$$

and $u_{\alpha q B}$ is the unique solution of problem (1.2) or (1.3). Let be the set

(3.2) $$S^+ = \{ v \in S / \ v \geq 0 \ \text{ in } \bar{\Omega} \} \, ,$$

and we define

(3.3) $$Q^+ = T^{-1}(S^+) = \{ q \in Q \ / \ T(q) \in S^+ \} = \{ q \in Q \ / \ u_{\alpha q B} \geq 0 \text{ in } \bar{\Omega} \} \, ,$$

then the whole material Ω is in the liquid phase if the heat flux $q \in Q^+$.

<u>Lemma</u> 3. (i) Application T can be decomposed in the form $T = T_1 + T_2$, where $T_2 : Q \to S_0$ is a linear and continuous application and $T_1 : Q \to S$ is a constant application defined by $T_1(q) = u_{\alpha B}$, with $u_{\alpha B}$ the unique solution of (2.11).

(ii) Q^+ is a convex set.

<u>Proof</u> (i) Let $u_2 = u_2(q) \in S_o$ be the unique solution of the variational equality [KiSt]

(3.4) $$a_\alpha(u_2, v) = -\int_{\Gamma_2} qv \ d\gamma \ , \ \forall v \in V \ , \quad u_2 \in V \ .$$

From the uniqueness of (1.3) we have that $u_{\alpha q B} = u_{\alpha B} + u_2(q)$. Therefore we can define $T_2(q) = u_2(q)$ and then part (i) is achieved.

(ii) It follows from the fact that T is an affine application (part (i)) and S^+ is a convex set.

Let $F : Q \to \mathbb{R}$ and $J : S \to \mathbb{R}$ be the functionals, defined by

(3.5) $$F(q)=\int_{\Gamma_2} q\, d\gamma\ , \qquad J(v)= -\int_{\Gamma_2} \frac{\partial v}{\partial n}\, d\gamma\ ,$$

which are linear and therefore convex functionals.

We consider the following optimization problem with state restrictions, defined by :

$$(\,P\,):\qquad \operatorname*{Sup}_{q\in Q^+}\ F(q)$$

that consist in finding the maximum total heat flow over Γ_2 so that the whole material is in the liquid phase.

The following optimization problem in S^+ is considered by

$$(\,NP\,):\qquad \operatorname*{Sup}_{v\in S^+}\ J(v)$$

which turns to be a new formulation of (P).

We will assume that the domain Ω, the boundary portions Γ_1 and Γ_2, and the function B on Γ_1 satisfy the necessary conditions to have the following regularity properties (The three examples we present to the end verify these properties) :

(i) $u_{\alpha qB} \in C^0(\bar{\Omega})$ (It is sufficient that $u_{\alpha qB} \in H^2(\Omega)$ for $n\leq 3$) ,

(ii) The element u^*, defined by (3.7), satifies that $\frac{\partial u^*}{\partial n}|_{\Gamma_2} \in Q$ (It is sufficient that $u^* \in H^2(\Omega)$).

(iii) The element v_o, defined by (3.16), satisfies that $\frac{\partial v_o}{\partial n}|_{\Gamma_2} \in Q$ and $\frac{\partial v_o}{\partial n}|_{\Gamma_2} > 0$ a.e on Γ_2 (It is sufficient that $v_o \in H^2(\Omega) \cap C^0(\bar{\Omega})$).

We have the following theorem of existence and uniqueness of solution for problems (P) and (NP) which follows the method developed in [GoTa].

<u>THEOREM</u> 4 (i) There exists an unique solution $q^* = q^*_{\alpha B} \in Q^+$ of the optimization problem (P) which is given by

(3.6) $$q^* = -\frac{\partial u^*}{\partial n}|_{\Gamma_2}$$

where u^* is the solution of the problem

(3.7) $$\Delta u^* = 0 \ \text{ in } \Omega\ , \ -\frac{\partial u^*}{\partial n}|_{\Gamma_1} = \alpha(u^* - B)\ ,\ u^*|_{\Gamma_2} = 0\ ,$$

whose variational formulation is given by

(3.8) $$a_\alpha(u^*,v) = \alpha\int_{\Gamma_1} Bv\, d\gamma\ ,\ \forall v \in V_2\ ,\quad u^* \in V_2\ ,$$

with

(3.9) $$V_2=\{v \in V/ \; v|_{\Gamma_2} =0\} .$$

(ii) The optimization problem (NP) has an unique solution which is given by u^* .

