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Optimization of Heat Flux in Domains with Temperature Constraints[1]

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Abstract. In this paper, we deal with a heat flux optimization problem. We maximize the heat output flow on a portion of a domain's boundary, while on the other portion the distribution of the temperature is fixed. The maximization is carried out under the condition that there are no phase changes.

The problem is solved using a convex-functional optimization technique, on Banach spaces, within restricted sets, yielding existence and uniqueness of the solution. The explicit form of the solution and the corresponding Lagrange multipliers associated to the problem are also given.

In addition, other optimization problems related to the maximum bound of the heat flux with no phase change are solved.

Key Words. Steady-state Stefan problem, mixed elliptic differential problem, constraint optimization problems, heat flux optimization problem, Lagrange multipliers theory, explicit solutions.

1. Introduction

We consider a regular bounded domain Ω of R^n ($n=1, 2, 3$ for the applications), with a boundary $\Gamma=\Gamma_1 \cup \Gamma_2$ [$\text{meas}(\Gamma_1)>0$, $\text{meas}(\Gamma_2)>0$], and it is assumed that the phase change temperature is 0°C. On the portion Γ_1 of the boundary, we have a temperature b; on the portion Γ_2 of the remaining boundary, a heat flow q is imposed.

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Considering in Ω a stationary heat conduction problem, we have the following properties.

(i) If the temperature $b = b(x)$ takes on Γ_1 positive and negative values, independently of the values of $q = q(x)$ on Γ_2, then there is a phase change in Ω.

(ii) If the temperature is a constant $b > 0$ on Γ_1 and the output heat flux is a constant $q > 0$ on Γ_2, then there is a phase change in Ω only when q is large enough. A sufficient condition for the existence of a phase change in Ω was obtained in Refs. 1-2 and is given by the following inequality:

$$q > q_1 \equiv (k_1 b/C)\,\text{meas}(\Gamma_2), \tag{1}$$

where k_1 is the thermal conductivity coefficient of the liquid phase and $C = C(\Omega, \Gamma_2, \Gamma_1) > 0$ is a constant that has the physical dimension $[C] = (cm)^n$, n being the dimension of the space R^n.

In this paper, the following optimization problems are studied considering the phase-invariance restriction, with $b > 0$ on Γ_1.

(P1) Maximize the output flow $F(q)$,

$$F(q) = \int_{\Gamma_2} q(s)\, ds,$$

under the restriction $u(x) \geq 0$, $\forall x \in \Omega$.

(P2) Find the maximum bound of the flow density which does not allow a phase change. That is, determine $\bar{\lambda}$,

$$\bar{\lambda} = \max\{\lambda / q(s) \leq \lambda, \forall s \in \Gamma_2 \Rightarrow u(x) \geq 0, \forall x \in \Omega\}.$$

(P3) Given a flux form $q_0(\cdot)$, determine the maximum flow of this form which does not permit a phase change. That is, find

$$Q_M = \max\{\mu / q(s) - \mu q_0(s), \forall s \in \Gamma_2 \Rightarrow u(x) \geq 0, \forall x \in \Gamma_2\}.$$

In each of these problems, we identify the solution and show the procedures for their computation.

2. Model of the Problem and Its Properties

If θ represents the temperature on Ω and the substitution $u = k_1\theta^+ - k_s\theta^-$ is made, where k_s is the thermal conductivity coefficient of the solid phase, then the variables u, b, q are related in the following way (Refs. 3 and 4):

$$\Delta u = 0, \qquad \text{in } \Omega, \tag{2a}$$

$$u|_{\Gamma_1} = b_0\,(\equiv k_1 b^+ - k_s b^-) = k_1 b, \tag{2b}$$

$$-\partial u/\partial n|_{\Gamma_2} = q, \tag{2c}$$

whose variational formulation is given by

$$a(u, v-u) = -\int_{\Gamma_2} q(v-u)\, d\gamma, \qquad \forall v \in K, \tag{3a}$$

$$u \in K, \tag{3b}$$

where

$$V = H^1(\Omega), \qquad K = \{v \in V / v|_{\Gamma_1} = b_0\}, \tag{4a}$$

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v\, dx, \qquad V_0 = \{v \in V / v|_{\Gamma_1} = 0\}. \tag{4b}$$

