# CONVERGENCE OF DISTRIBUTED OPTIMAL CONTROLS IN MIXED ELLIPTIC PROBLEMS BY THE PENALIZATION METHOD

CLAUDIA M. GARIBOLDI, DOMINGO A. TARZIA

Abstract. We consider a distributed optimal control problem $(P)$ and a family of distributed optimal control problems $(P_\alpha)$, for each $\alpha > 0$ (heat transfer coefficient). Both problems are related with steady-state heat conduction problems for the same Poisson equation with different mixed boundary condition for $(P)$ and $(P_\alpha)$ respectively. We use the penalization method in order to obtain a family of optimal control problems $(P_\epsilon)$ for each $\epsilon > 0$ and a family $(P_{\alpha\epsilon})$ for fixed $\alpha > 0$ with their correspondent cost functions $J_\epsilon$ and $J_{\alpha\epsilon}$. We prove strong convergence as $\epsilon \to 0$ of the optimal control $g_\epsilon$ of $(P_\epsilon)$ to the optimal control $g$ of $(P)$ and of the system state $U_\epsilon$ of problem $(P_\epsilon)$ to the system state $U$ of problem $(P)$ in suitable Sobolev spaces. We obtain similar results for fixed $\alpha > 0$ and $\epsilon \to 0$ in relation to the problems $(P_\alpha)$ and $(P_{\alpha\epsilon})$. Finally, we obtain weak convergence of solutions of the problems $(P_{\alpha\epsilon})$ to the solution of the problem $(P_\epsilon)$ when $\alpha \to \infty$, for fixed $\epsilon > 0$. This result can be considered as a new proof of the optimal controls convergence obtained in Gariboldi-Tarzia, Appl. Math. Optim., 47 (2003), 213-230 by using the variational equality theory.

Resumen: Se considera un problema de control óptimo distribuído $(P)$ y una familia de problemas de control óptimo distribuído $(P_\alpha)$, para cada $\alpha > 0$ (coeficiente de transferencia de calor). Ambos problemas están relacionados con problemas de conducción del calor estacionarios para la misma ecuación de Poisson con diferentes condiciones de frontera para $(P)$ y $(P_\alpha)$ respectivamente. Se usa el método de penalización de modo de obtener una familia de problemas de control óptimo $(P_\epsilon)$ para cada $\epsilon > 0$ y una familia $(P_{\alpha\epsilon})$ para cada $\alpha > 0$ fijo con sus correspondientes funciones costos $J_\epsilon$ y $J_{\alpha\epsilon}$. Se prueba la convegencia fuerte cuando $\epsilon \to 0$ del control óptimo $g_\epsilon$ de $(P_\epsilon)$ al control óptimo $g$ de $(P)$ y del estado del sistema $U_\epsilon$ del problema $(P_\epsilon)$ al estado del sistema $U$ del problema $(P)$ en espacios de Sobolev apropiados. Se obtiene resultados similares para $\alpha > 0$ fijos y $\epsilon \to 0$ en relación a los problemas $(P_\alpha)$ y $(P_{\alpha\epsilon})$. Finalmente, se obtiene la convergencia débil de soluciones de los problemas $(P_{\alpha\epsilon})$ a la solución del problema $(P_\epsilon)$ cuando $\alpha \to \infty$, para un $\epsilon > 0$ fijado. Este resultado puede considerarse como una nueva prueba de la convergencia de controles óptimos obtenida en Gariboldi-Tarzia, Appl. Math. Optim., 47 (2003), 213-230 usando la teoría de inecuaciones variacionales.

keywords: Elliptic variational inequality, Penalization method, Distributed optimal control problems, Mixed boundary conditions, Adjoint state, Steady-state Stefan problem, Optimality condition, Optimal controls convergence.

AMS Subject Classifications: 49J20, 35J85, 35R35.

## 1. Introduction

We consider two bounded regular domains $\Omega_1$ and $\Omega_2$ in $\mathbb{R}^n$. We define $\Omega = \Omega_2 - \overline{\Omega_1}$ where $\overline{\Omega_1} \subset \Omega_2$. The regular boundary $\Gamma = \partial\Omega$ consists of the union of three disjoint portions $\Gamma_1 = \partial\Omega_1$ and $\Gamma_2 \bigcup \Gamma_3 = \partial\Omega_2$, with meas$(\Gamma_i) > 0$, for $i = 1, 2, 3$, where meas$(\Gamma_i)$ is the $(n-1)-$dimensional Lebesgue measure of $\Gamma_i$. We assume that the boundary portion $\Gamma_1$ is divided in two disjoint portions $\Gamma_{1,0}$ and $\Gamma_{1,1}$.

We consider the following mixed elliptic problems

$$(1) \qquad \begin{cases} -\Delta u = g \text{ in } \Omega \\ u = b \text{ on } \Gamma_1 \\ -\frac{\partial u}{\partial \eta} = q \text{ on } \Gamma_2; \quad \frac{\partial u}{\partial \eta} = 0 \text{ on } \Gamma_3 \end{cases}$$

and

$$(2) \qquad \begin{cases} -\Delta u = g \text{ in } \Omega \\ u = b \text{ on } \Gamma_{1,0}; \quad -\frac{\partial u}{\partial \eta} = \alpha(u-b) \text{ on } \Gamma_{1,1} \\ -\frac{\partial u}{\partial \eta} = q \text{ on } \Gamma_2; \quad \frac{\partial u}{\partial \eta} = 0 \text{ on } \Gamma_3 \end{cases}$$

where $g \in H = L^2(\Omega)$ is the internal energy in $\Omega$, $b \in H^{\frac{3}{2}}(\Gamma_1)$ is the temperature on $\Gamma_1$ for (1) and the temperature of the external neighborhood of $\Gamma_{1,1}$ for (2), $q \in H^{\frac{1}{2}}(\Gamma_2)$ is the heat flux on $\Gamma_2$ and $\alpha > 0$ is the heat transfer coefficient on $\Gamma_{1,1}$ (Newton's law on $\Gamma_{1,1}$).

Problems (1) and (2) can be considered as the steady-state Stefan problem for suitable data $q$, $g$ and $b$ [10], [11].

If we consider $U = u - v_0$ with $v_0 \in H^2(\Omega)$ given and such that $v_0 = b$ on $\Gamma_1$ and $\frac{\partial v_0}{\partial \eta} = 0$ on $\Gamma_3$, the problems (1) and (2) are transformed into

$$(3) \qquad \begin{cases} -\Delta U = g + \Delta v_0 \text{ in } \Omega \\ U = 0 \text{ on } \Gamma_1 \\ -\frac{\partial U}{\partial \eta} = q + \frac{\partial v_0}{\partial \eta} \text{ on } \Gamma_2; \quad \frac{\partial U}{\partial \eta} = 0 \text{ on } \Gamma_3 \end{cases}$$

and

$$(4) \qquad \begin{cases} -\Delta U = g + \Delta v_0 \text{ in } \Omega \\ U = 0 \text{ on } \Gamma_{1,0}; \quad -\frac{\partial U}{\partial \eta} = \alpha U + \frac{\partial v_0}{\partial \eta} \text{ on } \Gamma_{1,1} \\ -\frac{\partial U}{\partial \eta} = q + \frac{\partial v_0}{\partial \eta} \text{ on } \Gamma_2; \quad \frac{\partial U}{\partial \eta} = 0 \text{ on } \Gamma_3 \end{cases}$$

respectively, whose variational equalities are given by [5], [7].

$$(5) \qquad a(U, v) = L_g(v) - a(v_0, v), \quad \forall v \in V_0, \quad U \in V_0$$

and

$$(6) \qquad a(U, v) = L_g(v) - a(v_0, v) - \alpha \int_{\Gamma_{1,1}} Uv \quad d\gamma, \quad \forall v \in V_{01}, \quad U \in V_{01}$$

where

$$V_0 = \left\{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1, \quad \frac{\partial v}{\partial \eta} = 0 \text{ on } \Gamma_3 \right\}$$

$$V_{01} = \left\{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_{1,0}, \quad \frac{\partial v}{\partial \eta} = 0 \text{ on } \Gamma_3 \right\}$$

$$a(u,v) = \int_\Omega \nabla u . \nabla v dx \quad , \quad L_g(v) = (g,v)_H - \int_{\Gamma_2} qv \ d\gamma.$$

We denote by $U$ and $U_\alpha$ the unique solution of the problems (3) and (4) respectively, and we have by [1], [9] that $U \in H^2(\Omega) \cap V_0$ and $U_\alpha \in H^2(\Omega) \cap V_{01}$. It is well known that the operators $T : H \to H^2(\Omega) \cap V_0$ and $T_\alpha : H \to H^2(\Omega) \cap V_{01}$ such that $T(g) = U$ and $T_\alpha(g) = U_\alpha$ respectively are affine and continuous applications.

