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DETERMINATION OF UNKNOWN COEFFICENTS OF A SEMI-INFINITE MATERIAL THROUGH A SIMPLE MUSHY ZONE MODEL FOR THE TWO-PHASE STEFAN PROBLEM

ADRIANA MARIA GONZALEZ

Depto. de Matemática, Fac. de Ciencias Exactas, Físico-Químicas y Naturales, Univ. Nac. de Río Cuarto, Ciudad Universitaria, Ruta 36, km 601, (5800) Río Cuarto, Argentina

DOMINGO ALBERTO TARZIA†

Depto. Matemática, FCE, Univ. Austral, Paraguay 1950, (2000) Rosario, Argentina

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Abstract—We use a simple mushy zone model in a two-phase solidification problem (Stefan problem) for the simultaneous determination of unknown coefficients of a semi-infinite material with an overspecified condition on the fixed face. We find the necessary and sufficient conditions for the existence of a solution and the corresponding formulae for the unknown coefficients. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

We consider a semi-infinite material with mass densities $\rho > 0$ equal in both solid and liquid phases and we can assume, without loss of generality, that the phase-change temperature is 0°C.

If the material is initially assumed to be liquid at the constant temperature $E > 0$ and a constant temperature $-D < 0$ is imposed on the fixed face $x = 0$, then three distinct regions can be distinguished (for a mathematical and properties description of this simple model see [1]; for the one-phase model see [2]):

(H_1) The liquid phase, at temperature $\theta_2 = \theta_2(x, t) > 0$, occupying the region $x > r(t)$, $t > 0$.

(H_2) The solid phase, at temperature $\theta_1 = \theta_1(x, t) < 0$, occupying the region $0 < x < s(t)$, $t > 0$.

(H_3) The mushy zone, at temperature 0, occupying the region $s(t) \le x \le r(t)$, $t > 0$. We make two assumptions on its structure:

(a) The material in the mushy zone contains a fixed fraction $\epsilon\lambda$ (with $0 < \epsilon < 1$) of the total latent heat $\lambda > 0$, i.e.

$$k_1 \theta_{1_x}(s(t), t) - k_2 \theta_{2_x}(r(t), t) = \lambda\rho(\epsilon \dot{s}(t) + (1 - \epsilon)\dot{r}(t)), \qquad t > 0. \tag{1.1}$$

(b) The width of the mushy zone is inversely proportional (with constant $\gamma > 0$) to the temperature gradient at the point $(s(t), t)$, i.e.

$$\theta_{1_x}(s(t), t)(r(t) - s(t)) = \gamma, \qquad t > 0. \tag{1.2}$$

We suppose that the temperature $\theta = \theta(x, t)$ of the material is defined by

$$\theta(x, t) = \begin{cases} \theta_1(x, t) < 0 & \text{if } 0 < x < s(t), t > 0 \\ 0 & \text{if } s(t) \le x \le r(t), t > 0 \\ \theta_2(x, t) > 0 & \text{if } x > r(t), t > 0. \end{cases} \tag{1.3}$$

† To whom all correspondence should be addressed. E-mail tarzia@uaufce.edu.ar.

The governing differential equations take the following forms for the solid and liquid phases:

$$\alpha_1 \theta_{1_{xx}}(x, t) = \theta_{1_t}(x, t), \qquad 0 < x < s(t), \qquad t > 0 \tag{1.4}$$

$$\alpha_2 \theta_{2_{xx}}(x, t) = \theta_{2_t}(x, t), \qquad x > r(t), \qquad t > 0 \tag{1.5}$$

where $c_i > 0$, $k_i > 0$ and $\alpha_i = a_i^2 = k_i/\rho c_i > 0$ are the specific heat, the thermal conductivity and the diffusion coefficient for the phase i ($i = 1$ denotes solid phase; $i = 2$ denotes liquid phase) respectively.

The conditions at the solid–mushy interface $x = s(t)$ and the mushy–liquid interface $x = r(t)$ are given by (1.1), (1.2) and the requirement of the continuity of the temperature, i.e.

$$\theta_1(s(t), t) = \theta_2(r(t), t) = 0, \qquad t > 0. \tag{1.6}$$

The initial and boundary conditions are given by

$$\theta_1(0, t) = -D < 0, \qquad t > 0 \tag{1.7}$$

$$\theta_2(x, 0) = \theta_2(+\infty, t) = E > 0, \qquad x > 0, \qquad t > 0, \tag{1.8}$$

$$s(0) = r(0) = 0. \tag{1.9}$$

We consider an overspecified heat flux condition [3, 4] on the fixed face $x = 0$ which is given by [1, 4–7]

$$k_1 \theta_{1_x}(0, t) = \frac{h_o}{\sqrt{t}}, \qquad t > 0, \quad \text{with} \quad h_o > 0. \tag{1.10}$$

If by means of a phase-change experiment we are able to measure certain quantities, then we shall find formulae for the simultaneous determination of the unknown coefficients (ϵ, γ denote parameters of the mushy zone; λ, ρ, c_1, c_2, k_1, k_2 denote thermal coefficients of the material).

We shall also prove that the different problems for determining several unknown coefficients, posed in the next sections, do not always have an explicit solution. Moreover, it does exist iff some complementary conditions for the corresponding data are verified. In this paper, we generalize the results obtained in [5] for the particular case $\epsilon = 1$ and $\gamma = 0$ (i.e. without mushy region) and those obtained in [7] for the one-phase case. In [4] several references on free-moving boundary problems and determination of physical coefficients are given.

In Section 2 we shall consider the simple mushy zone model for the two-phase Stefan problem for determining one unknown thermal coefficient of a semi-infinite material with an overspecified condition on the fixed face, supposing the free boundaries $x = s(t)$ and $x = r(t)$ are unknown. The results obtained for the eight possible cases are considered in Appendix C (Table 1) which shows both the necessary and sufficient conditions to be verified by the data for the existence and uniqueness of the solution and the expression of the corresponding unknown coefficient. Moreover, we shall also prove the respective properties for the determination of ϵ (case 3) and the determination of k_2 (case 7).

In Section 3 we shall consider the same model for determining two unknown thermal coefficients of a semi-infinite material with an overspecified condition on the fixed face, supposing known the expression for the moving boundary $x = s(t)$. The results obtained for the 28 possible cases are considered in Appendix D (Table 2) which shows both the necessary and sufficient conditions to be verified by the data for the existence of the solution and the expression of the corresponding unknown coefficients. There are several cases where the moving boundary problem has a unique solution iff some conditions are verified. Moreover, we shall also prove the respective properties for the determination of k_1 and k_2 (case 9), the determination of ϵ and k_2 (case 17), the determination of c_2 and k_2 (case 19) and, the determination of γ and k_2 (case 23).

The functions and the restrictions used in the text and, Appendices C and D are summarized in Appendix A and Appendix B respectively.

2. DETERMINATION OF ONE UNKNOWN THERMAL COEFFICIENT

Taking into account the hypotheses (H_1)–(H_3) we can formulate the following:

PROBLEM (P_1). *Find the free boundaries $x = s(t)$ and $x = r(t)$, defined for $t > 0$ with $0 < s(t) < r(t)$ and $s(0) = r(0) = 0$, the temperature $\theta = \theta(x, t)$, defined by* (1.3) *for $x > 0$ and $t > 0$, and one of the eight unknown thermal coefficients ϵ, γ, λ, ρ, c_1, c_2, k_1, k_2 such that they satisfy the conditions* (1.1), (1.2), (1.4)–(1.10) *where $D > 0$, $E > 0$ and $h_o > 0$ are data and they must be known or determined by an experience of phase-change* [8].

The solution of this problem is given [1, 6, 9–11] by

$$\theta_1(x, t) = -D + \frac{D}{f\left(\frac{\sigma}{a_1}\right)} f\left(\frac{x}{2a_1\sqrt{t}}\right) \tag{2.1}$$

$$\theta_2(x, t) = \frac{-Ef\left(\frac{\omega}{a_2}\right)}{1 - f\left(\frac{\omega}{a_2}\right)} + \frac{E}{1 - f\left(\frac{\omega}{a_2}\right)} f\left(\frac{x}{2a_2\sqrt{t}}\right) \tag{2.2}$$

$$s(t) = 2\sigma\sqrt{t}, \qquad \sigma > 0 \tag{2.3}$$

$$r(t) = 2\omega\sqrt{t}, \qquad \omega > \sigma \tag{2.4}$$

where f is the error function, the coefficient ω is given by

$$\omega = \omega(\sigma) = a_1 W\left(\frac{\sigma}{a_1}\right) \tag{2.5}$$

and the coefficient σ and the unknown thermal coefficient are obtained by solving the following system of equations:

$$\frac{h_o}{\lambda\rho a_1}\exp\left(-\frac{\sigma^2}{a_1^2}\right) - \frac{Ek_2}{\lambda\rho a_1 a_2\sqrt{\pi}} F_1\left(\frac{\omega(\sigma)}{a_1}\right) = G\left(\frac{\sigma}{a_1}\right) \tag{2.6a}$$

$$\frac{a_1}{k_1} f\left(\frac{\sigma}{a_1}\right) = \frac{D}{h_o\sqrt{\pi}}. \tag{2.6b}$$

The eight possible cases for Problem (P_1) are considered in Appendix C (Table 1) which shows both the necessary and sufficient conditions to be verified by the data for the existence and uniqueness of the solution of the problem and the expression of the coefficient σ together with the corresponding unknown coefficient. We remark here that the coefficient ω is always given by the expression (5) as a function of σ and a_1.

Now, we shall prove the following properties only for the determination of ϵ (case 3) and the determination of k_2 (case 7), which gave us different difficulties in all cases.