Proof. (a) Element u^* , defined by (3.7) or (3.8) verifies that $u^*>0$ in Ω and $u^* \geq 0$ in $\bar{\Omega}$, because if we choose $v=w \in V_2$ in (3.8) with $w=(u^*)^-$, we obtain

$$M_\alpha \, \| w\|^2 \leq a_\alpha(w,w)= - \alpha\int_{\Gamma_1} B \, w \, d\gamma \leq 0 \ ,$$

that is $w=0$ in $\bar{\Omega}$.

Let $u_q(=u_{\alpha qB}) \in S$ be the element that corresponds to $q \in Q^+$, that is $T(q)=u_{\alpha qB}$ Then, function $z=u_q-u^* \in S_o$ satisfies the problem

(3.10) $$\Delta z=0 \text{ in } \Omega \, , \; - \frac{\partial z}{\partial n}|_{\Gamma_1}= \alpha z \, , \; z|_{\Gamma_2} = u_q|_{\Gamma_2} \geq 0 \, ,$$

whose variational formulation is given by

(3.11) $$a_\alpha(z,v)= 0 \ , \forall v \in V_2 \ , \quad z \in u_q + V_2 \, ,$$

and verifies that $z\geq 0$ in $\bar{\Omega}$ by choosing $v=z^- \in V_2$ in (3.11).

Therefore, we deduce that

(3.12) $$F(q^*) -F(q) \ = \int_{\Gamma_2} (q^*-q) \, d\gamma \ = \int_{\Gamma_2} \frac{\partial z}{\partial n} \, d\gamma \ = -\int_{\Gamma_1} \frac{\partial z}{\partial n} \, d\gamma = \alpha\int_{\Gamma_1} z \, d\gamma \geq 0 \, ,$$

then q^* realizes the maximum of functional F.

(b) Let $\Psi = C^0 (\Gamma_2)$. Let $D : S \to \Psi$ be the application defined by

(3.13) $$D(v)= -v|_{\Gamma_2}$$

and the cone $P= \{ p\in\Psi \, / \, p\geq 0 \text{ on } \Gamma_2 \}$ which has a non empty interior.

Taking into account (2.6) [TaTa], problem (NP) may be reformulated as follows

(NPbis) $$\sup_{v\in S, \; D(v)\leq 0} J(v) \ .$$

(c) Let u be a solution of (NPbis). From [Be, EkTe] we deduce that there exists a Lagrange multiplier $\mu \in \Psi^*$ (dual of Ψ), $\mu \geq 0$ (i.e. $\langle\mu, p\rangle \geq 0, \forall p\in P$) so that the following conditions are satisfied

$$(3.14)\quad \begin{cases} \text{(i)} \quad -J(v) + \langle \mu, D(v) \rangle \geq -J(u), \ \forall v \in S \ , \\ \text{(ii)} \quad \langle \mu, D(u) \rangle = \int_{\Gamma_2} \mu\, D(u)\, d\gamma = 0 \ . \end{cases}$$

From (3.14 i,ii) and after elementary manipulations we obtain that

$$(3.15)\qquad \int_{\Gamma_2} \left(\frac{\partial w}{\partial n} - \mu\, w \right) d\gamma = 0, \ \forall w \in S_o.$$

Let $v_o \in S_o$ be the element which satisfies the problem

$$(3.16)\qquad \Delta v_o = 0 \text{ in } \Omega\ , \ -\frac{\partial v_o}{\partial n}\Big|_{\Gamma_1} = \alpha\, v_o \ , \ v_o|_{\Gamma_2} = 1 \ ,$$

whose variational formulation is given by

$$(3.17)\qquad a_\alpha(v_o,v) = 0 \ , \ \forall v \in V_2 \ , \ v_o \in 1 + V_2 \, .$$

Taking into account the equality

$$(3.18)\qquad \int_{\Gamma} \frac{\partial v_1}{\partial n}\, v_2\, d\gamma = \int_{\Gamma} \frac{\partial v_2}{\partial n}\, v_1\, d\gamma \ , \text{ with } \Delta v_1 = \Delta v_2 = 0 \text{ in } \Omega \ ,$$

we obtain that

$$\int_{\Gamma_2} \frac{\partial w}{\partial n} d\gamma = \int_{\Gamma_2} v_o \frac{\partial w}{\partial n} d\gamma = \int_{\Gamma} v_o \frac{\partial w}{\partial n} d\gamma - \int_{\Gamma_1} v_o \frac{\partial w}{\partial n} d\gamma = \int_{\Gamma_2} w \frac{\partial v_o}{\partial n} d\gamma + $$
$$+ \int_{\Gamma_1} w \frac{\partial v_o}{\partial n} d\gamma - \int_{\Gamma_1} v_o \frac{\partial w}{\partial n} d\gamma = \int_{\Gamma_2} w \frac{\partial v_o}{\partial n} d\gamma \ ,$$