Let the distributed optimal control problems be [2], [3], [4], [5], [6], [8]:

(7)
$$\min_{U \in K, g \in U_{ad}, U = T(g)} \quad J(U,g)$$

and

(8)
$$\min_{U \in K_1, g \in U_{ad}, U = T_\alpha(g)} \quad J(U,g) = \min_{U \in K_1, g \in U_{ad}} \quad J(T_\alpha(g), g)$$

where

$$J : H^2(\Omega) \cap V_{01} \times H \to \mathbb{R}_0^+$$

is given by

$$J(U,g) = \frac{1}{2} \|U - z_d\|_H^2 + \frac{M}{2} \|g\|_H^2$$

where $z_d \in H$ is given, $M = const. > 0$, $K$ is a closed, convex and nonempty subset of $H^2(\Omega) \cap V_0$, $K_1$ is a closed, convex and nonempty subset of $H^2(\Omega) \cap V_{01}$ and $U_{ad}$ is a closed convex and nonempty subset of $H$.

In [5] we study the convergence when $\alpha \to \infty$ of the optimal control problem (8) corresponding to the state system (4). We prove that the optimal state system, the optimal adjoint system and the optimal control of the problem (8) are strongly convergent in adequate Sobolev spaces to the corresponding state system, the optimal adjoint state and the optimal control of the problem (7) respectively, when $\alpha \to \infty$.

We use the classical notations

$$(f,g)_H = \int_\Omega fg \quad dx, \quad \forall f, g \in H, \quad \|g\|_H^2 = \int_\Omega g^2 dx,$$

$$\|(v,g)\|_{H \times H}^2 = \|v\|_H^2 + \|g\|_H^2 .$$

As $K \times U_{ad}$ and $K_1 \times U_{ad}$ are closed, convex and nonempty sets in $H^2(\Omega) \cap V_0 \times H$ and $H^2(\Omega) \cap V_{01} \times H$ respectively, then $D = \{(U,g) \in K \times U_{ad} : T(g) = U\}$ and $D_\alpha = \{(U,g) \in K_1 \times U_{ad} : T_\alpha(g) = U\}$ are closed and convex sets. Moreover, as $D$ and $D_\alpha$ are nonempy and $J$ is a strictly convex functional, the problems (7) and (8) have unique solution $(\overline{U}, \overline{g})$ and $(\overline{U_\alpha}, \overline{g_\alpha})$ respectively.

We consider, for each $\epsilon > 0$, as in [3] the following penalized problems

(9)
$$\min_{U \in K, g \in U_{ad}} \quad J_\epsilon(U,g)$$

and

(10)
$$\min_{U \in K_1, g \in U_{ad}} \quad J_{\alpha\epsilon}(U,g)$$

where $J_\epsilon : H^2(\Omega) \cap V_0 \times H \to \mathbb{R}_0^+$ is given by

$$J_\epsilon(U,g) = J(U,g) + \frac{1}{2\epsilon}\|\Delta U + g + \Delta v_0\|_H^2 + \frac{1}{2\epsilon}\|\frac{\partial U}{\partial n} + q + \frac{\partial v_0}{\partial n}\|_{L^2(\Gamma_2)}^2$$

and $J_{\alpha\epsilon} : H^2(\Omega) \cap V_{01} \times H \to \mathbb{R}_0^+$ is given by

$$\begin{aligned} J_{\alpha\epsilon}(U,g) &= J(U,g) + \frac{1}{2\epsilon}\|\Delta U + g + \Delta v_0\|_H^2 + \frac{1}{2\epsilon}\|\frac{\partial U}{\partial n} + q + \frac{\partial v_0}{\partial n}\|_{L^2(\Gamma_2)}^2 + \\ &\quad \frac{1}{2\epsilon}\|\frac{\partial U}{\partial n} + \alpha U + \frac{\partial v_0}{\partial n}\|_{L^2(\Gamma_{1,1})}^2. \end{aligned}$$

In Section 2 we prove that the problem (9), for each $\epsilon > 0$, has a unique solution and we give the corresponding optimality condition in terms of the adjoint state $p_\epsilon$ and suitable multipliers. We prove that for $\epsilon \to 0^+$ the sequence of solutions of the penalized problems (9) are strongly convergent to the solution of the problem (7) in suitable Sobolev spaces.

In Section 3 we obtain, in similar way to Section 2, that the problem (10), for each $\epsilon > 0$ and fixed $\alpha > 0$, has a unique solution and we give the corresponding optimality condition in terms of the adjoint state $p_{\alpha\epsilon}$ and suitable multipliers. We also prove that the solutions of the penalized problems (10) are strongly convergent to the solution of the problem (8) in suitable Sobolev spaces when $\epsilon \to 0^+$.

In Section 4 we prove that, for fixed $\epsilon > 0$, the solutions of the penalized problems (10) are weakly convergent to the solution of the penalized problem (9) in $H$ when $\alpha \to +\infty$, which is the same type of result obtained in [5] by using the variational equality theory.

## 2. PENALIZATION OF PROBLEM (7) AND STRONG CONVERGENCE WHEN $\epsilon \to 0^+$

**Proposition 1.** *a) $J$ is a strictly convex functional and there exists a unique solution $(\overline{U}, \overline{g})$ of the problem (7).*

*b) $J_\epsilon$ is a coercive and strictly convex functional. There exists a unique solution $(U_\epsilon, g_\epsilon)$ of the penalized problem (9).*

*c) $J_\epsilon$ is Gateaux-differentiable and the optimality condition is given by $\forall v \in K, \forall g \in U_{ad}$*

$$\begin{aligned} &(U_\epsilon - z_d, v - U_\epsilon)_H + M(g_\epsilon, g - g_\epsilon)_H \\ &+\frac{1}{\epsilon}(\Delta U_\epsilon + g_\epsilon + \Delta v_0, \Delta(v - U_\epsilon))_H \\ &+\frac{1}{\epsilon}(\Delta U_\epsilon + g_\epsilon + \Delta v_0, g - g_\epsilon)_H \\ &+\frac{1}{\epsilon}(\frac{\partial U_\epsilon}{\partial n} + q + \frac{\partial v_0}{\partial n}, \frac{\partial (v - U_\epsilon)}{\partial n})_{L^2(\Gamma_2)} \geq 0. \end{aligned} \tag{11}$$

*Proof.* a) This results as in [5], from the definition of $J$ and taking into account that the functions $x \to x^2$ and $x \to (x+c)^2$ are convex.