THEOREM 1 (Case 3). *The necessary and sufficient condition for Problem* (P_1), *with σ and ϵ unknown, to have a unique solution is that data $D>0$, $E>0$, $h_o>0$, mushy zone coefficient $\gamma>0$ and thermal coefficients of the phase-change material λ, ρ, c_1, c_2, k_1, $k_2>0$ do verify the conditions*

$$h_o > \frac{Ek_2}{a_2\sqrt{\pi}}, \qquad f(x_5) < \frac{Dk_1}{h_o a_1 \sqrt{\pi}} < f(x_4), \tag{2.7}$$

where x_4 and x_5 are unique positive zeros of functions H_4 and H_5 respectively. In such a case, the solution is given by (2.1)–(2.4) *with*

$$\epsilon = \frac{2D}{\gamma\sqrt{\pi}} F_2(\xi_1) H_5(\xi_1), \qquad \sigma = a_1 \xi_1, \qquad \omega = a_1 W(\xi_1) \tag{2.8}$$

where ξ_1 is the unique solution of the equation

$$f(x) = \frac{Dk_1}{h_o a_1 \sqrt{\pi}}, \qquad x>0. \tag{2.9}$$

PROOF. We define

$$\xi_1 = \frac{\sigma}{a_1}, \quad \text{with} \quad a_1 = \sqrt{\frac{k_1}{\rho c_1}}. \tag{2.10}$$

The coefficient σ is obtained from (2.10) and the element ξ_1 is given from (2.6b) as the solution of (2.9) iff the data verify the condition

$$\frac{Dk_1}{h_o a_1 \sqrt{\pi}} < 1. \tag{2.11}$$

From (2.6a) it follows that ξ_1 should verify

$$\frac{h_o}{\lambda \rho a_1} \exp(-\xi_1^2) - \frac{Ek_2}{\lambda \rho a_1 a_2 \sqrt{\pi}} F_1\left(\frac{a_1}{a_2} W(\xi_1)\right) = \xi_1 + \frac{(1-\epsilon)\gamma\sqrt{\pi}}{2D} f(\xi_1) \exp(\xi_1^2), \tag{2.12}$$

then we obtain the expression for ϵ in (2.8).

Therefore we have the following properties:

$$\epsilon < 1 \quad \text{iff} \quad H_4(\xi_1) > 0$$

$$\text{iff} \quad H_4(0^+) > 0 \left(\text{i.e. } h_o > \frac{Ek_2}{a_2\sqrt{\pi}}\right) \quad \text{and} \quad \xi_1 < x_4 \; (\text{i.e. } f(\xi_1) < f(x_4)), \tag{2.13}$$

where x_4 is the only positive root of H_4 (because H_4 is a decreasing function for $x>0$), and

$$\epsilon > 0 \quad \text{iff} \quad H_5(\xi_1) > 0 \quad \text{iff} \quad \xi_1 > x_5 \; (\text{i.e. } f(\xi_1) > f(x_5)), \tag{2.14}$$

where x_5 is the only positive root of H_5 (because H_5 is an increasing function for $x>0$, $H_5(0^+) = -\frac{h_o}{\lambda\rho a_1}\alpha_{20} < 0$ and $H_5(+\infty) = +\infty$). We can deduce (2.7) from (2.13) and (2.14) because $x_5 < x_4$.

THEOREM 2. (Case 7). *The necessary and sufficient condition for Problem* (P_1), *with* σ *and* k_2 *unknown, to have a unique solution is that data* $D>0$, $E>0$, $h_o>0$, *mushy zone coefficients* $0<\epsilon<1$ *and* $\gamma>0$, *and thermal coefficients of the phase-change material* $\lambda, \rho, c_1, c_2, k_1>0$ *do verify the condition*

$$\frac{Dk_1}{h_o a_1\sqrt{\pi}}<f(x_{17}) \tag{2.15}$$

where x_{17} *is the unique positive zero of function* H_{17}. *In such a case, the solution is given by* (2.1)–(2.4) *with*

$$k_2=\frac{k_1c_2}{c_1}\frac{W^2(\xi_1)}{B^2}, \qquad \sigma=a_1\xi_1, \qquad \omega=a_1W(\xi_1) \tag{2.16}$$

where ξ_1 *is the unique solution of* (2.9) *and* B *is the only solution of the equation*

$$\frac{1}{H_{16}(x)}=\frac{\lambda}{Ec_2}\frac{H_2(\xi_1)}{W(\xi_1)}, \qquad x>0. \tag{2.17}$$

PROOF. We obtain the coefficient σ as in Theorem 1. From (2.6a) it follows that ξ_1 should verify

$$\frac{Ek_2}{\lambda\rho a_1a_2\sqrt{\pi}}F_1\Big(\frac{a_1}{a_2}W(\xi_1)\Big)=H_2(\xi_1). \tag{2.18}$$

If we define

$$B=\frac{a_1}{a_2}W(\xi_1), \quad \text{with} \quad a_2=\frac{\sqrt{k_2}}{\sqrt{\rho c_2}}, \tag{2.19}$$

then equation (2.18) is equivalent to

$$\frac{F_1(B)}{B}=\frac{\lambda\sqrt{\pi}}{Ec_2}\frac{H_2(\xi_1)}{W(\xi_1)}, \qquad B>0, \tag{2.20}$$

that is, B is the solution of (2.17). Taking into account the properties of the function H_{16} we can deduce that there exists a unique solution of (2.17) if and only if

$$H_{17}(\xi_1)>0 \quad \text{iff} \quad \xi_1<x_{17}\ (\text{i.e. } (2.15)), \tag{2.21}$$

where x_{17} is the only positive root of H_{17} (because H_{17} is a decreasing function for $x>0$, $H_{17}(0^+)>0$ and $H_{17}(+\infty)=-\infty$). From (2.19) we obtain the coefficient k_2.

3. DETERMINATION OF TWO UNKNOWN THERMAL COEFFICIENTS

Taking into account the hypotheses (H_1)–(H_3) we can formulate the following:

PROBLEM (P_2). *Find the free boundary* $x=r(t)$, *defined for* $t>0$ *with* $r(0)=0$, *the temperature* $\theta=\theta(x,t)$, *defined by* (1.3) *for* $x>0$ *and* $t>0$, *and two of the eight unknown thermal coefficients* $\epsilon, \gamma, \lambda, \rho, c_1, c_2, k_1, k_2$ *such that they satisfy the conditions* (1.1), (1.2), (1.4)–(1.10) *where the moving boundary* $x=s(t)$, *defined for* $t>0$ *with* $s(0)=0$, *is given by* (2.3) *with a known coefficient* $\sigma>0$ *and* $D, E, h_o>0$ *are data and they must be known or determined by an experience of phase-change* [8].

The solution of that problem is given by (2.1), (2.2) and (2.4) where the coefficient ω and the unknown thermal coefficients are obtained by solving the system of equations (2.6).

The 28 cases for Problem (P_2) (cases 9 to 36) are considered in Appendix D (Table 2) which shows both the necessary and sufficient conditions to be verified by the data for the existence of the solution of the problem and the expression of the coefficient ω together with the corresponding unknown coefficients. There are several cases where the moving boundary problem has a unique solution iff some conditions are verified.

Now, we shall prove the properties corresponding only for the determination of k_1 and k_2 (case 9), the determination of ϵ and k_2 (case 17), the determination of c_2 and k_2 (case 19) and the determination of γ and k_2 (case 23), which give us different difficulties in all cases.

THEOREM 3 (Case 9). *The necessary and sufficient condition for Problem* (P_2), *with* ω, k_1 *and* k_2 *unknown, to have a unique solution is that data* $\sigma>0$, $D>0$, $E>0$, $h_o>0$, *mushy zone coefficients* $0<\epsilon<1$ *and* $\gamma>0$, *and thermal coefficients of the phase-change material* $\lambda, \rho, c_1, c_2>0$ *do verify the conditions*

$$\frac{h_o}{E\rho\sigma c_2}>1+\frac{\gamma}{D}+\frac{\lambda}{Ec_2}\left(1+\frac{(1-\varepsilon)\gamma}{D}\right), \qquad \frac{D\rho\sigma c_1}{h_o\sqrt{\pi}}<H_{20}(x_{23}), \tag{3.1}$$

where x_{23} *is the unique positive zero of function* H_{23}. *In such a case, the solution is given by* (2.1), (2.2) *and* (2.4) *with*

$$\omega=\sigma H_{25}(\xi_1), \qquad k_1=\rho\sigma^2c_1\frac{1}{\xi_1^2}, \qquad k_2=\rho\sigma^2c_2\frac{H_{25}^2(\xi_1)}{B^2} \tag{3.2}$$

where ξ_1 *is the unique solution of the equation*

$$H_{20}(x)=\frac{D\rho\sigma c_1}{h_o\sqrt{\pi}}, \qquad x>0 \tag{3.3}$$

and B is the only solution of the equation

$$\frac{1}{H_{16}(x)}=\frac{\lambda}{Ec_2}\frac{H_{21}(\xi_1)}{W(\xi_1)}, \qquad x>0. \tag{3.4}$$

PROOF. We define

$$\xi_1=\frac{\sigma}{a_1}, \quad \text{with} \quad a_1=\frac{\sqrt{k_1}}{\sqrt{\rho c_1}}. \tag{3.5}$$

The coefficients ω and k_1 are obtained using (3.5) and the element ξ_1 is given from (2.6b), as the solution of (3.3). From (2.6a) it follows that ξ_1 and k_2 should verify

$$\frac{Ek_2}{\lambda\rho a_1a_2\sqrt{\pi}}F_1\left(\frac{\sigma}{a_2}\frac{W(\xi_1)}{\xi_1}\right)=H_{21}(\xi_1). \tag{3.6}$$

If we define

$$B=\frac{\sigma}{a_2}\frac{W(\xi_1)}{\xi_1}=\frac{\sigma}{a_2}H_{25}(\xi_1), \quad \text{with} \quad a_2=\sqrt{\frac{k_2}{\rho c_2}} \tag{3.7}$$

then (3.6) is equivalent to

$$\frac{F_1(B)}{B}=\frac{\lambda\sqrt{\pi}}{Ec_2}\frac{H_{21}(\xi_1)}{W(\xi_1)}, \qquad B>0, \tag{3.8}$$

that is, B is the solution of (3.4). Taking into account the properties of the function H_{16} we can deduce that there exists a unique solution of (3.4) if and only if

$$H_{23}(\xi_1)>0 \quad \text{iff} \quad H_{23}(0^+)>0 \quad \text{and} \quad \xi_1<x_{23} \text{ (i.e. (3.1))},$$

where x_{23} is the only positive root of H_{23} (because H_{23} is a decreasing function for $x>0$ and $H_{23}(+\infty)=-\infty$). From (3.7) we obtain the coefficient k_2.