Consider the sets

$$Q = \{q : \Gamma_2 \to R / q(\cdot) \in L^2(\Gamma_2)\}, \tag{5a}$$

$$S = \{u \in K / \Delta u = 0, \text{ in } \Omega\}, \tag{5b}$$

$$S^+ = \{u \in S / u \geq 0, \text{ in } \bar{\Omega}\}, \tag{5c}$$

and the mapping $T : Q \to S$, defined by

$$T(q) = u_q \in S, \tag{6}$$

where u_q is the unique solution to (3) or (2) under the hypotheses $b \in H^{1/2}(\Gamma_1)$, $q \in L^2(\Gamma_2)$. We will also assume that the domain and the boundary function $b(\cdot)$ satisfy the following properties:

(i) $u_q \in C(\bar{\Omega})$.

(ii) The solution to the problem

$$\Delta u = 0, \qquad \text{in } \Omega,$$

$$u|_{\Gamma_1} = b_0,$$

$$u|_{\Gamma_2} = 0,$$

satisfies

$$\partial u/\partial n|_{\Gamma_2} \in L^2(\Gamma_2).$$

(iii) The solution of the problem

$$\Delta v = 0, \qquad \text{in } \Omega,$$

$$v|_{\Gamma_1} = 0,$$

$$v|_{\Gamma_2} = 1,$$

satisfies

$$\partial v/\partial n|_{\Gamma_2} \in L^2(\Gamma_2) \quad \text{and} \quad \partial v/\partial n|_{\Gamma_2} > 0, \qquad \text{a.e. } x \in \Gamma_2.$$

If

$$Q^+ = T^{-1}(S^+) = \{q \in Q / T(q) \in S^+\} = \{q \in Q / u_q \geq 0, \text{ in } \bar{\Omega}\}, \tag{7}$$

then, there is no phase change in Ω if the heat flux $q \in Q^+$.

Remark 2.1. The boundary portions Γ_1 and Γ_2 may be separated by a boundary portion Γ_3, disjoint from Γ_1 and Γ_2, that will behave like a heat-isolating wall, i.e., with a null heat flow over it. This new variant does not introduce any essential modification in the analysis of the problems to be formulated.

Some Preliminary Properties. The first property consists in the generalization of Property 1 in Ref. 1, proved there for the case $q = \text{const.} > 0$.

Lemma 2.1. Let $u = u_q$ be the solution to (3) for $q \in Q$. Then, we have

(i) $a(u^-, u^-) = \int_{\Gamma_2} q u^- \, d\gamma$;
(ii) $u^- \neq 0$, in $\bar{\Omega} \Leftrightarrow u^- \neq 0$, in Γ_2.

Proof. (i) It suffices to use $v = u^+ \in K$ in (3).
(ii) Since $u^-|_{\Gamma_1} = 0$, we have the following equivalences:

$$u^- \neq 0, \text{ in } \bar{\Omega} \Leftrightarrow a(u^-, u^-) \neq 0 \Leftrightarrow \int_{\Gamma_2} q u^- \, d\gamma > 0 \Leftrightarrow u^- \neq 0, \text{ in } \Gamma_2. \qquad \square$$

Remark 2.2. It follows from property (ii) of Lemma 2.1 that, for a given $q \in Q$, there will be a phase change in Ω (u_q takes positive and negative values in Ω) if and only if u_q takes negative values over boundary portion Γ_2.

Now, we will show some simple properties that result from the definition of the operator T.