b) Let $t \in [0,1]$, $(U_1, g_1)$, $(U_2, g_2) \in K \times U_{ad}$, then we have that

$$\begin{aligned} & (1-t)J(U_1,g_1) + tJ(U_2,g_2) - J((1-t)U_1 + tU_2, (1-t)g_1 + tg_2) \\ = \;& t\frac{(1-t)}{2}[\|U_1 - U_2\|_H^2 + M\|g_1 - g_2\|_H^2] \\ \geq \;& \frac{min(1,M)}{2}t(1-t)[\|U_1 - U_2\|_H^2 + \|g_1 - g_2\|_H^2] \\ = \;& \frac{min(1,M)}{2}t(1-t)\|(U_1,g_1) - (U_2,g_2)\|_{H\times H.}^2 \end{aligned}$$

On the other hand, we get that

$$\begin{aligned} & (1-t)\int_\Omega (\Delta U_1 + g_1 + \Delta v_0)^2 dx + t\int_\Omega (\Delta U_2 + g_2 + \Delta v_0)^2 dx \\ & -\int_\Omega [\Delta((1-t)U_1 + tU_2) + (1-t)g_1 + tg_2 + \Delta v_0]^2 dx \geq 0 \end{aligned}$$

and

$$\begin{aligned} & (1-t)\int_{\Gamma_2} (\frac{\partial U_1}{\partial n} + q + \frac{\partial v_0}{\partial n})^2 d\gamma + t\int_{\Gamma_2} (\frac{\partial U_2}{\partial n} + q + \frac{\partial v_0}{\partial n})^2 d\gamma \\ & -\int_{\Gamma_2} (\frac{\partial((1-t)U_1 + tU_2)}{\partial n} + q + \frac{\partial v_0}{\partial n})^2 d\gamma \geq 0 \end{aligned}$$

therefore there exists the constant $c = \frac{min(1,M)}{2} > 0$ such that

$$\begin{aligned} & (1-t)J_\epsilon(U_1,g_1) + tJ_\epsilon(U_2,g_2) - J_\epsilon((1-t)U_1 + tU_2, (1-t)g_1 + tg_2) \\ \geq \;& c\,(1-t)\,t\,\|(U_1,g_1) - (U_2,g_2)\|_H^2, \end{aligned}$$

and $J_\epsilon$ is a coercive and strictly convex functional.

c) Let $v \in K$, $g \in U_{ad}$, and $t \in (0,1]$, then we get

$$\begin{aligned} & J_\epsilon(U_\epsilon + t(v - U_\epsilon), g_\epsilon + t(g - g_\epsilon)) - J_\epsilon(U_\epsilon, g_\epsilon) \\ = \;& J(U_\epsilon + t(v - U_\epsilon), g_\epsilon + t(g - g_\epsilon)) \\ & + \frac{1}{2\epsilon}\int_\Omega [\Delta((1-t)U_\epsilon + tv) + ((1-t)g_\epsilon + tg) + \Delta v_0]^2 dx \\ & + \frac{1}{2\epsilon}\int_{\Gamma_2} (\frac{\partial((1-t)U_\epsilon + tv)}{\partial n} + q + \frac{\partial v_0}{\partial n})^2 d\gamma \\ & - J(U_\epsilon, g_\epsilon) - \frac{1}{2\epsilon}\int_\Omega [\Delta U_\epsilon + g_\epsilon + \Delta v_0]^2 dx - \frac{1}{2\epsilon}\int_{\Gamma_2} (\frac{\partial U_\epsilon}{\partial n} + q + \frac{\partial v_0}{\partial n})^2 d\gamma \\ = \;& t\int_\Omega (U_\epsilon - z_d)(v - U_\epsilon) dx + \frac{t^2}{2}\int_\Omega (v - U_\epsilon)^2 dx \\ & + Mt\int_\Omega g_\epsilon (g - g_\epsilon) dx + \frac{Mt^2}{2}\int_\Omega (g - g_\epsilon)^2 dx \\ & + \frac{t}{\epsilon}\int_\Omega (\Delta U_\epsilon + g_\epsilon + \Delta v_0)(\Delta(v - U_\epsilon) + (g - g_\epsilon)) dx \\ & + \frac{t^2}{2\epsilon}\int_\Omega (\Delta(v - U_\epsilon) + (g - g_\epsilon))^2 dx + \frac{t}{\epsilon}\int_{\Gamma_2} (\frac{\partial U_\epsilon}{\partial n} + q + \frac{\partial v_0}{\partial n})(\frac{\partial(v - U_\epsilon)}{\partial n}) d\gamma \\ & + \frac{t^2}{2\epsilon}\int_{\Gamma_2} (\frac{\partial(v - U_\epsilon)}{\partial n}) d\gamma. \end{aligned}$$

Now, dividing by $t$, and passing to the limit $t \to 0^+$, $\forall v \in K,\ \forall g \in U_{ad}$ we have

$$\begin{aligned} J'_\epsilon(U_\epsilon, g_\epsilon)(v - U_\epsilon, g - g_\epsilon) &= (U_\epsilon - z_d, v - U_\epsilon)_H + M(g_\epsilon, g - g_\epsilon)_H \\ &+ \frac{1}{\epsilon}(\Delta U_\epsilon + g_\epsilon + \Delta v_0, \Delta(v - U_\epsilon) + g - g_\epsilon)_H \\ &+ \frac{1}{\epsilon}(\frac{\partial U_\epsilon}{\partial n} + q + \frac{\partial v_0}{\partial n}, \frac{\partial(v - U_\epsilon)}{\partial n})_{L^2(\Gamma_2)} \geq 0 \end{aligned}$$

that is (11). □

Let us call, following [3], for each $\epsilon > 0$

$$\lambda_\epsilon = \frac{\Delta U_\epsilon + g_\epsilon + \Delta v_0}{\epsilon} \quad , \quad \delta_\epsilon = \frac{\frac{\partial U_\epsilon}{\partial n} + q + \frac{\partial v_0}{\partial n}}{\epsilon}$$

and we define $p_\epsilon \in V_0$ such that

$$\begin{cases} -\Delta p_\epsilon = U_\epsilon - z_d \text{ in } \Omega \\ p_\epsilon = 0 \text{ on } \Gamma_1 \\ \frac{\partial p_\epsilon}{\partial \eta} = 0 \text{ on } \Gamma_2; \quad \frac{\partial p_\epsilon}{\partial \eta} = 0 \text{ on } \Gamma_3. \end{cases}$$

We can prove the following property [3]

**Proposition 2.** *The optimality condition (11) can be decoupled:*

$$\begin{aligned} (12) \quad \forall v &\in K, (p_\epsilon - \lambda_\epsilon, -\Delta(v - U_\epsilon))_H + (p_\epsilon + \delta_\epsilon, \frac{\partial(v - U_\epsilon)}{\partial n})_{L^2(\Gamma_2)} \geq 0 \\ \forall g &\in U_{ad}, \quad (M g_\epsilon + \lambda_\epsilon, g - g_\epsilon)_H \geq 0 \end{aligned}$$

*Proof.* The optimality condition (11) can be decoupled in

$$\begin{aligned} \forall v &\in K,\ (U_\epsilon - z_d, v - U_\epsilon)_H + \frac{1}{\epsilon}(\Delta U_\epsilon + g_\epsilon + \Delta v_0, \Delta(v - U_\epsilon))_H \\ &+ \frac{1}{\epsilon}(\frac{\partial U_\epsilon}{\partial n} + q + \frac{\partial v_0}{\partial n}, \frac{\partial(v - U_\epsilon)}{\partial n})_{L^2(\Gamma_2)} \geq 0, \\ \forall g &\in U_{ad},\ (M g_\epsilon, g - g_\epsilon)_H + \frac{1}{\epsilon}(\Delta U_\epsilon + g_\epsilon + \Delta v_0, g - g_\epsilon)_H \geq 0. \end{aligned}$$