THEOREM 4 (Case 17). *The necessary and sufficient condition for Problem* (P_2), *with* ω, ϵ *and* k_2 *unknown, to have at least one solution is that data* $\sigma>0$, $D>0$, $E>0$, $h_o>0$, *mushy zone coefficient* $\gamma>0$, *and thermal coefficients of the phase-change material* $\lambda, \rho, c_1, c_2, k_1>0$ *do verify the conditions*

$$h_o=\frac{Dk_1}{a_1 f\left(\frac{\sigma}{a_1}\right)\sqrt{\pi}}, \qquad \xi_1=\frac{\sigma}{a_1}<x_{36} \tag{3.9}$$

where x_{36} *is the unique positive zero of function* H_{36}. *In such a case, there exist infinite solutions which have the form* (2.1), (2.2) *and* (2.4) *where*

$$\begin{gathered} \omega=a_1 W(\xi_1), \qquad k_2=\frac{k_1 c_2}{c_1}\frac{W^2(\xi_1)}{B^2}, \\ \epsilon=\frac{2D}{\gamma\sqrt{\pi}}F_2(\xi_1)\left(W(\xi_1)-\frac{Dc_1}{\lambda\sqrt{\pi}}F_2(\xi_1)+\frac{Ec_2}{\lambda}\frac{W(\xi_1)}{H_{16}(B)}\right), \end{gathered} \tag{3.10}$$

with B *an arbitrary parameter which is defined by*

$$B>H_{16}^{-1}\left(\frac{1}{A}\right) \quad \text{if} \quad x_{37}\le\xi_1<x_{36}\left(A=\frac{\lambda}{Ec_2}\frac{1}{W(\xi_1)}\left(\frac{Dc_1}{\lambda\sqrt{\pi}}F_2(\xi_1)-\xi_1\right)\right), \tag{3.11}$$

and

$$H_{16}^{-1}\left(\frac{1}{A}\right)<B<H_{16}^{-1}\left(\frac{1}{C}\right) \quad \text{if} \quad \xi_1<x_{37}\left(C=\frac{\lambda}{Ec_2}\frac{1}{W(\xi_1)}\left(\frac{Dc_1}{\lambda\sqrt{\pi}}F_2(\xi_1)-W(\xi_1)\right)\right), \tag{3.12}$$

where x_{36} *and* x_{37} *are the unique positive zeros of functions* H_{36} *and* H_{37} *respectively.*

PROOF. From (2.6b) it follows that $h_o=\dfrac{Dk_1}{a_1 f(\xi_1)\sqrt{\pi}}$. From (2.6a), ϵ and k_2 should verify

$$\frac{Dc_1}{\lambda\sqrt{\pi}}F_2(\xi_1)-\frac{E}{\lambda a_1}\sqrt{\frac{c_2k_2}{\rho\pi}}F_1\left(a_1\sqrt{\frac{\rho c_2}{k_2}}W(\xi_1)\right)=\xi_1+\frac{(1-\varepsilon)\gamma\sqrt{\pi}}{2D}f(\xi_1)\exp(\xi_1) \tag{3.13}$$

i.e. the expression for ϵ in (3.10), when we define $B=a_1\sqrt{\dfrac{\rho c_2}{k_2}}W(\xi_1)$. Then we have

$$\epsilon<1 \quad \text{iff} \quad \frac{1}{H_{16}(B)}<A \quad \text{and} \quad \epsilon<0 \quad \text{iff} \quad \frac{1}{H_{16}(B)}>C. \tag{3.14}$$

Taking into account the properties of the function H_{16} we can deduce

$$\begin{gathered} A>1 \Leftrightarrow H_{36}(\xi_1)>0 \Leftrightarrow \xi_1<x_{36} \\ C>1 \Leftrightarrow H_{37}(\xi_1)>0 \Leftrightarrow \xi_1<x_{37}. \end{gathered} \tag{3.15}$$

Then we obtain (3.11) and (3.12) from (3.14) and (3.15).

THEOREM 5 (Case 19). The necessary and sufficient condition for Problem (P_2), *with* ω, c_2 *and* k_2 *unknown, to have at least one solution is that data* $\sigma > 0$, $D > 0$, $E > 0$, $h_o > 0$, *mushy zone coefficients* $0 < \varepsilon < 1$ *and* $\gamma > 0$, *and thermal coefficients of the phase-change material* $\lambda, \rho, c_1, k_1 > 0$ *do verify the conditions*

$$h_o = \frac{Dk_1}{a_1 f\left(\frac{\sigma}{a_1}\right)\sqrt{\pi}}, \qquad H_{57}\left(\frac{\sigma}{a_1}\right) > 0 \qquad \left(or\ \xi_1 = \frac{\sigma}{a_1} < x_{57}\right) \tag{3.16}$$

where x_{57} *is the unique positive zero of function* H_{57}. *In such a case, there exist infinite solutions which have the form* (2.1), (2.2) *and* (2.4) *where*

$$\omega = a_1 W(\xi_1), \qquad c_2 = \frac{\lambda}{E}\frac{H_{16}(B)H_{57}(\xi_1)}{W(\xi_1)}, \qquad k_2 = \frac{\lambda k_1\sqrt{\pi}}{Ec_1}\frac{W(\xi_1)H_{57}(\xi_1)}{H_1(B)} \tag{3.17}$$

for any $B > 0$.

PROOF. From (2.6b) it follows that $h_o = \dfrac{Dk_1}{a_1 f(\xi_1)\sqrt{\pi}}$. We define

$$B = \sqrt{\frac{k_1 c_2}{c_1 k_2}}\, W(\xi_1). \tag{3.18}$$

From (2.6a), B and k_2 should verify

$$\frac{Ec_1 k_2}{\lambda k_1\sqrt{\pi}}\frac{H_1(B)}{W(\xi_1)} = H_{57}(\xi_1), \tag{3.19}$$

i.e. we deduce the expression for k_2 in (3.17). Then we obtain the coefficient c_2 from (3.18) and (3.19).

Thus we have

$$k_2 > 0 \Leftrightarrow H_{57}(\xi_1) > 0 \Leftrightarrow 0 < \xi_1 < x_{57} \tag{3.20}$$

where x_{57} is the only positive root of H_{57} (because H_{57} is a decreasing function for $x > 0$, with $H_{57}(0^+) = +\infty$ and $H_{57}(+\infty) = -\infty$).

THEOREM 6 (Case 23). *The necessary and sufficient condition for Problem* (P_2), *with* ω, γ *and* k_2 *unknown, to have at least one solution is that data* $\sigma > 0$, $D > 0$, $E > 0$, $h_o > 0$, *mushy zone coefficient* $0 < \varepsilon < 1$, *and thermal coefficients of the phase-change material* $\lambda, \rho, c_1, c_2, k_1 > 0$ *do verify the conditions*

$$h_o = \frac{Dk_1}{a_1 f\left(\frac{\sigma}{a_1}\right)\sqrt{\pi}}, \qquad H_{50}\left(\frac{\sigma}{a_1}\right) > 0 \quad \left(or\ \xi_1 = \frac{\sigma}{a_1} < x_{50}\right) \tag{3.21}$$

where x_{50} *is the unique positive zero of function* H_{50}. *In such a case, there exist infinite solutions which have the form* (2.1), (2.2) *and* (2.4) *where*

$$\omega = a_1 W(\xi_1, \gamma), \qquad k_2 = \frac{k_1 c_2}{c_1}\frac{W^2(\xi_1, \gamma)}{B^2} \tag{3.22}$$

for any $0<\gamma<\gamma_o$, *with*

$$\gamma_o=\frac{2D}{\left(1-\epsilon+\frac{Ec_2}{\lambda}\right)\sqrt{\pi}}F_2(\xi_1)H_{50}(\xi_1) \tag{3.23}$$

and $B=B(\gamma)$ *is the only solution of the equation*

$$\frac{1}{H_{16}(x)}=\frac{\lambda}{Ec_2}\frac{H_{57}(\xi_1,\gamma)}{W(\xi_1,\gamma)}, \qquad x>0. \tag{3.24}$$

PROOF. From (2.6b) it follows that $h_o=\dfrac{Dk_1}{a_1 f\left(\frac{\sigma}{a_1}\right)\sqrt{\pi}}$. We define

$$\xi_2=\frac{\sigma}{a_2}, \quad \text{with} \quad a_2=\sqrt{\frac{k_2}{\rho c_2}}. \tag{3.25}$$

From (2.6a), $\xi_2>0$ should verify

$$F_1\left(\frac{W(\xi_1,\gamma)}{\xi_1}\xi_2\right)\frac{1}{\xi_2}=\frac{\lambda\sqrt{\pi}}{Ec_2}\frac{1}{\xi_1}\left(\frac{h_o}{\lambda\rho\sigma}\xi_1\exp(-\xi_1^2)-G_1(\xi_1,\gamma)\right). \tag{3.26}$$

If we define

$$B=B(\gamma)=\frac{W(\xi_1,\gamma)}{\xi_1}\xi_2, \tag{3.27}$$

equation (3.26) for ξ_2 is equivalent to (3.24) for B. We obtain the coefficients ω and k_2 from (3.25) and (3.27).

Taking into account the properties of the function H_{16} we can deduce that there exists a unique solution of (3.24) if and only if

$$H_{50}(\xi_1)>0 \quad (\text{or equivalently } \xi_1<x_{50}), \tag{3.28}$$

for any $0<\gamma<\gamma_o$, where x_{50} is the only positive root of H_{50} (because H_{50} is a decreasing function for $x>0$ with $H_{50}(0^+)=+\infty$ and $H_{50}(+\infty)=-\infty$).

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APPENDIX A

The following real functions are defined, for $x > 0$, by

$$f(x) = \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt$$

$$F_1(x) = \frac{\exp(-x^2)}{1 - f(x)}$$

$$F_2(x) = \frac{\exp(-x^2)}{f(x)}$$

$$W(x) = W(x, \gamma) = x + \frac{\gamma\sqrt{\pi}}{2D} f(x)\exp(x^2)$$

$$G(x) = G(x, \gamma) = x + \frac{(1-\epsilon)\gamma\sqrt{\pi}}{2D} f(x)\exp(x^2)$$

$$H_1(x) = xF_1(x)$$

$$H_3(x) = \exp(-x^2) - \frac{Ek_2}{h_o a_2\sqrt{\pi}} F_1\left(\frac{a_1}{a_2} W(x)\right)$$

$$H_2(x) = \frac{h_o}{\lambda\rho a_1}\exp(-x^2) - G(x)$$

$$H_4(x) = \frac{h_o}{\lambda\rho a_1}\exp(-x^2) - x - \frac{Ek_2}{\lambda\rho a_1 a_2\sqrt{\pi}} F_1\left(\frac{a_1}{a_2} W(x)\right)$$

$$H_5(x) = \frac{\gamma\sqrt{\pi}}{2D} f(x)\exp(x^2) - H_4(x)$$

$$H_6(x) = (1-\epsilon)\frac{a_2}{a_1} x + \frac{Ek_2}{\lambda\rho a_1 a_2\sqrt{\pi}} F_1(x)$$

$$H_7(x) = \frac{h_o}{\lambda\rho a_1}\exp(-x^2) - x - \frac{Ek_2}{\lambda\rho a_1 a_2\sqrt{\pi}} F_1\left(\frac{a_1}{a_2} x\right)$$

$$H_8(x) = \frac{W(x)}{f(x)}$$

$$H_9(x) = \frac{G(x)}{f(x)}$$

$$H_{10}(x) = \frac{h_o^2\sqrt{\pi}}{D\lambda\rho k_1}\exp(-x^2) - \frac{Eh_o k_2}{D\lambda\rho k_1 a_2} F_1\left(\frac{Dk_1}{h_o a_2\sqrt{\pi}} H_8(x)\right)$$