and therefore, from (3.15), we deduce that the Lagrange multiplier μ is given by

$$(3.19)\qquad \mu = \frac{\partial v_0}{\partial n}\Big|_{\Gamma_2} \in Q \ .$$

Element $v_0 \in S_0$ verifies $0 \leq v_0 \leq 1$ in $\bar{\Omega}$, $0 < v_0 < 1$ in Ω and $\mu > 0$ on Γ_2. From (3.14ii) we deduce that $u|_{\Gamma_2} = 0$, that is $u = u^*$.

d) Let $u^* \in S$, $v_0 \in S_0$ and $\mu \in Q$ be defined by (3.8), (3.17) and (3.19) respectively. Let v be any element that verifies $v \in S$, then we have

$$(3.20)\qquad -J(v) + \langle \mu, D(v) \rangle + J(u^*) = \int_{\Gamma_2} \left(\frac{\partial v}{\partial n} - \frac{\partial u^*}{\partial n} - \mu\, v\right) d\gamma =$$
$$= \int_{\Gamma_2} \left[\frac{\partial v_0}{\partial n}(v-u^*) - \mu\, v\right] d\gamma + \int_{\Gamma_1} \frac{\partial v_0}{\partial n}(v-u^*)\, d\gamma - \int_{\Gamma_1} v_0 \frac{\partial}{\partial n}(v-u^*)\, d\gamma = 0$$

then, by virtue of the theory of Lagrange multipliers [Be,EkTe], the element u^* realizes the optimum of the problem (NPbis) because it satisfies the sufficient conditions of optimality.

Taking into account (a), (b), (c) and (d) the thesis is achieved.

IV. EXAMPLES

We shall give three examples in which the solution of the different problems presented is explicitely known for α, q, B=const.>0 (We note $B=k_2\, b > 0$)

1) Example 1. The following data are considered:

$$n = 2 \quad , \Omega = (0,x_0) \times (0,y_0) \quad , x_0 > 0 \ , \ y_0 > 0 \ ,$$

$$\Gamma_1 = \{0\} \times [0,y_0] \quad , \quad \Gamma_2 = \{x_0\} \times [0,y_0] \quad ,$$

$$\Gamma_3 = (0,x_0) \times \{0\} \ \cup \ (0,x_0) \times \{y_0\}$$

obtaining

$$u_{\alpha q B}(x,y) = B - \frac{q}{\alpha} - q\,x \, , \quad U_\alpha(x,y) = \frac{1}{\alpha} + x \ , \qquad u_{\alpha B}(x,y) = B \ ,$$

$$u^* = u^*_{\alpha B}(x,y) = \frac{B\,\alpha}{1+\alpha\, x_0}(x_0 - x) \ , \ v_0 = v_{0\alpha}(x,y) = \frac{1+\alpha\, x}{1+\alpha\, x_0} \ ,$$

$$q^* = q^*_{\alpha B}(x,y) = \frac{B\,\alpha}{1+\alpha\, x_0} \ ((x,y) \in \Gamma_2) \ , \ \mu = \mu_\alpha(x,y) = \frac{\alpha}{1+\alpha\, x_0}((x,y) \in \Gamma_2) \ ,$$

$$F(q^*) = J(u^*) = \frac{B\,\alpha\, y_0}{1+\alpha\, x_0} \ , \ A(\alpha) = y_0\,(x_0 + \frac{1}{\alpha}) \ , \qquad C = x_0\, y_0 \ ,$$

$$q_1(\alpha,B) = q_m(\alpha,B) = \frac{B\,\alpha}{1+\alpha\, x_0} \ , \qquad q_2(\alpha,B) = q_M(\alpha,B) = B\,\alpha \ .$$

2) Example 2. The following data are considered:

$$n = 2 \ , 0 < r_1 < r_2 \ , \ \Gamma_3 = \emptyset \ ,$$

$$\Omega = \left\{(x,y) \ / \ r_1 < r = (\,x^2 + y^2\,)^{1/2} < r_2 \right\} \ ,$$

$$\Gamma_1 = \left\{(x,y) \ / \ r = r_1 \right\} , \ \Gamma_2 = \left\{(x,y) \ / \ r = r_2 \right\} \ ,$$

obtaining

$$u_{\alpha q B}(r) = B - \frac{q\, r_2}{\alpha\, r_1} - q\, r_2 \log(\tfrac{r}{r_1}) \ , \qquad U_\alpha(r) = r_2 \left(\frac{1}{\alpha\, r_1} + \log(\tfrac{r}{r_1}) \right) ,$$