By the Green's formula, $\forall v \in K,\ \forall g \in U_{ad}$ we have

$$(U_\epsilon - z_d, v - U_\epsilon)_H = (p_\epsilon, -\Delta(v - U_\epsilon))_H + (p_\epsilon, \frac{\partial(v - U_\epsilon)}{\partial \eta})_{L^2(\Gamma_2)}$$

and the optimality condition (11) is now decoupled and it is given by (12). □

**Theorem 1.** *Let $(\overline{U}, \overline{g})$ be the solution of the problem (7), then we have that $g_\epsilon \longrightarrow \overline{g}$ in $H$ and $U_\epsilon \longrightarrow \overline{U}$ in $H^2(\Omega) \cap V_0$ when $\epsilon \to 0^+$.*

*Proof.* Let $\epsilon > 0$ be given. In similar way to [3], we have

$$(13) \qquad 0 \leq J_\epsilon(U_\epsilon, g_\epsilon) \leq J_\epsilon(\overline{U}, \overline{g}) = J(\overline{U}, \overline{g}) < \infty$$

next, there exists a constant $k > 0$ such that

$$\sup_{\epsilon > 0} \|U_\epsilon\|_H \leq k \text{ and } \sup_{\epsilon > 0} \|g_\epsilon\|_H \leq k.$$

Therefore, there exists $U_0,\, g_0 \in H$ and subcollection $\{U_\epsilon\}$ and $\{g_\epsilon\}$ such that $U_\epsilon \rightharpoonup U_0$ weakly in $H$ and $g_\epsilon \rightharpoonup g_0$ weakly in $H$. From (13) we get

$$\int_\Omega [\Delta U_\epsilon + g_\epsilon + \Delta v_0]^2\, dx \longrightarrow 0 \text{ when } \epsilon \to 0^+$$

and

$$\int_{\Gamma_2} (\frac{\partial U_\epsilon}{\partial n} + q + \frac{\partial v_0}{\partial n})^2 d\gamma \longrightarrow 0 \text{ when } \epsilon \to 0^+.$$

Now by using the weak lower semicontinuity of the function $u \to \int u^2$ we get

$$\int_\Omega [\Delta U_0 + g_0 + \Delta v_0]^2\, dx = 0 \text{ and } \int_{\Gamma_2} (\frac{\partial U_0}{\partial n} + q + \frac{\partial v_0}{\partial n})^2 d\gamma = 0.$$

Therefore $-\Delta U_0 = g_0 + \Delta v_0$ in $\Omega$ and $-\frac{\partial U_0}{\partial n} = q + \frac{\partial v_0}{\partial n}$ in $\Gamma_2$; moreover as $K$ and $U_{ad}$ are convex and closed sets in the strong topology then they are closed sets in the weak topology, next we have that $U_0 \in K$, $g_0 \in U_{ad}$ and $U_0 = T(g_0)$. From (13) we obtain that

(14) $$\liminf_{\epsilon\to 0^+} J_\epsilon(U_\epsilon, g_\epsilon) \le \liminf_{\epsilon\to 0^+} J_\epsilon(\overline{U}, \overline{g}) = J(\overline{U}, \overline{g})$$

and taking into account that

(15) $$J_\epsilon(U,g) \ge J(U,g) \quad \forall (U,g) \in K \times U_{ad}$$

we have

(16) $$J(\overline{U}, \overline{g}) \ge \liminf_{\epsilon\to 0^+} J_\epsilon(U_\epsilon, g_\epsilon) \ge \liminf_{\epsilon\to 0^+}\ J(U_\epsilon, g_\epsilon) \ge J(U_0, g_0).$$

As $(\overline{U}, \overline{g})$ is the solution of the problem (7) we have $J(\overline{U}, \overline{g}) \le J(U_0, g_0)$ and by the uniqueness of the solution we get $\overline{U} = U_0$, $\overline{g} = g_0$; therefore we prove that $U_\epsilon \rightharpoonup \overline{U}$ weakly in $H$ and $g_\epsilon \rightharpoonup \overline{g}$ weakly in $H$ when $\epsilon \to 0^+$. Moreover, we prove that

(17) $$\lim_{\epsilon\to 0^+}\ J_\epsilon(U_\epsilon, g_\epsilon) = J(\overline{U}, \overline{g})$$

(18) $$\lim_{\epsilon\to 0^+}\ J(U_\epsilon, g_\epsilon) = J(\overline{U}, \overline{g}).$$

From (17) and (18) we obtain that

$$\frac{1}{2\epsilon}\int_\Omega [\Delta U_\epsilon + g_\epsilon + \Delta v_0]^2\, dx \longrightarrow 0 \text{ when } \epsilon \to 0^+$$

and

$$\frac{1}{2\epsilon}\int_{\Gamma_2} (\frac{\partial U_\epsilon}{\partial n} + q + \frac{\partial v_0}{\partial n})^2 d\gamma \longrightarrow 0 \text{ when } \epsilon \to 0^+$$

then

$$\lim_{\epsilon\to 0^+}\ \|U_\epsilon - z_d\|_H^2 + M\|g_\epsilon\|_H^2 = \|\overline{U} - z_d\|_H^2 + M\|\overline{g}\|_H^2.$$

From the weak convergence and the convergence in norm in $H \times H$ we have the strong convergence, that is

$$U_\epsilon \longrightarrow \overline{U} \text{ in } H \text{ and } \quad g_\epsilon \longrightarrow \overline{g} \text{ in } H \text{ when } \epsilon \to 0^+.$$

On the other hand $\lim_{\epsilon\to 0^+} \Delta U_\epsilon + g_\epsilon + \Delta v_0 = 0$, i.e. $\lim_{\epsilon\to 0^+} -\Delta U_\epsilon = \overline{g} + \Delta v_0 = -\Delta\overline{U}$, and taking into account that $-\Delta : H^2(\Omega)\cap V_0 \to H$ is a isomorphism, the thesis holds. □

If we define $\overline{p}$ as the solution of the following elliptic problem

$$\begin{cases} -\Delta p = \overline{U} - z_d \text{ in } \Omega \\ p = 0 \text{ on } \Gamma_1 \\ \frac{\partial p}{\partial \eta} = 0 \text{ on } \Gamma_2; \quad \frac{\partial p}{\partial \eta} = 0 \text{ on } \Gamma_3 \end{cases}$$

we get two corollaries from the previous theorem.

**Corollary 1.** *When $\epsilon \to 0^+$, we have that*

*a) $p_\epsilon \to \overline{p}$ in $H^2(\Omega)\cap V_0$.*

*b) $\frac{\partial U_\epsilon}{\partial n} \to \frac{\partial \overline{U}}{\partial n}$ in $L^2(\Gamma_2)$; therefore there exists $c' = Const. > 0$ such that $\| \frac{\partial U_\epsilon}{\partial n}\|_{L^2(\Gamma_2)} \le c'$.*

*c) $p_\epsilon \to \overline{p}$ in $L^2(\Gamma_2)$; therefore there exists $c'' = Const. > 0$ such that $\|p_\epsilon\|_{L^2(\Gamma_2)} \le c''$.*

*Proof.* a) We know that $\| -\Delta p_\epsilon + \Delta\overline{p}\|_H = \| U_\epsilon - \overline{U}\|_H$. From the Theorem 1 we obtain that $-\Delta p_\epsilon \to \Delta\overline{p}$ in $H$ when $\epsilon \to 0^+$ and the fact that $-\Delta : H^2(\Omega)\cap V_0 \to H$ is a isomorphism we get that $p_\epsilon \to \overline{p}$ in $H^2(\Omega)\cap V_0$ when $\epsilon \to 0^+$.

b) Because the Trace Theorem there exists a positive constant $C > 0$ such that $\| \frac{\partial U_\epsilon}{\partial n} - \frac{\partial \overline{U}}{\partial n}\|_{L^2(\Gamma_2)} \le C\| U_\epsilon - \overline{U}\|_{H^2(\Omega)}$ and the thesis holds by Theorem 1.