$$H_{11}(x) = \frac{\beta_2}{\beta_1} x + F_1(x)$$

$$H_{12}(x) = \beta_1\beta_3\frac{1}{x}$$

$$H_{13}(x) = f(x)W(x)$$

$$\beta_1 = \frac{Dk_1}{h_o a_2\sqrt{\pi}}\left(\frac{2}{\sqrt{\pi}} + \frac{\gamma\sqrt{\pi}}{2D}\right)$$

$$\beta_2 = \frac{D\lambda\rho k_1 a_2}{Eh_o k_2}\left(\frac{2}{\sqrt{\pi}} + \frac{(1-\epsilon)\gamma\sqrt{\pi}}{2D}\right)$$

$$\beta_3 = \frac{h_0 a_2\sqrt{\pi}}{Ek_2}$$

$$H_{14}(x) = f(x)G(x)$$

$$H_{15}(x) = \frac{Dc_1}{\lambda\sqrt{\pi}}\exp(-x^2) - \frac{DEc_1k_2}{\lambda\pi h_o a_2} F_1\left(\frac{h_o\sqrt{\pi}}{D\rho c_1 a_2} H_{13}(x)\right)$$

$$H_{16}(x) = \sqrt{\pi}\, x\exp(x^2)(1 - f(x))$$

$$H_{17}(x) = \frac{h_o}{E\rho a_1 c_2}\exp(-x^2) - \left(1 + \frac{(1-\epsilon)\lambda}{Ec_2}\right) \times \frac{\gamma\sqrt{\pi}}{2D} f(x)\exp(x^2) - \left(1 + \frac{\lambda}{Ec_2}\right)x$$

$$H_{18}(x) = \exp(x^2)\left(G(x) + \frac{E}{\lambda}\sqrt{\frac{c_1c_2k_2}{\pi k_1}} F_1\left(\sqrt{\frac{k_1c_2}{c_1k_2}} W(x)\right)\right)$$

$$H_{19}(x) = f(x)H_{18}(x)$$

$$H_{20}(x) = xf(x)$$

$$H_{21}(x) = H_{21}(x, \gamma) = \frac{h_o}{\lambda\rho\sigma} x\exp(-x^2) - G(x, \gamma)$$

$$H_{22}(x) = \frac{h_o}{E\rho\sigma c_2} x\exp(-x^2) - \left(1 + \frac{(1-\epsilon)\lambda}{Ec_2}\right) \times \frac{\gamma\sqrt{\pi}}{2D} f(x)\exp(x^2) - \left(1 + \frac{\lambda}{Ec_2}\right)x$$

$$H_{23}(x) = \frac{H_{22}(x)}{x}$$

$$H_{24}(x) = \frac{f(x)}{x}$$

$$H_{25}(x) = \frac{W(x)}{x}$$

$$H_{26}(x) = x\exp(-x^2) - \frac{Ek_2}{h_o a_2\sqrt{\pi}} xF_1\left(\frac{\sigma}{a_2} H_{25}(x)\right)$$

$$H_{27}(x) = \frac{H_{26}(x)}{x}$$

$$H_{28}(x) = \frac{H_{21}(x)}{x}$$

$$H_{29}(x) = \frac{h_o}{\lambda\rho\sigma}\exp(-x^2) - 1 - \frac{Ek_2}{\lambda\rho\sigma a_2\sqrt{\pi}} F_1\left(\frac{\sigma}{a_2} H_{25}(x)\right)$$

$$H_{30}(x) = \frac{\gamma\sqrt{\pi}}{2D}\frac{1}{xF_2(x)} - H_{29}(x)$$

$$H_{31}(x) = (1-\epsilon)\frac{a_2}{\sigma} x + \frac{Ek_2}{\lambda\rho\sigma a_2\sqrt{\pi}} F_1(x)$$

$$H_{32}(x) = \frac{h_o}{\lambda\rho\sigma}\exp(-x^2) - 1 - \frac{Ek_2}{\lambda\rho\sigma a_2\sqrt{\pi}} F_1\left(\frac{\sigma}{a_2}\right)$$

$$H_{33}(x) = \frac{\sigma h_o c_1}{\lambda k_1}\exp(-x^2) - xG(x) - \frac{Ec_2}{\lambda} xW(x)$$

$$H_{34}(x) = \frac{Ec_2\sqrt{\pi}}{Dc_1}\frac{W(x)}{F_2(x)}$$

$$H_{35}(x) = 1 - H_{34}(x)$$

$$H_{36}(x) = \frac{Dc_1}{\lambda\sqrt{\pi}} F_2(x) - x - \frac{Ec_2}{\lambda} W(x)$$

$$H_{37}(x) = H_{36}(x) - \frac{\gamma\sqrt{\pi}}{2D}\frac{1}{F_2(x)}$$

$$H_{38}(x) = x + \frac{Ec_2}{\lambda\sqrt{\pi}} F_1(x)$$

$$H_{39}(x)=\sqrt{H_{20}(x)}\left(1+\frac{\gamma\sqrt{\pi}}{2D}\frac{1}{xF_2(x)}\right)$$

$$H_{40}(x)=G(x)+\frac{E}{\lambda}\frac{1}{\sqrt{H_{24}(x)}}\sqrt{\frac{Dc_1c_2k_2}{\sigma h_o\sqrt{\pi^3}}}\times F_1\left(\sqrt{\frac{\sigma h_o c_2\sqrt{\pi}}{Dc_1k_2}}H_{39}(x)\right)$$

$$H_{41}(x)=\frac{Dc_1}{\lambda\sqrt{\pi}}F_2(x)$$

$$H_{42}(x)=\frac{G(x)}{x}$$

$$H_{43}(x)=\frac{h_o}{\lambda\rho\sigma}\exp(-x^2)-\left(1+\frac{Ec_2}{\lambda}\right)$$

$$H_{44}(x)=\frac{Ec_2}{\lambda\sqrt{\pi}}W(\xi_1)H_1(x)+G(\xi_1)x^2\ (\xi_1>0)$$

$$H_{45}(x)=\frac{\sigma h_o c_1}{\lambda k_1}\exp(-x^2)-xG(x)$$

$$H_{46}(x)=\frac{Dc_1}{\lambda\sqrt{\pi}}F_2(x)-W(x)$$

$$H_{47}(x)=H_{46}(x)+\frac{\gamma\sqrt{\pi}}{2D}\frac{1}{F_2(x)}$$

$$H_{48}(x)=\frac{h_o}{\lambda\rho\sigma}\exp(-x^2)-1$$

$$H_{49}(x)=\frac{\exp(-x^2)}{x}-\frac{E}{\sigma h_o}\sqrt{\frac{k_1c_2k_2}{\pi c_1}}F_1\left(\sqrt{\frac{k_1c_2}{c_1k_2}}W(x)\right)$$

$$H_{50}(x)=\frac{Dc_1}{\lambda\sqrt{\pi}}F_2(x)-\left(1+\frac{Ec_2}{\lambda}\right)x$$

$$H_{51}(x)=\frac{\sigma h_o c_1}{\lambda k_1}\exp(-x^2)-x^2-\frac{E}{\lambda}\sqrt{\frac{c_1c_2k_2}{\pi k_1}}\times xF_1\left(\sqrt{\frac{k_1c_2}{c_1k_2}}W(x)\right)$$

$$H_{52}(x)=H_{51}(x)-\frac{\gamma\sqrt{\pi}}{2D}\frac{x}{F_2(x)}$$

$$H_{53}(x)=(1-\epsilon)\sqrt{\frac{c_1k_2}{k_1c_2}}x+\frac{E}{\lambda}\sqrt{\frac{c_1c_2k_2}{\pi k_1}}F_1(x)$$

$$H_{54}(x)=\frac{\sigma h_o c_1}{\lambda k_1}\exp(-x^2)-\left(\epsilon+(1-\epsilon)\sqrt{\frac{c_1k_2}{k_1c_2}}\right)x^2-\frac{E}{\lambda}\sqrt{\frac{c_1c_2k_2}{\pi k_1}}xF_1\left(\sqrt{\frac{k_1c_2}{c_1k_2}}x\right),$$

$$H_{55}(x)=xf(x)\exp(x^2)=\frac{x}{F_2(x)}$$

$$H_{56}(x)=\frac{1}{H_{55}(x)}-\frac{Ek_2}{D\rho\sigma c_1a_2}F_1\left(\frac{\sigma}{a_2}H_{25}(x)\right)$$

$$H_{57}(x)=H_{57}(x,\gamma)=\frac{Dc_1}{\lambda\sqrt{\pi}}F_2(x)-G(x,\gamma).$$

$$H_{58}(x)=xF_2(x)$$

The principal properties of some of these functions, for $x>0$, are

$f(0^+)=0$	$f(+\infty)=1$	$f'(x)>0$
$F_1(0^+)=1$	$F_1(+\infty)=+\infty$	$F_1'(x)>0$
$F_2(0^+)=+\infty$	$F_2(+\infty)=0$	$F_2'(x)<0$
$W(0^+)=0$	$W(+\infty)=+\infty$	$W'(x)>0$
$G(0^+)=0$	$G(+\infty)=+\infty$	$G'(x)>0$
$H_1(0^+)=0$	$H_1(+\infty)=+\infty$	$H_1'(x)>0$
$H_2(0^+)=\dfrac{h_o}{\lambda\rho a_1}$	$H_2(+\infty)=-\infty$	$H_2'(x)<0$
$H_3(0^+)=\alpha_{20}$	$H_3(+\infty)=-\infty$	$H_3'(x)<0$
$H_4(0^+)=\dfrac{h_o}{\lambda\rho a_1}\alpha_{20}$	$H_4(+\infty)=-\infty$	$H_4'(x)<0$
$H_5(0^+)=-\dfrac{h_o}{\lambda\rho a_1}\alpha_{20}$	$H_5(+\infty)=+\infty$	$H_5'(x)>0$
$H_6(0^+)=\dfrac{Ek_2}{\lambda\rho a_1a_2\sqrt{\pi}}$	$H_6(+\infty)=+\infty$	$H_6'(x)>0$
$H_7(0^+)=\dfrac{h_o}{\lambda\rho a_1}\alpha_{20}$	$H_7(+\infty)=-\infty$	$H_7'(x)<0$
$H_8(0^+)=\alpha_3$	$H_8(+\infty)=+\infty$	$H_8'(x)>0$
$H_9(0^+)=\alpha_4$	$H_9(+\infty)=+\infty$	$H_9'(x)>0$
$H_{10}(0^+)=\alpha_5$	$H_{10}(+\infty)=-\infty$	$H_{10}'(x)<0$
$H_{11}(0^+)=1$	$H_{11}(+\infty)=+\infty$	$H_{11}'(x)>0$
$H_{12}(0^+)=+\infty$	$H_{12}(+\infty)=0$	$H_{12}'(x)<0$
$H_{13}(0^+)=0$	$H_{13}(+\infty)=+\infty$	$H_{13}'(x)>0$
$H_{14}(0^+)=0$	$H_{14}(+\infty)=+\infty$	$H_{14}'(x)>0$