Lemma 2.2. The operator T, defined by (6), can be decomposed in the following form:

$$T = T_1 + T_2,$$

where

$$T_1 : Q \to S, \qquad T_1(q) = u_1 \text{ (independent of } q), \qquad \forall q \in Q, \tag{8a}$$

$$T_2 : Q \to V_0, \qquad T_2 \in L(Q, V_0). \tag{8b}$$

Proof. Let $u_1 \in K$ and $u_2 = u_2\{q\} \in V_0$ be the unique solution of the following problems (Ref. 5):

$$a(u_1, v - u_1) = 0, \qquad \forall v \in K, \tag{9a}$$

$$u_1 \in K, \tag{9b}$$

$$a(u_2, v) = -\int_{\Gamma_2} qv\, d\gamma, \qquad \forall v \in V_0, \tag{10a}$$

$$u_2 \in V_0. \tag{10b}$$

From (6), the uniqueness of u yields

$$u = u_1 + u_2.$$

T_1 and T_2 can be defined as follows:

$$T_1(q) = u_1, \qquad T_2(q) = u_2, \qquad \forall q \in Q. \tag{11}$$

The operator T_2 is continuous and linear, because it verifies the relation

$$\| T_2(q) \|_{V_0} \leq (\| \gamma_0 \| / \alpha) \| q \|_{L^2(\Gamma_2)}, \qquad \forall q \in L^2(\Gamma_2), \tag{12}$$

where $\gamma_0 : V \to V_0$ is the trace function and α is the coercive constant of the bilinear form a over V_0. □

Lemma 2.3. The set Q^+ is convex.

Proof. This follows from the fact that T is an affine operator and S^+ is a convex set. □

Next, we will show a monotone property that turns out to be valid for the case $b = b(x)$ and $q = q(x)$ with any sign.

Lemma 2.4. Let u_{bq} be the solution to (3) for $b \in H^{1/2}(\Gamma_1)$ and $q \in L^2(\Gamma_2)$. Thus, we have the following monotone property:

$$b_1 \leq b_2, \text{ on } \Gamma_1, \quad \text{and} \quad q_2 \leq q_1, \text{ on } \Gamma_2.$$

Then,

$$u_{b_1 q_1} \leq u_{b_2 q_2}, \qquad \text{in } \Omega. \tag{13}$$

Proof. To verify (13), the following equivalence will be taken into account (the notation is $u_i = u_{b_i q_i}$, $i = 1, 2$):

$$u_1 \leq u_2, \text{ in } \Omega \Leftrightarrow w = 0, \text{ in } \Omega, \qquad w = (u_2 - u_1)^-. \tag{14}$$

Since $w \in V_0$, if we use $v = u_1 + w \in K_{b_1}$ in the variational equality corresponding to u_1 and $v = u_2 + w \in K_{b_2}$ in the variational equality corresponding to u_2, and then substract both equalities, we have

$$0 \le a(w, w) = -\int_{\Gamma_2} (q_1 - q_2)\, d\gamma \le 0,$$

that is, $w = 0$ in Ω. □

3. Some Optimation Problems and Their Solutions

Next, some optimization problems with their respective solutions will be analyzed.

3.1. Total Maximum Heat Flow. Consider the functional $F: Q \to R$, defined by

$$F(q) = \int_{\Gamma_2} q\, d\gamma, \tag{15}$$

which turns to be linear and therefore convex. We consider the following optimization problem in Q^+:

$$\text{(P1)} \qquad \max_{q \in Q^+} F(q), \tag{16}$$

which consists in finding the total maximum heat flow over Γ_2 so that there will be no phase change in the material Ω. We have the following theorem of existence of solution for Problem (P1).

Theorem 3.1. There exists an element $\bar{q} \in Q^+$ so that we have

$$F(\bar{q}) = \max_{q \in Q^+} F(q). \tag{17}$$

Moreover, $\bar{q}$ is defined in Γ_2 in the following way:

$$\bar{q} = -\partial u_0 / \partial n, \tag{18}$$

where u_0 satisfies

$$u_0 \in S, \qquad u_0|_{\Gamma_2} = 0. \tag{19}$$

Proof. The element u_0, defined by (19), verifies that $u_0 > 0$ in Ω, $u_0 \ge 0$ in $\bar{\Omega}$; and, by the maximum principle (Ref. 6), we have that $\bar{q} > 0$ in Γ_2.