c) We obtain the proof in similar way to (b). □

**Corollary 2.** *i) There exists $\epsilon_0 > 0$ and $k_0 > 0$ such that:*

$$\sup_{0<\epsilon<\epsilon_0} \|g_\epsilon\|_H \le k_0; \quad \sup_{0<\epsilon<\epsilon_0} \|U_\epsilon\|_{H^2(\Omega)} \le k_0; \quad \sup_{0<\epsilon<\epsilon_0} \|p_\epsilon\|_{H^2(\Omega)} \le k_0$$

$$\sup_{0<\epsilon<\epsilon_0} \|\sqrt{\epsilon}\lambda_\epsilon\|_H \le k_0; \quad \sup_{0<\epsilon<\epsilon_0} \|\sqrt{\epsilon}\delta_\epsilon\|_{L^2(\Gamma_2)} \le k_0$$

*ii) There exists a constant $C' > 0$ such that $\sup_{0<\epsilon<\epsilon_0} \|\lambda_\epsilon\|_H \le C'$.*

*iii) There exists a constant $C'' > 0$ such that $\sup_{0<\epsilon<\epsilon_0} \|\delta_\epsilon\|_{L^2(\Gamma_2)} \le C''$.*

*Proof.* i) When $\epsilon \to 0^+$ we have that: $g_\epsilon \longrightarrow \overline{g}$ in $H$, $U_\epsilon \longrightarrow \overline{U}$ in $H^2(\Omega)\bigcap V_0$, and $p_\epsilon \to \overline{p}$ in $H^2(\Omega)\cap V_0$ by Corollary 1, therefore there exist:

$$\begin{aligned} \epsilon_1 &> 0 \text{ and } C_1 > 0 \text{ such that if } \epsilon < \epsilon_1 \text{ then } \|g_\epsilon\|_H \le C_1 \\ \epsilon_2 &> 0 \text{ and } C_2 > 0 \text{ such that if } \epsilon < \epsilon_2 \text{ then } \|U_\epsilon\|_{H^2(\Omega)} \le C_2 \\ \epsilon_3 &> 0 \text{ and } C_3 > 0 \text{ such that if } \epsilon < \epsilon_3 \text{ then } \|p_\epsilon\|_{H^2(\Omega)} \le C_3. \end{aligned}$$

On the other hand, we have that

$$\text{(19)} \quad \begin{aligned} \|\sqrt{\epsilon}\lambda_\epsilon\|_H^2 &= \int_\Omega \epsilon\lambda_\epsilon^2 = \frac{\epsilon}{\epsilon^2}\int_\Omega (\Delta U_\epsilon + g_\epsilon + \Delta v_0)^2 dx \\ &= \frac{1}{\epsilon}\int_\Omega (\Delta U_\epsilon + g_\epsilon + \Delta v_0)^2 dx \longrightarrow 0 \text{ in } H \text{ when } \epsilon \to 0^+ \end{aligned}$$

next, there exist $\epsilon_4 > 0$ and $C_4 > 0$ such that if $\epsilon < \epsilon_4$ then $\|\sqrt{\epsilon}\lambda_\epsilon\|_H \le C_4$. In similar way to (19) we prove that there exist $\epsilon_5 > 0$ and $C_5 > 0$ such that if $\epsilon < \epsilon_5$ then $\|\sqrt{\epsilon}\delta_\epsilon\|_{L^2(\Gamma_2)} \leq C_5$. Finally taking $\epsilon_0 = min(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5)$ and $k_0 = max\,(C_1, C_2, C_3, C_4, C_5)$ the thesis holds.

ii) Let an arbitrary $s \in B^2(\Omega)$, with $B^2(\Omega)$ the unit ball in $H$, by adding the inequalities (12) $\forall v \in K$, $\forall g \in U_{ad}$ we have:

$$(p_\epsilon - \lambda_\epsilon, -\Delta(v - U_\epsilon))_H + (p_\epsilon + \delta_\epsilon, \frac{\partial(v - U_\epsilon)}{\partial n})_{L^2(\Gamma_2)} + (Mg_\epsilon + \lambda_\epsilon, g - g_\epsilon)_H \geq 0.$$

If in the preceding inequality we take $g = g_1 \in U_{ad}$ and $v = U_s \in K$ such that $T(g_1 + s) = U_s$, we get

$$(p_\epsilon - \lambda_\epsilon, -\Delta(U_s - U_\epsilon))_H + (p_\epsilon + \delta_\epsilon, \frac{\partial(U_s - U_\epsilon)}{\partial n})_{L^2(\Gamma_2)} + (Mg_\epsilon + \lambda_\epsilon, g_1 - g_\epsilon)_H \geq 0.$$

Now, by using that $T(g_1 + s) = U_s$, we have

$$\begin{aligned}&(p_\epsilon, g_1 + s + \Delta v_0 + g_\epsilon - g_\epsilon + \Delta U_\epsilon)_H - (\lambda_\epsilon, g_1 + s + \Delta v_0 + g_\epsilon - g_\epsilon + \Delta U_\epsilon)_H \\ &+(p_\epsilon + \delta_\epsilon, -q - \frac{\partial v_0}{\partial n} - \frac{\partial U_\epsilon}{\partial n})_{L^2(\Gamma_2)} + (Mg_\epsilon + \lambda_\epsilon, g_1 - g_\epsilon)_H \geq 0\end{aligned}$$

i.e.

$$\begin{aligned}(\lambda_\epsilon, s)_H \leq (p_\epsilon, g_1 - g_\epsilon + s)_H + (p_\epsilon, \epsilon\lambda_\epsilon)_H \quad &-(\lambda_\epsilon, \epsilon\lambda_\epsilon)_H + (Mg_\epsilon, g_1 - g_\epsilon)_H \\ &-(p_\epsilon, \epsilon\delta_\epsilon)_{L^2(\Gamma_2)} - (\delta_\epsilon, \epsilon\delta_\epsilon)_{L^2(\Gamma_2)}.\end{aligned}$$

Taking into account that $(\lambda_\epsilon, \epsilon\lambda_\epsilon)_H \geq 0$ and $(\delta_\epsilon, \epsilon\delta_\epsilon)_{L^2(\Gamma_2)} \geq 0$ we have

$$(\lambda_\epsilon, s)_H \leq (p_\epsilon, g_1 - g_\epsilon + s)_H + (p_\epsilon, \epsilon\lambda_\epsilon)_H + (Mg_\epsilon, g_1 - g_\epsilon)_H - (p_\epsilon, \epsilon\delta_\epsilon)_{L^2(\Gamma_2)}$$

then $\forall s \in B^2(\Omega)$, $\forall \epsilon \in (0, \epsilon_0)$ we get

$$\begin{aligned}|(\lambda_\epsilon, s)_H| &\leq \|p_\epsilon\|_H \|g_1 - g_\epsilon + s\|_H + \sqrt{\epsilon}\|p_\epsilon\|_H \|\sqrt{\epsilon}\lambda_\epsilon\|_H + M\|g_\epsilon\|_H \|g_1 - g_\epsilon\|_H \\ &\quad +\sqrt{\epsilon}\|p_\epsilon\|_{L^2(\Gamma_2)} \|\sqrt{\epsilon}\delta_\epsilon\|_{L^2(\Gamma_2)} \\ &\leq k_0(\|g_1\|_H + \|g_\epsilon\|_H + 1) + \sqrt{\epsilon_0}k_0^2 + Mk_0(\|g_1\|_H + k_0) + \sqrt{\epsilon_0}c'k_0 \\ &\equiv C'\end{aligned}$$

and therefore $\sup_{s \in B^2(\Omega)} |(\lambda_\epsilon, s)_H| \leq C'$, $\forall \epsilon \in (0, \epsilon_0)$; then $\|\lambda_\epsilon\| \leq C'$ $\forall \epsilon \in (0, \epsilon_0)$, i.e. $\sup_{0<\epsilon<\epsilon_0} \|\lambda_\epsilon\|_H \leq C'$.