$H_{15}(0^+) = \alpha_6$	$H_{15}(+\infty) = -\infty$	$H'_{15}(x) < 0$
$H_{16}(0^+) = 0$	$H_{16}(+\infty) = 1$	$H'_{16}(x) > 0$
$H_{17}(0^+) = \alpha_7$	$H_{17}(+\infty) = -\infty$	$H'_{17}(x) < 0$
$H_{18}(0^+) = \alpha_8$	$H_{18}(+\infty) = +\infty$	$H'_{18}(x) > 0$
$H_{19}(0^+) = 0$	$H_{19}(+\infty) = +\infty$	$H'_{19}(x) > 0$
$H_{20}(0^+) = 0$	$H_{20}(+\infty) = +\infty$	$H'_{20}(x) > 0$
$H_{23}(0^+) = \alpha_9$	$H_{23}(+\infty) = -\infty$	$H'_{23}(x) < 0$
$H_{24}(0^+) = \dfrac{2}{\sqrt{\pi}}$	$H_{24}(+\infty) = 0$	$H'_{24}(x) < 0$
$H_{25}(0^+) = 1 + \dfrac{\gamma}{D}$	$H_{25}(+\infty) = +\infty$	$H'_{25}(x) > 0$
$H_{27}(0^+) = \alpha_{10}$	$H_{27}(+\infty) = -\infty$	$H'_{27}(x) < 0$
$H_{28}(0^+) = \alpha_{11}$	$H_{28}(+\infty) = -\infty$	$H'_{28}(x) < 0$
$H_{29}(0^+) = \alpha_{12}$	$H_{29}(+\infty) = -\infty$	$H'_{29}(x) < 0$
$H_{30}(0^+) = \alpha_{13}$	$H_{30}(+\infty) = +\infty$	$H'_{30}(x) > 0$
$H_{31}(0^+) = \alpha_{14}$	$H_{31}(+\infty) = +\infty$	$H'_{31}(x) > 0$
$H_{32}(0^+) = \alpha_{15}$	$H_{32}(+\infty) = -1 - \alpha_{14}$	$H'_{32}(x) < 0$
$H_{33}(0^+) = \dfrac{\sigma h_o c_1}{\lambda k_1}$	$H_{33}(+\infty) = -\infty$	$H'_{33}(x) < 0$
$H_{34}(0^+) = 0$	$H_{34}(+\infty) = +\infty$	$H'_{34}(x) > 0$
$H_{35}(0^+) = 1$	$H_{35}(+\infty) = -\infty$	$H'_{35}(x) < 0$
$H_{36}(0^+) = +\infty$	$H_{36}(+\infty) = -\infty$	$H'_{36}(x) < 0$
$H_{37}(0^+) = +\infty$	$H_{37}(+\infty) = -\infty$	$H'_{37}(x) < 0$
$H_{38}(0^+) = \dfrac{Ec_2}{\lambda\sqrt{\pi}}$	$H_{38}(+\infty) = +\infty$	$H'_{38}(x) > 0$
$H_{39}(0^+) = 0$	$H_{39}(+\infty) = +\infty$	$H'_{39}(x) > 0$
$H_{40}(0^+) = \alpha_{16}$	$H_{40}(+\infty) = +\infty$	$H'_{40}(x) > 0$
$H_{41}(0^+) = +\infty$	$H_{41}(+\infty) = 0$	$H'_{41}(x) < 0$
$H_{42}(0^+) = \alpha_{17}$	$H_{42}(+\infty) = +\infty$	$H'_{42}(x) > 0$
$H_{43}(0^+) = \alpha_{18}$	$H_{43}(+\infty) = -1 - \dfrac{Ec_2}{\lambda}$	$H'_{43}(x) < 0$
$H_{44}(0^+) = 0$	$H_{44}(+\infty) = +\infty$	$H'_{44}(x) > 0$
$H_{45}(0^+) = \dfrac{\sigma h_o c_1}{\lambda k_1}$	$H_{45}(+\infty) = -\infty$	$H'_{45}(x) < 0$
$H_{46}(0^+) = +\infty$	$H_{46}(+\infty) = -\infty$	$H'_{46}(x) < 0$
$H_{47}(0^+) = +\infty$	$H_{47}(+\infty) = -\infty$	$H'_{47}(x) < 0$
$H_{48}(0^+) = \alpha_{19}$	$H_{48}(+\infty) = -1$	$H'_{48}(x) < 0$
$H_{49}(0^+) = +\infty$	$H_{49}(+\infty) = -\infty$	$H'_{49}(x) < 0$
$H_{50}(0^+) = +\infty$	$H_{50}(+\infty) = -\infty$	$H'_{50}(x) < 0$
$H_{51}(0^+) = \dfrac{\sigma h_o c_1}{\lambda k_1}$	$H_{51}(+\infty) = -\infty$	$H'_{51}(x) < 0$
$H_{52}(0^+) = \dfrac{\sigma h_o c_1}{\lambda k_1}$	$H_{52}(+\infty) = -\infty$	$H'_{52}(x) < 0$
$H_{53}(0^+) = \alpha_8$	$H_{53}(+\infty) = +\infty$	$H'_{53}(x) > 0$
$H_{54}(0^+) = \dfrac{\sigma h_o c_1}{\lambda k_1}$	$H_{54}(+\infty) = -\infty$	$H'_{54}(x) < 0$
$H_{55}(0^+) = 0$	$H_{55}(+\infty) = +\infty$	$H'_{55}(x) > 0$
$H_{56}(0^+) = +\infty$	$H_{56}(+\infty) = -\infty$	$H'_{56}(x) < 0$
$H_{57}(0^+) = +\infty$	$H_{57}(+\infty) = -\infty$	$H'_{57}(x) < 0$
$H_{58}(0^+) = \dfrac{\sqrt{\pi}}{2}$	$H_{58}(+\infty) = 0$	$H'_{58}(x) < 0$

with

$$\alpha_3 = \frac{2}{\sqrt{\pi}} + \frac{\gamma\sqrt{\pi}}{2D} \qquad \alpha_4 = \frac{2}{\sqrt{\pi}} + \frac{(1-\epsilon)\gamma\sqrt{\pi}}{2D}$$

$$\alpha_6 = \frac{Dc_1}{\lambda\sqrt{\pi}}\left(1 - \frac{Ek_2}{h_o a_2\sqrt{\pi}}\right)$$

$$\alpha_7 = \frac{h_o}{E\rho a_1 c_2}$$

$$\alpha_9 = \frac{h_o}{E\rho\sigma c_2} - \left(1 + \frac{\gamma}{D} + \frac{\lambda}{Ec_2}\left(1 + \frac{(1-\epsilon)\gamma}{D}\right)\right)$$

$$\alpha_{11} = \frac{h_o}{\lambda\rho\sigma} - 1 - \frac{(1-\epsilon)\gamma}{D}$$

$$\alpha_{14} = \frac{Ek_2}{\lambda\rho\sigma a_2\sqrt{\pi}}$$

$$\alpha_{16} = \frac{E}{\lambda}\sqrt{\frac{Dc_1c_2k_2}{2\sigma h_o\sqrt{\pi}}}$$

$$\alpha_{17} = 1 + \frac{(1-\epsilon)\gamma}{D}$$

$$\alpha_{19} = \frac{h_o}{\lambda\rho\sigma} - 1$$

$$\alpha_5 = \frac{Eh_o k_2}{D\lambda\rho k_1 a_2}\left(\frac{h_o a_2\sqrt{\pi}}{Ek_2} - F_1\left(\frac{Dk_1}{h_o a_2\sqrt{\pi}}\alpha_3\right)\right)$$

$$\alpha_8 = \frac{E}{\lambda}\sqrt{\frac{c_1c_2k_2}{\pi k_1}}$$

$$\alpha_{10} = 1 - \frac{Ek_2}{h_o a_2\sqrt{\pi}}F_1\left(\frac{\sigma}{a_2}\left(1 + \frac{\gamma}{D}\right)\right)$$

$$\alpha_{12} = \frac{h_o}{\lambda\rho\sigma} - 1 - \frac{Ek_2}{\lambda\rho\sigma a_2\sqrt{\pi}}F_1\left(\frac{\sigma}{a_2}\left(1 + \frac{\gamma}{D}\right)\right)$$

$$\alpha_{13} = \frac{\gamma}{D} - \frac{h_o}{\lambda\rho\sigma} + 1 + \frac{Ek_2}{\lambda\rho\sigma a_2\sqrt{\pi}}F_1\left(\frac{\sigma}{a_2}\left(1 + \frac{\gamma}{D}\right)\right)$$

$$\alpha_{15} = \frac{h_o}{\lambda\rho\sigma} - 1 - \frac{Ek_2}{\lambda\rho\sigma a_2\sqrt{\pi}}F_1\left(\frac{\sigma}{a_2}\right)$$

$$\alpha_{18} = \frac{h_o}{\lambda\rho\sigma} - 1 - \frac{Ec_2}{\lambda}$$

$$\alpha_{20} = 1 - \frac{Ek_2}{h_o a_2\sqrt{\pi}}.$$

APPENDIX B

The restrictions used in the text are the following:

(R1) $h_o > \dfrac{Ek_2}{a_2\sqrt{\pi}}$

(R2) $\dfrac{Dk_1}{h_o a_1\sqrt{\pi}} < f(x_2)$, where x_2 is the unique positive zero of H_2

(R3) $\dfrac{Dk_1}{h_o a_1\sqrt{\pi}} < f(x_3)$, where x_3 is the unique positive zero of H_3

(R4) $\dfrac{Dk_1}{h_o a_1\sqrt{\pi}} < f(x_4)$, where x_4 is the unique positive zero of H_4

(R5) $\dfrac{Dk_1}{h_o a_1\sqrt{\pi}} > f(x_5)$, where x_5 is the unique positive zero of H_5

(R6) $\dfrac{Dk_1}{h_o a_1\sqrt{\pi}} < f(x_7)$, where x_7 is the unique positive zero of H_7