Let $u = u_q \in S$ be the element that corresponds to $q \in Q^+$. Let $v = u - u_0$ verify the following conditions:

$$\Delta v = 0, \qquad \text{in } \Omega, \tag{20a}$$

$$v|_{\Gamma_1} = 0, \tag{20b}$$

$$v|_{\Gamma_2} = u|_{\Gamma_2} \geq 0. \tag{20c}$$

Taking into account that v has its minimum in Γ_1 and if the differential equation which v satisfies is integrated in Ω, from the maximum principle we deduce that

$$F(\bar{q}) - F(q) = \int_{\Gamma_2} (\bar{q} - q)\, d\gamma = \int_{\Gamma_2} (\partial v/\partial n)\, d\gamma = -\int_{\Gamma_1} (\partial v/\partial n)\, d\gamma \geq 0.$$

Therefore, $\bar{q}$ yields the maximum of the functional F. □

Next, a new approach to the optimization problem (16) will be stated. Define a linear (and therefore convex) functional $J: S \to R$, by

$$J(v) = -\int_{\Gamma_2} (\partial v/\partial n)\, d\gamma. \tag{21}$$

The following optimization problem in s^+ is considered:

$$\max_{v \in S^+} J(v), \tag{22}$$

which turns to be a new formulation of (16).

Let $\Psi = C^0(\Gamma_2)$. Define the linear operator $B: S \to \Psi$ by

$$B(v) = -v|_{\Gamma_2};$$

and let

$$\text{cone } P = \{p \in \Psi \mid p \geq 0, \text{ on } \Gamma_2\}$$

which has a nonempty interior. Taking into account Lemma 2.1, we may formulate the optimization problem (22), in an equivalent way, as follows:

$$\max_{v \in S,\, B(v) \leq 0} J(v). \tag{23}$$

We have the following existence and uniqueness theorem for the solution of problem (23).

Theorem 3.2. The optimization problem (23) has a unique solution. Moreover, the element that produces the optimum is u_0, defined by (19).

Proof. (i) Let u be a solution to (23). From Ref. 7, we deduce that there exists a Lagrange multiplier $\mu \in \Psi^*$ (dual of Ψ), $\mu \geq 0$ [i.e., $\langle \mu, p \rangle \geq 0$,

$\forall p \in P]$, so that the following conditions are satisfied:

(a) $$-J(v)+\langle \mu, B(v)\rangle \geq -J(u), \qquad \forall v \in S; \tag{24a}$$

(b) $$\langle \mu, B(u)\rangle = \int_{\Gamma_2} \mu B(u)\, d\gamma = 0. \tag{24b}$$

From (24) we deduce that

$$\int_{\Gamma_2} \{(\partial u/\partial n - \partial v/\partial n) - \mu(u-v)\}\, d\gamma \leq 0, \qquad \forall v \in S.$$

Calling $w = u - v$, with $v \in S$, we have that

$$\int_{\Gamma_2} (\partial w/\partial n - \mu w)\, d\gamma \leq 0, \qquad \forall w \in V_0, \text{ with } \Delta w = 0, \text{ in } \Omega.$$

Then,

$$\int_{\Gamma_2} (\partial w/\partial n - \mu w)\, d\gamma = 0, \qquad \forall w \in V_0, \text{ with } \Delta w = 0, \text{ in } \Omega. \tag{25}$$

Let $v_0 \in V_0$ be the element that satisfies

$$\Delta v_0 = 0, \qquad \text{in } \Omega, \tag{26a}$$

$$v_0|_{\Gamma_1} = 0, \tag{26b}$$

$$v_0|_{\Gamma_2} = 1. \tag{26c}$$

Taking into account v_0 and using Green's formula, we obtain that

$$\int_{\Gamma_2} (\partial w/\partial n)\, d\gamma = \int_{\Gamma_2} v_0(\partial w/\partial n)\, d\gamma = \int_{\Gamma_2} w(\partial v_0/\partial n)\, d\gamma. \tag{27}$$

Therefore, from (25), we deduce that the Lagrange multiplier is given by

$$\mu = \partial v_0/\partial n|_{\Gamma_2} \in L^2(\Gamma_2). \tag{28}$$

Moreover, using the orthogonality condition given by (24b), we deduce that

$$\int_{\Gamma_2} u(\partial v_0/\partial n)\, d\gamma = 0.$$

Taking into account the fact that $u|_{\Gamma_2} \geq 0$ and the maximum principle for v_0, we obtain that $u|_{\Gamma_2} = 0$. We deduce the uniqueness of the solution to problem (23) because $u = u_0$.