iii) Let an arbitrary $\tau \in B^2(\Gamma_2)$, with $B^2(\Gamma_2)$ the unit ball of $L^2(\Gamma_2)$, by adding the inequalities (12) $\forall v \in K$, $\forall g \in U_{ad}$ we have:

$$(p_\epsilon - \lambda_\epsilon, -\Delta(v - U_\epsilon))_H + (p_\epsilon + \delta_\epsilon, \frac{\partial(v - U_\epsilon)}{\partial n})_{L^2(\Gamma_2)} + (Mg_\epsilon + \lambda_\epsilon, g - g_\epsilon)_H \geq 0$$

next, we take $g = g_2 \in U_{ad}$ and $v = U_\tau \in K$ such that $U_\tau$ satisfy

$$-\Delta U_\tau = g_2 + \Delta v_0 \text{ in } \Omega, \quad -\frac{\partial U_\tau}{\partial n} = q + \frac{\partial v_0}{\partial n} + \tau \text{ on } \Gamma_2 \tag{20}$$

therefore the preceding inequality we get

$$(p_\epsilon - \lambda_\epsilon, -\Delta(U_\tau - U_\epsilon))_H + (p_\epsilon + \delta_\epsilon, \frac{\partial(U_\tau - U_\epsilon)}{\partial n})_{L^2(\Gamma_2)} + (Mg_\epsilon + \lambda_\epsilon, g_2 - g_\epsilon)_H \geq 0$$

and from (20) we have

$$(p_\epsilon - \lambda_\epsilon, g_2 - g_\epsilon + \epsilon\lambda_\epsilon)_H + (p_\epsilon + \delta_\epsilon, -\tau - \epsilon\delta_\epsilon)_{L^2(\Gamma_2)} + (Mg_\epsilon + \lambda_\epsilon, g_2 - g_\epsilon)_H \geq 0$$

i. e.

$$\begin{aligned}(\delta_\epsilon,\tau)_{L^2(\Gamma_2)} &\le (p_\epsilon, g_2-g_\epsilon)_H+(p_\epsilon,\epsilon\lambda_\epsilon)_H+(Mg_\epsilon,g_2-g_\epsilon)_H-(p_\epsilon,\tau)_{L^2(\Gamma_2)}\\ &\quad -(p_\epsilon,\epsilon\lambda_\epsilon)_{L^2(\Gamma_2)}-(\delta_\epsilon,\epsilon\delta_\epsilon)-(\lambda_\epsilon,\epsilon\lambda_\epsilon).\end{aligned}$$

Now, we have

$$\begin{aligned}\left|(\delta_\epsilon,\tau)_{L^2(\Gamma_2)}\right| &\le \|p_\epsilon\|_H\|g_2-g_\epsilon\|_H+\epsilon\|p_\epsilon\|_H\|\lambda_\epsilon\|_H+M\|g_\epsilon\|_H\|g_2-g_\epsilon\|_H\\ &\quad +\sqrt{\epsilon}\|p_\epsilon\|_{L^2(\Gamma_2)}\|\sqrt{\epsilon}\delta_\epsilon\|_{L^2(\Gamma_2)}+\|p_\epsilon\|_{L^2(\Gamma_2)}\|\tau\|_{L^2(\Gamma_2)}\\ &\le k_0(\|g_2\|_H+k_0)+\epsilon_0k_0C'+c'+Mk_0(\|g_2\|_H+k_0)+\sqrt{\epsilon_0}c'k_0\\ &\equiv C''.\end{aligned}$$

Then $\sup_{\tau\in B^2(\Gamma_2)}\left|(\delta_\epsilon,\tau)_{L^2(\Gamma_2)}\right|\le C''$ $\forall\epsilon\in(0,\epsilon_0)$, and therefore $\|\delta_\epsilon\|_{L^2(\Gamma_2)}\le C''$ $\forall\epsilon\in(0,\epsilon_0)$ and finally $\sup_{0<\epsilon<\epsilon_0}\|\delta_\epsilon\|_{L^2(\Gamma_2)}\le C''$. □

**Theorem 2.** *$(\overline{U},\overline{g})\in D$ is the optimal solution of (7) if and only if there exists $\overline{\lambda}\in H$, $\overline{\delta}\in L^2(\Gamma_2)$ such that*

$$\forall v\in K,\quad (\overline{p}-\overline{\lambda},-\Delta(v-\overline{U}))_H+(\overline{p}+\overline{\delta},\frac{\partial(v-\overline{U})}{\partial n})_{L^2(\Gamma_2)}\ge 0. \tag{21}$$

$$\forall g\in U_{ad},(M\overline{g}+\overline{\lambda},g-\overline{g})_H\ge 0. \tag{22}$$

*Proof.* From the Corollary 2, we deduce that there exists a subcollection $\{\lambda_\epsilon\}$ and $\overline{\lambda}$ in $H$ such that $\lambda_\epsilon\rightharpoonup\overline{\lambda}$ weakly in $H$ when $\epsilon\to 0^+$and a subcollection $\{\delta_\epsilon\}$ and $\overline{\delta}$ in $L^2(\Gamma_2)$ such that $\delta_\epsilon\rightharpoonup\overline{\delta}$ weakly in $L^2(\Gamma_2)$ when $\epsilon\to 0^+$. Moreover, from the Corollary 1 we have that $\frac{\partial U_\epsilon}{\partial n}\to\frac{\partial\overline{U}}{\partial n}$ in $L^2(\Gamma_2)$ and $p_\epsilon\to\overline{p}$ in $L^2(\Gamma_2)$ when $\epsilon\to 0^+$, therefore $\forall\, v\in K$ we get

$$\begin{aligned}&\lim_{\epsilon\to 0^+}\ (p_\epsilon-\lambda_\epsilon,-\Delta(v-U_\epsilon))_H+(p_\epsilon+\delta_\epsilon,\frac{\partial(v-U_\epsilon)}{\partial n})_{L^2(\Gamma_2)}\\ &= (\overline{p}-\overline{\lambda},-\Delta(v-\overline{U}))_H+(\overline{p}+\overline{\delta},\frac{\partial(v-\overline{U})}{\partial n})_{L^2(\Gamma_2)}\end{aligned}$$

and $\forall\, g\in U_{ad}$ we get

$$\lim_{\epsilon\to 0^+}\ (Mg_\epsilon+\lambda_\epsilon,g-g_\epsilon)_H=(M\overline{g}+\overline{\lambda},g-\overline{g})_H.$$

Conversely, let $(v,g)\in D$ be, by adding (21) and (22) we have

$$(\overline{p}+M\overline{g},g-\overline{g})_H+(\overline{p}+\overline{\delta},\frac{\partial(v-\overline{U})}{\partial n})_{L^2(\Gamma_2)}\ge 0 \tag{23}$$

and for the Green's formula, we get

$$\begin{aligned}(\overline{p},-\Delta(v-\overline{U}))_H &= (-\Delta\overline{p},v-\overline{U})_H-(\overline{p},\frac{\partial(v-\overline{U})}{\partial n})_{L^2(\Gamma_2)}\\ &= (\overline{U}-z_d,v-\overline{U})_H-(\overline{p},\frac{\partial(v-\overline{U})}{\partial n})_{L^2(\Gamma_2)}.\end{aligned}$$

Then the inequality (23) is given by $(\overline{U}-z_d,v-\overline{U})_H+(M\overline{g},g-\overline{g})_H\ge 0$, that is $J'(\overline{U},\overline{g})(v-\overline{U},g-\overline{g})\ge 0$ and therefore $(\overline{U},\overline{g})$ is the optimal solution of (7). □