(R7) $h_0 > \dfrac{Dk_1}{a_2\sqrt{\pi}}\left(\dfrac{2}{\sqrt{\pi}} + \dfrac{\gamma\sqrt{\pi}}{2D}\right)\dfrac{1}{\eta}$, where η is the unique positive solution of the equation $H_{11}(x) = H_{12}(x)$, $x > 0$

(R8) $\dfrac{Dk_1}{h_o a_1\sqrt{\pi}} < f(x_{17})$, where x_{17} is the unique positive zero of H_{17}

(R9) $\dfrac{h_o}{E\rho c_2} > 1 + \dfrac{\gamma}{D} + \dfrac{\lambda}{Ec_2}\left(1 + \dfrac{(1-\epsilon)\gamma}{D}\right)$

(R10) $h_o > \dfrac{Dk_1}{2\sigma}$

(R11) $\dfrac{D\rho\sigma c_1}{h_o\sqrt{\pi}} < H_{20}(x_{23})$, where x_{23} is the unique positive zero of H_{23}

(R12) $\dfrac{Dk_1}{\sigma h_o\sqrt{\pi}} > H_{24}(x_{23})$, where x_{23} is the unique positive zero of H_{23}

(R13) $h_o > \dfrac{Ek_2}{a_2\sqrt{\pi}}F_1\left(\dfrac{\sigma}{a_2}\left(1 + \dfrac{\gamma}{D}\right)\right)$

(R14) $\dfrac{h_o}{\lambda\rho\sigma} > 1 + \dfrac{(1-\epsilon)\gamma}{D}$

(R15) $\dfrac{h_o}{\lambda\rho\sigma} > 1 + \dfrac{Ek_2}{\lambda\rho\sigma a_2\sqrt{\pi}}F_1\left(\dfrac{\sigma}{a_2}\left(1 + \dfrac{\gamma}{D}\right)\right)$

(R16) $\dfrac{h_o}{\lambda\rho\sigma} > 1 + \dfrac{\gamma}{D} + \dfrac{Ek_2}{\lambda\rho\sigma a_2\sqrt{\pi}} F_1\left(\dfrac{\sigma}{a_2}\left(1 + \dfrac{\gamma}{D}\right)\right)$

(R17) $\dfrac{h_o}{\lambda\rho\sigma} > 1 + \dfrac{Ek_2}{\lambda\rho\sigma a_2\sqrt{\pi}} F_1\left(\dfrac{\sigma}{a_2}\right)$

(R18) $\dfrac{D\rho\sigma c_1}{h_o\sqrt{\pi}} < H_{20}(x_{27})$, where x_{27} is the unique positive zero of H_{27}

(R19) $\dfrac{D\rho\sigma c_1}{h_o\sqrt{\pi}} < H_{20}(x_{28})$, where x_{28} is the unique positive zero of H_{28}

(R20) $H_{20}(x_{29}) > \dfrac{D\rho\sigma c_1}{h_o\sqrt{\pi}} > H_{20}(x_{30})$, where x_{29} is the unique positive zero of H_{29}, x_{30} is the unique positive zero of H_{30}

(R21) $\dfrac{D\rho\sigma c_1}{h_o\sqrt{\pi}} < H_{20}(x_{32})$, where x_{32} is the unique positive zero of H_{32}

(R22) $\dfrac{Dk_1}{\sigma h_o\sqrt{\pi}} > H_{24}(x_{33})$, where x_{33} is the unique positive zero of H_{33}

(R23) $h_o = \dfrac{Dk_1}{a_1 f\left(\dfrac{\sigma}{a_1}\right)\sqrt{\pi}}$

(R24) $H_{35}\left(\dfrac{\sigma}{a_1}\right) > 0$ or $\dfrac{\sigma}{a_1} < x_{35}$, where x_{35} is the unique positive zero of H_{35}

(R25) $H_{36}\left(\dfrac{\sigma}{a_1}\right) > 0$ or $\dfrac{\sigma}{a_1} < x_{36}$, where x_{36} is the unique positive zero of H_{36}

(R26) $\dfrac{Dk_1}{E\rho a_1 a_2 c_2} F_2\left(\dfrac{\sigma}{a_1}\right) \leq 1$

(R27) $\dfrac{\lambda\rho\sigma a_2\sqrt{\pi}}{Ek_2}\left(\dfrac{Dk_1}{\lambda\rho\sigma a_1\sqrt{\pi}} F_2\left(\dfrac{\sigma}{a_1}\right) - 1\right) > 1$

(R28) $F_1\left(\dfrac{\sigma}{a_2}\right) < \dfrac{\lambda\rho\sigma a_2\sqrt{\pi}}{Ek_2}\left(\dfrac{Dk_1}{\lambda\rho\sigma a_1\sqrt{\pi}} F_2\left(\dfrac{\sigma}{a_1}\right) - 1\right)$

(R29) $H_{57}\left(\dfrac{\sigma}{a_1}\right) > 0$ or $\dfrac{\sigma}{a_1} < x_{57}$, where x_{57} is the unique positive zero of H_{57}

(R30) $\dfrac{h_o}{\lambda\rho\sigma} > 1 + \dfrac{(1-\epsilon)\gamma}{D} + \dfrac{Ek_2}{\lambda\rho\sigma a_2\sqrt{\pi}} F_1\left(\dfrac{\sigma}{a_2}\left(1 + \dfrac{\gamma}{D}\right)\right)$

(R31) $\dfrac{Dk_1}{\sigma h_o\sqrt{\pi}} > H_{24}(x_{27})$, where x_{27} is the unique positive zero of H_{27}

(R32) $h_o > \rho\sigma(\lambda + Ec_2)$

(R33) $H_{43}\left(\dfrac{\sigma}{a_1}\right) > 0$ or $\dfrac{\sigma}{a_1} < x_{43}$, where x_{43} is the unique positive zero of H_{43} when (R32) is verified

(R34) $\dfrac{Dk_1}{\sigma h_o\sqrt{\pi}} > H_{24}(x_{28})$, where x_{28} is the unique positive zero of H_{28}

(R35) $\dfrac{Dk_1}{\sigma h_o\sqrt{\pi}} > H_{24}(x_{29})$, where x_{29} is the unique positive zero of H_{29}

(R36) $\dfrac{Dk_1}{\sigma h_o\sqrt{\pi}} < H_{24}(x_{30})$, where x_{30} is the unique positive zero of H_{30}

(R37) $\dfrac{Dk_1}{\sigma h_o\sqrt{\pi}} > H_{24}(x_{32})$, where x_{32} is the unique positive zero of H_{32}

(R38) $\dfrac{Dk_1}{\sigma h_o\sqrt{\pi}} > H_{24}(x_{45})$, where x_{45} is the unique positive zero of H_{45}

(R39) $H_{47}\left(\dfrac{\sigma}{a_1}\right) > 0$ or $\dfrac{\sigma}{a_1} < x_{47}$, where x_{47} is the unique positive zero of H_{47}

(R40) $H_{55}\left(\dfrac{\sigma}{a_1}\right) < \dfrac{Dc_1}{\lambda\sqrt{\pi}}$

(R41) $\dfrac{Dk_1}{\sigma h_o\sqrt{\pi}} > H_{24}(x_{54})$, where x_{54} is the unique positive zero of H_{54}

(R42) $\dfrac{Dk_1}{\sigma h_o\sqrt{\pi}} > H_{24}(x_{49})$, where x_{49} is the unique positive zero of H_{49}

(R43) $H_{50}\left(\dfrac{\sigma}{a_1}\right) > 0$ or $\dfrac{\sigma}{a_1} < x_{50}$, where x_{50} is the unique positive zero of H_{50}

(R44) $\dfrac{Dk_1a_2}{Ea_1k_2}F_2\left(\dfrac{\sigma}{a_1}\right) > 1$

(R45) $F_1\left(\dfrac{\sigma}{a_2}\right) < \dfrac{Dk_1a_2}{Ea_1k_2}F_2\left(\dfrac{\sigma}{a_1}\right)$

(R46) $\dfrac{Dk_1}{\sigma h_o\sqrt{\pi}} > H_{24}(x_{51})$, where x_{51} is the unique positive zero of H_{51}

(R47) $\dfrac{Dk_1}{\sigma h_o\sqrt{\pi}} < H_{24}(x_{52})$, where x_{52} is the unique positive zero of H_{52}

(R48) $H_{56}\left(\dfrac{\sigma}{a_1}\right) > 0$ or $\dfrac{\sigma}{a_1} < x_{56}$, where x_{56} is the unique positive zero of H_{56}.

APPENDIX C

Table 1

Case	Unknown coefficients	Restrictions	Solution
1	c_2, σ, ω	(R2)	$\sigma = a_1\xi_1, \quad c_2 = \dfrac{c_1k_2}{k_1}\dfrac{B^2}{W^2(\xi_1)}, \quad \omega = a_1w(\xi_1)$ where ξ_1 is the unique positive solution of the equation $f(x) = \dfrac{Dk_1}{h_oa_1\sqrt{\pi}}, \quad x > 0$ and B is the only positive solution of the equation $H_1(x) = \dfrac{\lambda k_1\sqrt{\pi}}{Ec_1k_2}W(\xi_1)H_2(\xi_1), \quad x > 0.$
2	λ, σ, ω	(R1) (R3)	$\sigma = a_1\xi_1, \quad \lambda = \dfrac{h_o}{\rho a_1}\dfrac{H_3(\xi_1)}{G(\xi_1)}, \quad \omega = a_1W(\xi_1)$ where ξ_1 is given as in case 1.
3	ϵ, σ, ω	(R1) (R4) (R5)	$\sigma = a_1\xi_1, \quad \epsilon = \dfrac{2D}{\gamma\sqrt{\pi}}F_2(\xi_1)H_5(\xi_1), \quad \omega = a_1W(\xi_1)$ where ξ_1 is given as in case 1.
4	γ, σ, ω	(R1) (R6)	$\sigma = a_1\xi_1, \quad \gamma = \dfrac{2D}{\sqrt{\pi}}\left(\dfrac{a_2}{a_1}B - \xi_1\right)F_2(\xi_1), \quad \omega = a_1W(\xi_1)$ where ξ_1 is given as in case 1 and B is the only positive solution of the equation $H_6(x) = \dfrac{h_o}{\lambda\rho a_1}\exp(-\xi_1^2) - \epsilon\xi_1, \quad x > \dfrac{a_1}{a_2}\xi_1.$
5	c_1, σ, ω	(R7)	$\sigma = \dfrac{Dk_1}{h_o\sqrt{\pi}}\dfrac{1}{H_{24}(\xi_1)}, \quad c_1 = \dfrac{\pi h_o^2}{D^2\rho k_1}f^2(\xi_1), \quad \omega = \dfrac{Dk_1}{h_o\sqrt{\pi}}H_8(\xi_1)$ where ξ_1 is the unique positive solution of the equation $H_9(x) = H_{10}(x), \quad x > 0.$
6	k_1, σ, ω	(R1)	$\sigma = \dfrac{h_o\sqrt{\pi}}{D\rho c_1}H_{20}(\xi_1), \quad k_1 = \dfrac{\pi h_o^2}{D^2\rho c_1}f^2(\xi_1), \quad \omega = \dfrac{h_o\sqrt{\pi}}{D\rho c_1}H_{13}(\xi_1)$ where ξ_1 is the unique positive solution of the equation $H_{14}(x) = H_{15}(x), \quad x > 0.$
7	k_2, σ, ω	(R8)	$\sigma = a_1\xi_1, \quad k_2 = \dfrac{k_1c_2}{c_1}\dfrac{W^2(\xi_1)}{B^2}, \quad \omega = a_1W(\xi_1)$ where ξ_1 is given as in case 1 and B is the only positive solution of the equation $\dfrac{1}{H_{16}(x)} = \dfrac{\lambda}{Ec_2}\dfrac{H_2(\xi_1)}{W(\xi_1)}, \quad x > 0.$
8	ρ, σ, ω	—	$\sigma = \dfrac{\lambda k_1}{h_oc_1}\xi_1H_{18}(\xi_1), \quad \rho = \dfrac{h_o^2c_1}{\lambda^2k_1}\dfrac{1}{H_{18}^2(\xi_1)}, \quad \omega = \dfrac{\lambda k_1}{h_oc_1}w(\xi_1)H_{18}(\xi_1)$ where ξ_1 is the unique positive solution of the equation $H_{19}(x) = \dfrac{Dc_1}{\lambda\sqrt{\pi}}, \quad x > 0.$