$$u_{\alpha B}(r) = B \quad , \ u^* = u^*_{\alpha B}(r) = \frac{B\,\alpha\, r_1}{1 + \alpha\, r_1 \log(\frac{r_2}{r_1})} \log(\tfrac{r_2}{r}) \ ,$$

$$v_0 = v_{0\alpha}(r) = \frac{1 + \alpha\, r_1 \log(\frac{r}{r_1})}{1 + \alpha\, r_1 \log(\frac{r_2}{r_1})}\ ,\quad q^* = q^*_{\alpha B}(x,y) = \frac{B\, \alpha\, r_1}{r_2\, [1 + \alpha\, r_1 \log(\frac{r_2}{r_1})]}\ ((x,y) \in \Gamma_2)\ ,$$

$$\mu = \mu_\alpha(x,y) = \frac{\alpha\, r_1}{r_2 \left(1 + \alpha\, r_1 \log(\frac{r_2}{r_1})\right)}\ ((x,y) \in \Gamma_2),\quad F(q^*) = J(u^*) = \frac{2\pi\, B\, \alpha\, r_1}{1 + \alpha\, r_1 \log(\frac{r_2}{r_1})}\ ,$$

$$A(\alpha) = 2\pi\, r_2^2 \left(\frac{1}{\alpha\, r_1} + \log(\frac{r_2}{r_1}) \right)\ ,\qquad C = 2\pi\, r_2^2 \log(\frac{r_2}{r_1})\ ,$$

$$q_1(\alpha,B) = q_m(\alpha,B) = \frac{B}{r_2 \left(\frac{1}{\alpha\, r_1} + \log(\frac{r_2}{r_1}) \right)}\ ,\qquad q_2(\alpha,B) = q_M(\alpha,B) = \frac{B\, \alpha\, r_1}{r_2}\ .$$

3) Example 3. We take into account the same information of Example 2 but now for the case n=3; by doing this, we reach the following results ($r = (x^2 + y^2 + z^2)^{1/2}$):

$$u_{\alpha q B}(r) = B - \frac{q\, r_2^2}{\alpha\, r_1^2} - q\, r_2^2 \left(\frac{1}{r_1} - \frac{1}{r} \right)\ ,\quad U_\alpha(r) = r_2^2 \left(\frac{1}{\alpha\, r_1^2} + \frac{1}{r_1} - \frac{1}{r} \right)\ ,$$

$$u_{\alpha B}(r) = B\ ,\qquad u^* = u^*_{\alpha B}(r) = \frac{B}{\frac{1}{\alpha\, r_1^2} + \frac{1}{r_1} - \frac{1}{r_2}} \left(\frac{1}{r} - \frac{1}{r_2} \right)\ ,$$

$$v_0 = v_{0\alpha}(r) = \frac{\frac{1}{\alpha\, r_1^2} + \frac{1}{r_1} - \frac{1}{r}}{\frac{1}{\alpha\, r_1^2} + \frac{1}{r_1} - \frac{1}{r_2}}\ ,\qquad C = 4\pi\, \frac{r_2^3\, (r_2 - r_1)}{r_1}\ ,$$

$$q^* = q^*_{\alpha B}(r) = \frac{B}{r_2^2 \left(\frac{1}{\alpha\, r_1^2} + \frac{1}{r_1} - \frac{1}{r_2} \right)}\ ((x,y,z) \in \Gamma_2)\ ,$$

$$\mu = \mu_\alpha(r) = \frac{1}{r_2^2 \left(\frac{1}{\alpha\, r_1^2} + \frac{1}{r_1} - \frac{1}{r_2} \right)}\ ((x,y,z) \in \Gamma_2)\ ,$$

$$F(q^*) = J(u^*) = \frac{4\pi\, B}{\frac{1}{\alpha\, r_1^2} + \frac{1}{r_1} - \frac{1}{r_2}}\ ,\qquad A(\alpha) = 4\pi\, r_2^4 \left(\frac{1}{\alpha\, r_1^2} + \frac{1}{r_1} - \frac{1}{r_2} \right)\ ,$$

$$q_1(\alpha,B) = q_m(\alpha,B) = \frac{B}{r_2^2 \left(\frac{1}{\alpha\, r_1^2} + \frac{1}{r_1} - \frac{1}{r_2} \right)}\ ,\quad q_2(\alpha,B) = q_M(\alpha,B) = \frac{B\, \alpha\, r_1^2}{r_2^2}\ .$$

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