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Convergence of boundary optimal controls in mixed elliptic problems

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We consider a steady-state heat conduction problem P_α with mixed boundary conditions for the Poisson equation in a bounded multidimensional domain Ω depending of a positive parameter α which represents the heat transfer coefficient on a portion Γ_1 of the boundary of Ω. We consider, for each $\alpha > 0$, a cost function J_α and we formulate boundary optimal control problems with restrictions over the heat flux q on a complementary portion Γ_2 of the boundary of Ω . We obtain that the optimality conditions are given by a complementary free boundary problem in Γ_2 in terms of the adjoint state. We prove that the optimal control q_{op_α} and its corresponding system state $u_{q_{op_\alpha}\alpha}$ and adjoint state $p_{q_{op_\alpha}\alpha}$ for each α are strongly convergent to q_{op}, $u_{q_{op}}$ and $p_{q_{op}}$ in $L^2(\Gamma_2)$, $H^1(\Omega)$, and $H^1(\Omega)$ respectively when $\alpha \to \infty$. We also prove that these limit functions are respectively the optimal control, the system state and the adjoint state corresponding to another boundary optimal control problem with restrictions for the same Poisson equation with a different boundary condition on the portion Γ_1 . We use the elliptic variational inequality theory in order to prove all the strong convergences. In this paper, we generalize the convergence result obtained in Ben Belgacem-El Fekih-Metoui, ESAIM:M2AN, 37 (2003), 833-850 by considering boundary optimal control problems with restrictions on the heat flux q defined on Γ_2 and the parameter α (which goes to infinity) is defined on Γ_1.

1 Introduction

We consider a bounded domain Ω in R^n whose regular boundary Γ consists of the union of two disjoint portions Γ_1 and Γ_2 with meas($\Gamma_i > 0$) for $i = 1, 2$. We consider the following two steady-state heat conduction problems P and P_α (for each heat transfer parameter $\alpha > 0$) respectively with mixed boundary conditions:

$$-\Delta u = g \text{ in } \Omega, \quad u\big|_{\Gamma_1} = b, \quad -\frac{\partial u}{\partial n}\big|_{\Gamma_2} = q \tag{1}$$

$$-\Delta u = g \text{ in } \Omega, \quad -\frac{\partial u}{\partial n}\big|_{\Gamma_1} = \alpha(u-b), \quad -\frac{\partial u}{\partial n}\big|_{\Gamma_2} = q \tag{2}$$

where $g \in H = L^2(\Omega)$, $q \in Q = L^2(\Gamma_2)$ and $b \in H^{\frac{1}{2}}(\Gamma_1)$. These problems can be considerer as steady-state two-phase Stefan problems for suitable data [6], [9].

We denote with u_q and $u_{q\alpha}$ the unique solutions of the mixed elliptic problems (1) and (2) respectively for each $q \in Q$ and $\alpha > 0$ whose variational formulations are given as in [3], [4] and [7].

We define the boundary optimal control problems with restrictions as following [2] and [5]:

$$\text{Find } q_{op} \in U_{ad} \text{ such that } J(q_{op}) = \min_{q \in U_{ad}} J(q), \tag{3}$$

$$\text{Find } q_{op_\alpha} \in U_{ad} \text{ such that } J_\alpha(q_{op_\alpha}) = \min_{q \in U_{ad}} J_\alpha(q), \tag{4}$$

with $J : Q \to R_0^+$ and $J_\alpha : Q \to R_0^+$ are given by:

$$J(q) = \frac{1}{2}\left\|u_q - z_d\right\|_H^2 + \frac{M}{2}\left\|q\right\|_Q^2 \quad \text{and} \quad J_\alpha(q) = \frac{1}{2}\left\|u_{q\alpha} - z_d\right\|_H^2 + \frac{M}{2}\left\|q\right\|_Q^2$$

with $z_d \in H$, M is a positive constant and $U_{ad} = \{q \in Q : q \geq 0 \text{ en } \Gamma_2\}$ is a non empty, closed and convex subset of Q.