APPENDIX D

Table 2

Case	Unknown coefficients	Restrictions	Solution
9	ω, k_1, k_2	(R9) (R11)	$\omega = \sigma H_{25}(\xi_1), \quad k_1 = \rho\sigma^2 c_1 \frac{1}{\xi_1^2}, \quad k_2 = \rho\sigma^2 c_2 \frac{H_{25}^2(\xi_1)}{B^2}$ where ξ_1 is the unqiue solution of the equation $H_{20}(x) = \frac{D\rho\sigma c_1}{h_o\sqrt{\pi}}, \quad x > 0$ and B is the only solution of the equation $\frac{1}{H_{16}(x)} = \frac{\lambda}{Ec_2}\frac{H_{21}(\xi_1)}{W(\xi_1)}, \quad x > 0.$
10	ω, c_1, k_2	(R9) (R10) (R12)	$\omega = \sigma H_{25}(\xi_1), \quad c_1 = \frac{k_1}{\rho\sigma^2}\xi_1^2, \quad k_2 = \rho\sigma^2 c_2 \frac{H_{25}^2(\xi_1)}{B^2}$ where ξ_1 is the unique solution of the equation $H_{24}(x) = \frac{Dk_1}{\sigma h_o\sqrt{\pi}}, \quad x > 0$ and B is given as in case 9.
11	ω, λ, k_1	(R13) (R18)	$\omega = \sigma H_{25}(\xi_1), \quad \lambda = \frac{h_o}{\rho\sigma}\frac{H_{26}(\xi_1)}{G(\xi_1)}, \quad k_1 = \rho\sigma^2 c_1 \frac{1}{\xi_1^2}$ where ξ_1 is given as in case 9.
12	ω, k_1, c_2	(R14) (R19)	$\omega = \sigma H_{25}(\xi_1), \quad k_1 = \rho\sigma^2 c_1 \frac{1}{\xi_1^2}, \quad c_2 = \frac{k_2}{\rho\sigma^2}\frac{B^2}{H_{25}^2(\xi_1)}$ where ξ_1 is given as in case 9 and B is the only solution of the equation $H_1(x) = \frac{\lambda\rho\sigma^2\sqrt{\pi}}{Ek_2} H_{25}(\xi_1)H_{28}(\xi_1), \quad x > 0.$
13	ω, ϵ, k_1	(R16) (R20)	$\omega = \sigma H_{25}(\xi_1), \quad k_1 = \rho\sigma^2 c_1 \frac{1}{\xi_1^2}, \quad \epsilon = 1 - \frac{2D}{\gamma\sqrt{\pi}} H_{29}(\xi_1)H_{58}(\xi_1)$ where ξ_1 is given as in case 9.
14	ω, γ, k_1	(R17) (R21)	$\omega = Ba_2, \quad \gamma = \frac{2D}{\sqrt{\pi}}\left(\frac{a_2}{\sigma}B - 1\right)H_{58}(\xi_1), \quad k_1 = \rho\sigma^2 c_1 \frac{1}{\xi_1^2}$ where ξ_1 is given as in case 9 and B is the only solution of the equation $H_{31}(x) = \frac{h_o}{\lambda\rho\sigma}\exp(-\xi_1^2) - \epsilon, \quad x > \frac{\sigma}{a_2}.$
15	ω, ρ, k_2	(R10) (R22)	$\omega = \sigma H_{25}(\xi_1), \quad \rho = \frac{k_1}{\sigma^2 c_1}\xi_1^2, \quad k_2 = \frac{k_1 c_2}{c_1}\frac{W^2(\xi_1)}{B^2}$ where ξ_1 is given as in case 10 and B is the only solution of the equation $\frac{1}{H_{16}(x)} = \frac{\lambda}{Ec_2}\frac{H_{45}(\xi_1)}{\xi_1 W(\xi_1)}, \quad x > 0.$
16	ω, λ, k_2	(R23) (R24)	$\omega = a_1 W\left(\frac{\sigma}{a_1}\right), \quad k_2 = \frac{k_1 c_2}{c_1}\frac{W^2\left(\frac{\sigma}{a_1}\right)}{B^2}, \quad \lambda = \frac{1}{G\left(\frac{\sigma}{a_1}\right)}\left(\frac{Dc_1}{\sqrt{\pi}}F_2\left(\frac{\sigma}{a_1}\right) - Ec_2\frac{W\left(\frac{\sigma}{a_1}\right)}{H_{16}(B)}\right)$ for any $B > H_{16}^{-1}\left(\frac{Ec_2\sqrt{\pi}}{Dc_1}\frac{W\left(\frac{\sigma}{a_1}\right)}{F_2\left(\frac{\sigma}{a_1}\right)}\right).$

Table 2. contd.

Case	Unknown coefficients	Restrictions	Solution
17	ω, ϵ, k_2	(R23) (R25)	$\omega = a_1 W\left(\frac{\sigma}{a_1}\right), \qquad k_2 = \frac{k_1 c_2}{c_1} \frac{W^2\left(\frac{\sigma}{a_1}\right)}{B^2},$ $\epsilon = \frac{2D}{\gamma\sqrt{\pi}} W\left(\frac{\sigma}{a_1}\right) F_2\left(\frac{\sigma}{a_1}\right) - \frac{2D}{\lambda\gamma\sqrt{\pi}} F_2\left(\frac{\sigma}{a_1}\right)\left(\frac{Dc_1}{\sqrt{\pi}} F_2\left(\frac{\sigma}{a_1}\right) - Ec_2 \frac{W\left(\frac{\sigma}{a_1}\right)}{H_{16}(B)}\right)$ for any $H_{16}^{-1}\left(\frac{1}{A}\right) < B < H_{16}^{-1}\left(\frac{1}{C}\right)$ if $\frac{\sigma}{a_1} < x_{37}$ or for any $B > H_{16}^{-1}\left(\frac{1}{A}\right)$ if $x_{37} \le \frac{\sigma}{a_1} < x_{36}$ where $A = \frac{\lambda}{Ec_2 W\left(\frac{\sigma}{a_1}\right)}\left(H_{41}\left(\frac{\sigma}{a_1}\right) - \frac{\sigma}{a_1}\right)$ $C = \frac{\lambda}{Ec_2 W\left(\frac{\sigma}{a_1}\right)} H_{46}\left(\frac{\sigma}{a_1}\right).$
18	ω, ϵ, γ	(R23) (R26) (R27) (R28)	$\omega = \omega(\gamma) = \sigma\left(1 + \frac{\gamma\sqrt{\pi}}{2D} \frac{1}{H_{58}\left(\frac{\sigma}{a_1}\right)}\right),$ $\epsilon = \epsilon(\gamma) = 1 - \frac{\sigma}{\omega(\gamma) - \sigma}\left(\frac{Dk_1}{\lambda\rho\sigma\sqrt{\pi}} H_{58}\left(\frac{\sigma}{a_1}\right) - 1\right) - \frac{1}{\omega(\gamma) - \sigma} \frac{Ek_2}{\lambda\rho a_2\sqrt{\pi}} F_1\left(\frac{\omega(\gamma)}{a_2}\right)$ for any $0 < \gamma < \frac{2Da_2}{a_1\sqrt{\pi}} F_2\left(\frac{\sigma}{a_1}\right)\left(F_1^{-1}(B) - \frac{\sigma}{a_2}\right)$ where $B = \frac{\lambda\rho\sigma_2\sqrt{\pi}}{Ek_2}\left(\frac{Dk_1}{\lambda\rho\sigma a_1\sqrt{\pi}} F_2\left(\frac{\sigma}{a_1}\right) - 1\right).$
19	ω, c_2, k_2	(R23) (R29)	$\omega = a_1 W\left(\frac{\sigma}{a_1}\right), \qquad c_2 = \frac{\lambda}{E} \frac{H_{16}(B) H_{57}\left(\frac{\sigma}{\sigma_1}\right)}{W\left(\frac{\sigma}{a_1}\right)}, \qquad k_2 = \frac{\lambda k_1\sqrt{\pi}}{Ec_1} \frac{W\left(\frac{\sigma}{a_1}\right) H_{57}\left(\frac{\sigma}{a_1}\right)}{H_1(B)}$ for any $B > 0$.
20	ω, ρ, k_1	—	$\omega = \sigma H_{25}(\xi_1), \qquad \rho = \frac{h_o\sqrt{\pi}}{D\sigma c_1} H_{20}(\xi_1), \qquad k_1 = \frac{\sigma h_o\sqrt{\pi}}{D} H_{24}(\xi_1)$ where ξ_1 is the unique solution of the equation $H_{40}(x) = H_{41}(x), \qquad x > 0.$
21	ω, c_1, k_1	(R30)	$\omega = \sigma H_{25}(\xi_1), \qquad c_1 = \frac{h_o\sqrt{\pi}}{D\rho\sigma} H_{20}(\xi_1), \qquad k_1 = \frac{\sigma h_o\sqrt{\pi}}{D} H_{24}(\xi_1)$ where ξ_1 is the unique solution of the equation $\frac{h_o}{\lambda\rho\sigma} H_{27}(x) = H_{42}(x), \qquad x > 0.$
22	ω, λ, c_1	(R10) (R13) (R31)	$\omega = \sigma H_{25}(\xi_1), \qquad \lambda = \frac{h_o}{\rho\sigma} \frac{H_{26}(\xi_1)}{G(\xi_1)}, \qquad c_1 = \frac{k_1}{\rho\sigma^2} \xi_1^2$ where ξ_1 is given as in case 10.