(ii) Let u_0, v_0, μ be defined by (19), (26), (28), respectively. Let v be any element that satisfies $v \in S$. Then, we have

$$-J(v)+\langle\mu, B(v)\rangle+J(u_0)=\int_{\Gamma_2}\{v_0(\partial v/\partial n-\partial u_0/\partial n)-\mu v\}\,d\gamma$$

$$=\int_{\Gamma_2}\{(v-u_0)(\partial v_0/\partial n)-\mu v\}\,d\gamma$$

$$=-\int_{\Gamma_2}u_0(\partial v_0/\partial n)\,d\gamma=0. \tag{29}$$

Then, by virtue of the theory of Lagrange multipliers (Refs. 7 and 8), u_0 realizes the optimum because it satisfies the sufficient conditions of optimality. □

Taking into account Theorem 3.2 and the relationship existing between the optimization problems (17) and (23), we have the following corollary.

Corollary 3.1. The element $\bar{q}$, defined by (18), is the unique solution to the optimization problem (17).

3.2. Maximum Bound of the Heat Flow Density That Does Not Permit a Phase Change. Consider the original problem (2) or (3) with $b=b(x)>0$ over Γ_1 and $q=q(x)>0$ over Γ_2. Now, we must find $q_M>0$ so that, if $q\leq q_M$, we do not have a phase change in domain Ω, that is: Find

$$q_M>0/u_q\geq 0,\ \text{in }\Omega,\quad \forall q\leq q_M. \tag{30}$$

For the case $q=\text{const}>0$, we obtain the following property.

Lemma 3.1. The element q_M, defined by

$$q_M=\inf_{x\in\Gamma_2}[u_b(x)/u_3(x)], \tag{31}$$

satisfies the condition (30), where u_b and u_3 are respectively defined by (32) and (33), with

$$\Delta u_b=0,\qquad \text{in }\Omega, \tag{32a}$$

$$u_b|_{\Gamma_1}=b_0, \tag{32b}$$

$$\partial u_b/\partial n|_{\Gamma_2}=0, \tag{32c}$$

$$\Delta u_3=0,\qquad \text{in }\Omega, \tag{33a}$$

$$u_3|_{\Gamma_1}=0, \tag{33b}$$

$$\partial u_3/\partial n|_{\Gamma_2}=1. \tag{33c}$$

Proof. We have

$$u_b \geq \min_{x\in\Gamma_1} b_0(x) > 0, \qquad \text{in } \bar{\Omega},$$

$$u_3 > 0, \qquad \text{in } \Omega,$$

with

$$u_3|_{\Gamma_2} \leq M_3, \qquad M_3 > 0,$$

such that element q_M, defined by (31), verifies

$$q_M \geq \left[\min_{x\in\Gamma_1} b_0(x)\right] \Big/ M_3 > 0. \tag{34}$$

Taking into account the definition of $u_q = u_q(x)$, we deduce that $u_q = u_b - qu_3$, with which we obtain

$$u_q \geq 0, \text{ in } \bar{\Omega} \Leftrightarrow u_b - qu_3 \geq 0, \text{ on } \Gamma_2 \Leftrightarrow q \leq q_M. \tag{35}$$

□

If we now consider the case $q = q(x) > 0$ on Γ_2, we can deduce the following property that generalizes Lemma 3.1 to the nonconstant q case.

Theorem 3.3. If $q = q(x) > 0$ over Γ_2 satisfies

$$\max_{x\in\Gamma_2} q(x) \leq q_M, \tag{36}$$

where q_M is defined by (31), then there is no phase change in Ω, that is, $q_M \geq 0$ in Ω.