## 3. PENALIZATION OF PROBLEM (8) AND STRONG CONVERGENCE WHEN $\epsilon \to 0^+$

**Proposition 3.** *a) $J_{\alpha\epsilon}$ is a coercive and strictly convex functional. There exists a unique solution $(U_{\alpha\epsilon}, g_{\alpha\epsilon})$ of the penalized problem (10).*

*b) $J_{\alpha\epsilon}$ is Gateaux-differentiable and the optimality condition is given by $\forall v \in K_1, \forall g \in U_{ad}$*

$$\begin{aligned}(24)\qquad &(U_{\alpha\epsilon} - z_d, v - U_{\alpha\epsilon})_H + M(g_{\alpha\epsilon}, g - g_{\alpha\epsilon})_H \\ &+\frac{1}{\epsilon}(\Delta U_{\alpha\epsilon} + g_{\alpha\epsilon} + \Delta v_0, \Delta(v - U_{\alpha\epsilon}))_H \\ &+\frac{1}{\epsilon}(\Delta U_{\alpha\epsilon} + g_{\alpha\epsilon} + \Delta v_0, g - g_{\alpha\epsilon})_H \\ &+\frac{1}{\epsilon}(\frac{\partial U_{\alpha\epsilon}}{\partial n} + q + \frac{\partial v_0}{\partial n}, \frac{\partial (v - U_{\alpha\epsilon})}{\partial n})_{L^2(\Gamma_2)} \\ &+\frac{1}{\epsilon}(\frac{\partial U_{\alpha\epsilon}}{\partial n} + \alpha U_{\alpha\epsilon} + \frac{\partial v_0}{\partial n}, \frac{\partial (v - U_{\alpha\epsilon})}{\partial n} + \alpha(v - U_{\alpha\epsilon}))_{L^2(\Gamma_{1,1})} \geq 0.\end{aligned}$$

*Proof.* a) It results in similar way that Proposition 1 and taking into account that for $t \in [0,1]$, $(U_{1\alpha}, g_{1\alpha})$, $(U_{2\alpha}, g_{2\alpha}) \in K_1 \times U_{ad}$ we have

$$\begin{aligned}&(1-t)\int_{\Gamma_{1,1}}(\frac{\partial U_{1\alpha}}{\partial n} + \alpha U_{1\alpha} + \frac{\partial v_0}{\partial n})^2 d\gamma + t\int_{\Gamma_{1,1}}(\frac{\partial U_{2\alpha}}{\partial n} + \alpha U_{2\alpha} + \frac{\partial v_0}{\partial n})^2 d\gamma \\ &-\int_{\Gamma_{1,1}}(\frac{\partial((1-t)U_{1\alpha} + tU_{2\alpha})}{\partial n} + \alpha((1-t)U_{1\alpha} + tU_{2\alpha}) + \frac{\partial v_0}{\partial n})^2 d\gamma \geq 0.\end{aligned}$$

b) Let $v \in K_1$, $g \in U_{ad}$ and $t \in (0,1]$ be, then we get

$$\begin{aligned}&J_{\alpha\epsilon}(U_{\alpha\epsilon} + t(v - U_{\alpha\epsilon}), g_{\alpha\epsilon} + t(g - g_{\alpha\epsilon})) - J_{\alpha\epsilon}(U_{\alpha\epsilon}, g_{\alpha\epsilon}) \\ =\ & t\int_\Omega (U_{\alpha\epsilon} - z_d)(v - U_{\alpha\epsilon})dx + \frac{t^2}{2}\int_\Omega (v - U_{\alpha\epsilon})^2 dx \\ &+Mt\int_\Omega g_{\alpha\epsilon}(g - g_{\alpha\epsilon})dx + \frac{Mt^2}{2}\int_\Omega (g - g_{\alpha\epsilon})^2 dx \\ &+\frac{t}{\epsilon}\int_\Omega (\Delta U_{\alpha\epsilon} + g_{\alpha\epsilon} + \Delta v_0)(\Delta(v - U_{\alpha\epsilon}) + (g - g_{\alpha\epsilon}))dx \\ &+\frac{t^2}{2\epsilon}\int_\Omega (\Delta(v - U_{\alpha\epsilon}) + (g - g_{\alpha\epsilon}))^2 dx \\ &+\frac{t}{\epsilon}\int_{\Gamma_2}(\frac{\partial U_{\alpha\epsilon}}{\partial n} + q + \frac{\partial v_0}{\partial n})(\frac{\partial (v - U_{\alpha\epsilon})}{\partial n})d\gamma + \frac{t^2}{2\epsilon}\int_{\Gamma_2}(\frac{\partial (v - U_{\alpha\epsilon})}{\partial n})d\gamma \\ &+\frac{t}{\epsilon}\int_{\Gamma_{1,1}}(\frac{\partial U_{\alpha\epsilon}}{\partial n} + \alpha U_{\alpha\epsilon} + \frac{\partial v_0}{\partial n})(\frac{\partial (v - U_{\alpha\epsilon})}{\partial n} + \alpha(v - U_{\alpha\epsilon}))d\gamma \\ &+\frac{t^2}{2\epsilon}\int_{\Gamma_{1,1}}(\frac{\partial (v - U_{\alpha\epsilon})}{\partial n} + \alpha(v - U_{\alpha\epsilon}))d\gamma.\end{aligned}$$

Now, dividing by $t$, and passing to the limit $t \to 0^+$, $\forall v \in K_1$, $\forall g \in U_{ad}$ we have

$$\begin{aligned} J'_{\alpha\epsilon}(U_{\alpha\epsilon}, g_{\alpha\epsilon})(v - U_{\alpha\epsilon}, g - g_{\alpha\epsilon}) &= (U_{\alpha\epsilon} - z_d, v - U_{\alpha\epsilon})_H + M(g_{\alpha\epsilon}, g - g_{\alpha\epsilon})_H \\ &+ \frac{1}{\epsilon}(\Delta U_{\alpha\epsilon} + g_{\alpha\epsilon} + \Delta v_0, \Delta(v - U_{\alpha\epsilon}) + g - g_{\alpha\epsilon})_H \\ &+ \frac{1}{\epsilon}(\frac{\partial U_{\alpha\epsilon}}{\partial n} + q + \frac{\partial v_0}{\partial n}, \frac{\partial (v - U_{\alpha\epsilon})}{\partial n})_{L^2(\Gamma_2)} \\ &+ \frac{1}{\epsilon}(\frac{\partial U_{\alpha\epsilon}}{\partial n} + \alpha U_{\alpha\epsilon} + \frac{\partial v_0}{\partial n}, \frac{\partial (v - U_{\alpha\epsilon})}{\partial n})_{L^2(\Gamma_{1,1})} \\ &+ \frac{1}{\epsilon}(\frac{\partial U_{\alpha\epsilon}}{\partial n} + \alpha U_{\alpha\epsilon} + \frac{\partial v_0}{\partial n}, \alpha(v - U_{\alpha\epsilon}))_{L^2(\Gamma_{1,1})} \geq 0 \end{aligned}$$

that is (24). □

Let us call, for each fixed $\epsilon > 0$, $\alpha > 0$:

$$\lambda_{\alpha\epsilon} = \frac{\Delta U_{\alpha\epsilon} + g_{\alpha\epsilon} + \Delta v_0}{\epsilon}, \quad \delta_{\alpha\epsilon} = \frac{\frac{\partial U_{\alpha\epsilon}}{\partial n} + q + \frac{\partial v_0}{\partial n}}{\epsilon}, \quad \gamma_{\alpha\epsilon} = \frac{\frac{\partial U_{\alpha\epsilon}}{\partial n} + \alpha U_{\alpha\epsilon} + \frac{\partial v_0}{\partial n}}{\epsilon}$$

and we define $p_{\alpha\epsilon} \in V_{01}$ such that

$$\begin{cases} -\Delta p_{\alpha\epsilon} = U_{\alpha\epsilon} - z_d \text{ in } \Omega \\ -\frac{\partial p_{\alpha\epsilon}}{\partial \eta} = \alpha p_{\alpha\epsilon} \text{ on } \Gamma_{1,1}; \quad p_{\alpha\epsilon} = 0 \text{ on } \Gamma_{1,0} \\ \frac{\partial p_{\alpha\epsilon}}{\partial \eta} = 0 \text{ on } \Gamma_2; \quad \frac{\partial p_{\alpha\epsilon}}{\partial \eta} = 0 \text{ on } \Gamma_3. \end{cases}$$

Now, we can prove the following property.