Table 2. contd.

Case	Unknown coefficients	Restrictions	Solution
23	ω, γ, k_2	(R23) (R43)	$\omega = a_1 W\left(\frac{\sigma}{a_1}, \gamma\right), \quad k_2 = \frac{k_1 c_2}{c_1} \frac{W^2\left(\frac{\sigma}{a_1}, \gamma\right)}{B^2}$ for any $0 < \gamma < \frac{2DF_2\left(\frac{\sigma}{a_1}\right)}{\left(1 - \epsilon + \frac{Ec_2}{\lambda}\right)\sqrt{\pi}} H_{50}\left(\frac{\sigma}{a_1}\right)$ where $B = B(\gamma)$ is the unique solution of the equation $\frac{1}{H_{16}(x)} = \frac{\lambda}{Ec_2} \frac{H_{57}\left(\frac{\sigma}{a_1}, \gamma\right)}{W\left(\frac{\sigma}{a_1}, \gamma\right)}, \quad x > 0.$
24	ω, ϵ, c_1	(R16) (R35)	$\omega = \sigma H_{25}(\xi_1), \quad c_1 = \frac{k_1}{\rho\sigma^2} \xi_1^2, \quad \epsilon = 1 - \frac{2D}{\gamma\sqrt{\pi}} H_{29}(\xi_1) H_{58}(\xi_1)$ where ξ_1 is given as in case 10.
25	ω, c_1, c_2	(R10) (R14) (R34)	$\omega = \sigma H_{25}(\xi_1), \quad c_1 = \frac{k_1}{\rho\sigma^2} \xi_1^2, \quad c_2 = \frac{k_2}{\rho\sigma^2} \frac{B^2}{H_{25}^2(\xi_1)}$ where ξ_1 is given as in case 10 and B is given as in case 12.
26	ω, ρ, c_1	(R10)	$\omega = \sigma H_{25}(\xi_1), \quad \rho = \frac{k_2}{\sigma^2 c_2} \frac{B^2}{H_{25}^2(\xi_1)}, \quad c_1 = \frac{k_1 c_2}{k_2} \frac{W^2(\xi_1)}{B^2}$ where ξ_1 is given as in case 10 and B is the only solution of the equation $H_{44}(x) = \frac{\sigma h_o c_2}{\lambda k_2} \frac{W^2(\xi_1)}{\xi_1 \exp(\xi_1^2)}, \quad x > 0.$
27	ω, γ, c_1	(R10) (R17) (R37)	$\omega = Ba_2, \quad \gamma = \frac{2D}{\sqrt{\pi}}\left(\frac{a_2}{\sigma} B - 1\right) H_{58}(\xi_1), \quad c_1 = \frac{k_1}{\rho\sigma^2} \xi_1^2$ where ξ_1 is given as in case 10 and B is given as in case 14.
28	ω, λ, c_2	(R23)	$\omega = a_1 W\left(\frac{\sigma}{a_1}\right), \quad c_2 = \frac{c_1 k_2}{k_1} \frac{B^2}{W^2\left(\frac{\sigma}{a_1}\right)},$ $\lambda = \frac{1}{G\left(\frac{\sigma}{a_1}\right)} \left(\frac{Dc_1}{\sqrt{\pi}} F_2\left(\frac{\sigma}{a_1}\right) - \frac{Ec_1 k_2}{k_1\sqrt{\pi}} \frac{H_1(B)}{W\left(\frac{\sigma}{a_1}\right)} \right)$ for any $0 < B < H_1^{-1}\left(\frac{Dk_1}{Ek_2} W\left(\frac{\sigma}{a_1}\right) F_2\left(\frac{\sigma}{a_1}\right)\right)$.
29	ω, ρ, c_2	(R10) (R38)	$\omega = \sigma H_{25}(\xi_1), \quad \rho = \frac{k_1}{\sigma^2 c_1} \xi_1^2, \quad c_2 = \frac{c_1 k_2}{k_1} \frac{B^2}{W^2(\xi_1)}$ where ξ_1 is given as in case 10 and B is the only solution of the equation $H_1(x) = \frac{\lambda k_1 \sqrt{\pi}}{Ec_1 k_2} H_{25}(\xi_1) H_{45}(\xi_1), \quad x > 0.$

Table 2. contd.

Case	Unknown coefficients	Restrictions	Solution
30	ω, ϵ, c_2	(R23) (R39)	$\omega = a_1 W\left(\frac{\sigma}{a_1}\right), \qquad c_2 = \frac{c_1 k_2}{k_1} \frac{B^2}{W^2\left(\frac{\sigma}{a_1}\right)},$ $\epsilon = \frac{2D}{\gamma\sqrt{\pi}} W\left(\frac{\sigma}{a_1}\right) F_2\left(\frac{\sigma}{a_1}\right) - \frac{2D}{\lambda\gamma\sqrt{\pi}} F_2\left(\frac{\sigma}{a_1}\right)\left(\frac{Dc_1}{\sqrt{\pi}} F_2\left(\frac{\sigma}{a_1}\right) - \frac{Ec_1 k_2}{k_1\sqrt{\pi}} \frac{H_1(B)}{W\left(\frac{\sigma}{a_1}\right)}\right)$ for any $H_1^{-1}(A) < B < H_1^{-1}(C)$ if $\frac{\sigma}{a_1} < x_{46}$ or for any $0 < B < H_1^{-1}(C)$ if $x_{46} \le \frac{\sigma}{a_1} < x_{47}$ where $A = \frac{\lambda k_1 \sqrt{\pi}}{Ec_1 k_2} W\left(\frac{\sigma}{a_1}\right) H_{46}\left(\frac{\sigma}{a_1}\right)$ $C = \frac{\lambda k_1 \sqrt{\pi}}{Ec_1 k_2} W\left(\frac{\sigma}{a_1}\right) H_{47}\left(\frac{\sigma}{a_1}\right).$
31	ω, γ, c_2	(R23) (R40)	$\omega = a_1 W\left(\frac{\sigma}{a_1}, \gamma\right), \qquad c_2 = \frac{k_2}{\rho a_1^2} \frac{B^2}{W^2\left(\frac{\sigma}{a_1}, \gamma\right)},$ for any $0 < \gamma < \frac{2DH_{58}\left(\frac{\sigma}{a_1}\right)}{(1-\epsilon)\sqrt{\pi}}\left(\frac{Dc_1}{\lambda\sqrt{\pi}} \frac{1}{H_{55}\left(\frac{\sigma}{a_1}\right)} - 1\right)$ where $B = B(\gamma)$ is the unique solution of the equation $H_1(x) = \frac{\lambda k_1 \sqrt{\pi}}{Ec_1 k_2} W\left(\frac{\sigma}{a_1}, \gamma\right) H_{57}\left(\frac{\sigma}{a_1}, \gamma\right), \qquad x > 0.$
32	ω, λ, ρ	(R10) (R42)	$\omega = \sigma H_{25}(\xi_1), \qquad \rho = \frac{k_1}{\sigma^2 c_1} \xi_1^2, \qquad \lambda = \frac{\sigma h_o c_1}{k_1} \frac{H_{49}(\xi_1)}{G(\xi_1)}$ where ξ_1 is given as in case 10.
33	$\omega, \lambda, \epsilon$	(R23) (R48)	$\omega = a_1 W\left(\frac{\sigma}{a_1}\right), \qquad \epsilon = 1 - \frac{2D}{\gamma\sqrt{\pi}} H_{58}\left(\frac{\sigma}{a_1}\right)\left(\frac{Dc_1}{\lambda\sqrt{\pi}} H_{56}\left(\frac{\sigma}{a_1}\right) - 1\right)$ for any $\frac{\frac{Dc_1}{\sqrt{\pi}} H_{56}\left(\frac{\sigma}{a_1}\right)}{1 + \frac{\gamma\sqrt{\pi}}{2D} \frac{1}{H_{58}\left(\frac{\sigma}{a_1}\right)}} < \lambda < \frac{Dc_1}{\sqrt{\pi}} H_{56}\left(\frac{\sigma}{a_1}\right)$
34	ω, λ, γ	(R23) (R44) (R45)	$\omega = a_1 W\left(\frac{\sigma}{a_1}, \gamma\right),$ $\lambda = \frac{Dc_1}{G\left(\frac{\sigma}{a_1}, \gamma\right), \sqrt{\pi}}\left(F_2\left(\frac{\sigma}{a_1}\right) - \frac{E}{D}\sqrt{\frac{c_2 k_1}{c_1 k_1}} F_1\left(\frac{\sigma}{a_1} H_{25}\left(\frac{\sigma}{a_1}\right)\right)\right)$ for any $0 < \gamma < \frac{2DH_{58}\left(\frac{\sigma}{a_1}\right)}{(1-\epsilon)\sqrt{\pi}}\left(\frac{a_2}{\sigma} F_1^{-1}\left(\frac{Dk_1 a_2}{Ea_1 k_2} F_2\left(\frac{\sigma}{a_1}\right)\right) - 1\right).$
35	ω, ϵ, ρ	(R46) (R47)	$\omega = \sigma H_{25}(\xi_1), \qquad \rho = \frac{k_1}{\sigma^2 c_1} \xi_1^2, \qquad \epsilon = 1 - \frac{2D}{\gamma\sqrt{\pi}} \frac{H_{51}(\xi_1)}{H_{55}(\xi_1)}$ where ξ_1 is given as in case 10.
36	ω, γ, ρ	(R10) (R41)	$\omega = \sigma\sqrt{\frac{c_1 k_2}{k_1 c_2} \frac{B}{\xi_1}}, \qquad \rho = \frac{k_1}{\sigma^2 c_1} \xi_1^2,$ $\gamma = \frac{2D}{\sqrt{\pi}}\left(\sqrt{\frac{c_1 k_2}{k_1 c_2}} B - \xi_1\right) F_2(\xi_1)$ where ξ_1 is given as in case 10 and B is the only solution of the equation $H_{53}(x) = \frac{\sigma h_o c_1}{\lambda k_1} \frac{\exp(-\xi_1^2)}{\xi_1} - \epsilon\xi_1, \qquad x > \sqrt{\frac{k_1 c_2}{c_1 k_2}}\, \xi_1.$