Proof. It suffices to use the monotony property (13) and the fact that $q \leq q_M$ on Γ_2. □

Next, we will consider some examples in which the constant q_M may explicitly be calculated.

Example 3.1. The following data are considered:

$$n = 2, \qquad \Omega = (0, x_0) \times (0, y_0), \quad \text{with } x_0 > 0, y_0 > 0, \tag{37a}$$

$$\Gamma_1 = \{0\} \times [0, y_0], \qquad \Gamma_2 = \{x_0\} \times [0, y_0], \tag{37b}$$

$$\Gamma_3 = (0, x_0) \times \{0\} \cup (0, x_0) \times \{y_0\}, \tag{37c}$$

obtaining

$$u_b(x, y) = k_2 b, \qquad u_3(x, y) = x, \tag{38a}$$

$$q_M = k_2 b / x_0. \tag{38b}$$

Example 3.2. The following data are considered:

$$n=2, \qquad 0<r_1<r_2, \tag{39a}$$

$$\Omega=\{(x,y)/r_1<r=(x^2+y^2)^{1/2}<r_2\}, \tag{39b}$$

$$\Gamma_1=\{(x,y)/r=r_1\}, \qquad \Gamma_2=\{(x,y)/r=r_2\}, \tag{39c}$$

obtaining

$$u_b(x,y)=k_2 b, \qquad u_3(x,y)=r_2\log(r/r_1), \tag{40a}$$

$$q_M=(k_2,b)/[r_2\log(r_2/r_1)]. \tag{40b}$$

Example 3.3. For the case $n=3$, the same data as in Example 3.2 are considered. Taking

$$r=(x^2+y^2+z^2)^{1/2},$$

we obtain

$$u_b(x,y,z)=k_2 b, \qquad u_3(x,y,z)=r_2^2(1/r_1-1/r), \tag{41a}$$

$$q_M=k_1 b r_1/[r_2(r-r_1)]. \tag{41b}$$

Remark 3.1. It can be observed that, in the three previous examples, the obtained element q_M agrees, for the case $q=\text{const.}>0$, with the element that produces the necessary and sufficient condition so that there is a phase change in Ω (Ref. 9).

3.3. Maximum Bound of the Constant Involved in the Form of the Heat Flow Density That Does Not Permit a Phase Change. We consider the original problem (2) or (3) with $b=b(x)>0$ on Γ_1, with a given form for the heat flow density,

$$q(x)=Qq_0(x), \tag{42}$$

where $Q>0$ and $q_0=q_0(x)>0$ on Γ_2. The problem consists in finding $Q_M>0$ so that, if $Q\le Q_M$, we do not have a phase change in the domain Ω, that is: Find

$$Q_M>0/u_q\ge 0, \text{ in } \Omega, \qquad \forall Q\le Q_M. \tag{43}$$

We get the following property.

Theorem 3.4. The element Q_M, defined by

$$Q_M=\inf_{x\in\Gamma_2}[u_b(x)/u_4(x)], \tag{44}$$

satisfies condition (43), where u_b and u_4 are defined respectively by (32) and (45), being

$$\Delta u_4 = 0, \qquad \text{in } \Omega, \tag{45a}$$

$$u_4|_{\Gamma_1} = 0, \tag{45b}$$

$$\partial u_4/\partial n|_{\Gamma_2} = q_0. \tag{45c}$$

Proof. In an analogous way to what was done in Lemma 3.1, we have that

$$u_q = u_b - Qu_4, \quad u_4 > 0, \qquad \text{in } \Omega,$$

with

$$Q_M \geq \left[\min_{x \in \Gamma_1} b_0(x)\right] \Big/ m_4 > 0, \tag{46}$$

and the following equivalence holds:

$$u_q \geq 0 \text{ in } \bar{\Omega} \Leftrightarrow Q \leq Q_M \tag{47}$$

so that the element Q_M, defined by (44), satisfies condition (43). □

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