2 Complementary Conditions and Convergence When the Parameter α Goes to Infinity

We obtain the following results:

Theorem 2.1. i) There exists a unique boundary optimal control with restriction q_{op} for the problem (3) and the corresponding optimality condition is given as a complementary free boundary problem

$$q_{op} \geq 0 \text{ on } \Gamma_2, \quad Mq_{op} - p_{q_{op}} \geq 0 \text{ on } \Gamma_2, \quad q_{op}(Mq_{op} - p_{q_{op}}) = 0 \text{ on } \Gamma_2$$

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where p_q is the corresponding adjoint state defined by the variational equality:

$$a(p_q, v) = (u_q - z_d, v)_H, \quad \forall v \in V_0, \quad p_q \in V_0.$$

with $V = H^1(\Omega)$, $V_0 = \{v \in V / v\big|_{\Gamma_1} = 0\}$, $(g, h)_H = \int_\Omega gh\,dx$ and $a(u, v) = \int_\Omega \nabla u.\nabla v\,dx$.

ii) For each $\alpha > 0$, there exists a unique boundary optimal control with restriction q_{op_α} for the problem (4) and the corresponding optimality condition is given as a complementary free boundary problem

$$q_{op_\alpha} \geq 0 \text{ on } \Gamma_2, \quad Mq_{op_\alpha} - p_{q_{op_\alpha}\alpha} \geq 0 \text{ on } \Gamma_2, \quad q_{op_\alpha}(Mq_{op_\alpha} - p_{q_{op_\alpha}\alpha}) = 0 \text{ on } \Gamma_2$$

where p_{q_α} is the corresponding adjoint state defined by the variational equality:

$$a(p_{q\alpha}, v) + \alpha(p_{q\alpha}, v)_{L^2(\Gamma_1)} = (u_{q\alpha} - z_d, v), \quad \forall v \in V, \quad p_{q\alpha} \in V. \tag{5}$$

iii) If there is not restriction on the control, i.e. $U_{ad} = Q$ then we have the relations: $Mq_{op} = p_{q_{op}}$ on Γ_2 and $Mq_{op_\alpha} = p_{q_{op_\alpha}\alpha}$ on Γ_2.

In similar way that [6], [7] and [8] we prove the following result:

Theorem 2.2. For all $\alpha > 0$, $q \in Q$, $b \in H^{\frac{1}{2}}(\Gamma_1)$ we have:

$$\lim_{\alpha\to\infty} \|u_{q\alpha} - u_q\|_V = 0 \quad \text{and} \quad \lim_{\alpha\to\infty} \|p_{q\alpha} - p_q\|_V = 0 \quad \forall q \in Q.$$

In [1], it was considered a boundary optimal control problem with $\Gamma = \Gamma_1$ and the Dirichlet control variable is the temperature b which is defined in the same boundary where the penalization parameter $\epsilon = \frac{1}{\alpha}$ is given. In this case, the boundary optimal control is proportional to the corresponding adjoint state. In this paper, we generalize the results obtained in [1] by considering a Neumann boundary optimal control with restrictions on the heat flux q on Γ_2 and the parameter $\alpha\ (= \frac{1}{\varepsilon})$ which goes to infinity is defined on a complementary boundary portion Γ_1.

Finally, we prove that the optimal state system $u_{q_{op_\alpha}\alpha}$ and the optimal adjoint states $p_{q_{op_\alpha}\alpha}$ of the problems (4) are strongly convergent in V to the corresponding $u_{q_{op}}$ and $p_{q_{op}}$ for the boundary optimal control problem with restrictions (3). Moreover, the strong convergence in Q of the optimal controls q_{op_α} of problems (4) to the optimal control q_{op} of problem (3) is also proved, that is:

Theorem 2.3. If q_{op_α} is the unique solution of the boundary optimal control problem (4) for each $\alpha > 0$ and q_{op} is the unique solution of the boundary optimal control problem (3), then we have:

$$\lim_{\alpha\to\infty} \left\|u_{q_{op_\alpha}\alpha} - u_{q_{op}}\right\|_V = \lim_{\alpha\to\infty} \left\|p_{q_{op_\alpha}\alpha} - p_{q_{op}}\right\|_V = \lim_{\alpha\to\infty} \|q_{op_\alpha} - q_{op}\|_Q = 0.$$

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