**Proposition 4.** *The optimality condition (24) can be decoupled in*

(25)
$$\begin{aligned} &\forall v \in K_1 \quad , (p_{\alpha\epsilon} - \lambda_{\alpha\epsilon}, -\Delta(v - U_{\alpha\epsilon}))_H + (p_{\alpha\epsilon} + \delta_{\alpha\epsilon}, \frac{\partial (v - U_{\alpha\epsilon})}{\partial n})_{L^2(\Gamma_2)} \\ &\qquad + (p_{\alpha\epsilon} + \gamma_{\alpha\epsilon}, \frac{\partial (v - U_{\alpha\epsilon})}{\partial n} + \alpha(v - U_{\alpha\epsilon}))_{L^2(\Gamma_{1,1})} \geq 0. \\ &\forall g \in U_{ad} \quad , \quad (M g_{\alpha\epsilon} + \lambda_{\alpha\epsilon}, g - g_{\alpha\epsilon})_H \geq 0 \end{aligned}$$

*Proof.* From the Green's formula, $\forall v \in K_1, \forall g \in U_{ad}$ we have that

$$\begin{aligned} (U_{\alpha\epsilon} - z_d, v - U_{\alpha\epsilon})_H &= (p_{\alpha\epsilon}, -\Delta(v - U_{\alpha\epsilon}))_H + (p_{\alpha\epsilon}, \frac{\partial (v - U_{\alpha\epsilon})}{\partial \eta})_{L^2(\Gamma_2)} \\ &+ (p_{\alpha\epsilon}, \frac{\partial (v - U_{\alpha\epsilon})}{\partial n} + \alpha(v - U_{\alpha\epsilon}))_{L^2(\Gamma_{1,1})} \end{aligned}$$

and the optimality condition (24) is now decoupled and it is given by (25). □

**Theorem 3.** *Let* $(\overline{U_\alpha}, \overline{g_\alpha})$ *be the solution of the problem* (8) *for fixed* $\alpha > 0$, *then we have that* $g_{\alpha\epsilon} \longrightarrow \overline{g_\alpha}$ *in* $H$ *and* $U_{\alpha\epsilon} \longrightarrow \overline{U_\alpha}$ *in* $H^2(\Omega) \cap V_{01}$, *when* $\epsilon \to 0^+$.

*Proof.* Let us give $\epsilon > 0$, and fixed $\alpha > 0$. In similar way to the Theorem 1 we have

(26)
$$0 \leq J_{\alpha\epsilon}(U_{\alpha\epsilon}, g_{\alpha\epsilon}) \leq J_{\alpha\epsilon}(\overline{U_\alpha}, \overline{g_\alpha}) = J(\overline{U_\alpha}, \overline{g_\alpha}) < \infty$$

next, there exists a constant $k_\alpha > 0$ such that

$$\sup_{\epsilon > 0} \|U_{\alpha\epsilon}\|_H \leq k_\alpha \text{ and } \sup_{\epsilon > 0} \|g_{\alpha\epsilon}\|_H \leq k_\alpha.$$

Therefore, there exists $U_{\alpha 0},\, g_{\alpha 0}\ \in H$ and subcollection $\{U_{\alpha\epsilon}\}$ and $\{g_{\alpha\epsilon}\}$ such that $U_{\alpha\epsilon}\rightharpoonup U_{\alpha 0}$ weakly in $H$ and $g_{\alpha\epsilon}\rightharpoonup g_{\alpha 0}$ weakly in $H$. From (26) and the weak lower semicontinuity of $u \to \int u^2$ we get

$$\int_\Omega (\Delta U_{\alpha 0} + g_{\alpha 0} + \Delta v_0)^2 dx = 0; \quad \int_{\Gamma_2} (\frac{\partial U_{\alpha 0}}{\partial n} + q + \frac{\partial v_0}{\partial n})^2 d\gamma = 0$$

and

$$\int_{\Gamma_{1,1}} (\frac{\partial U_{\alpha 0}}{\partial n} + \alpha U_{\alpha 0} + \frac{\partial v_0}{\partial n})^2 d\gamma = 0.$$

Therefore $-\Delta U_{\alpha 0} = g_{\alpha 0} + \Delta v_0$ in $\Omega$, $-\frac{\partial U_{\alpha 0}}{\partial n} = q + \frac{\partial v_0}{\partial n}$ on $\Gamma_2$ and $-\frac{\partial U_{\alpha 0}}{\partial n} = \alpha U_{\alpha 0} + \frac{\partial v_0}{\partial n}$ on $\Gamma_{1,1}$; moreover as $K_1$ and $U_{ad}$ are convex and closed sets in the strong topology then they are closed sets in the weak topology, next we have that $U_{\alpha 0} \in K_1$, $g_{\alpha 0} \in U_{ad}$ and $U_{\alpha 0} = T_\alpha(g_{\alpha 0})$. From (26) we obtain that

(27) $$\liminf_{\epsilon\to 0^+} J_{\alpha\epsilon}(U_{\alpha\epsilon}, g_{\alpha\epsilon}) \le \liminf_{\epsilon\to 0^+} J_{\alpha\epsilon}(\overline{U_\alpha}, \overline{g_\alpha}) = J(\overline{U_\alpha}, \overline{g_\alpha})$$

and taking into account that

(28) $$J_{\alpha\epsilon}(U_\alpha, g_\alpha) \ge J(U_\alpha, g_\alpha) \quad \forall (U_\alpha, g_\alpha) \in K_1 \times U_{ad}$$

we have

(29) $$J(\overline{U_\alpha}, \overline{g_\alpha}) \ge \liminf_{\epsilon\to 0^+} J_{\alpha\epsilon}(U_{\alpha\epsilon}, g_{\alpha\epsilon}) \ge \liminf_{\epsilon\to 0^+} \ J(U_{\alpha\epsilon}, g_{\alpha\epsilon}) \ge J(U_{\alpha 0}, g_{\alpha 0}).$$

As $(\overline{U_\alpha}, \overline{g_\alpha})$ is the solution of the problem (8) we have $J(\overline{U_\alpha}, \overline{g_\alpha}) \le J(U_{\alpha 0}, g_{\alpha 0})$ and by the uniqueness of the solution we get $\overline{U_\alpha} = U_{\alpha 0}$, $\overline{g_\alpha} = g_{\alpha 0}$; therefore we prove that $U_{\alpha\epsilon}\rightharpoonup\overline{U_\alpha}$ weakly in $H$ and $g_{\alpha\epsilon}\rightharpoonup\overline{g_\alpha}$ weakly in $H$ when $\epsilon \to 0^+$. Moreover, we prove that

(30) $$\lim_{\epsilon\to 0^+} J_{\alpha\epsilon}(U_{\alpha\epsilon}, g_{\alpha\epsilon}) = J(\overline{U_\alpha}, \overline{g_\alpha})$$

(31) $$\lim_{\epsilon\to 0^+} J(U_{\alpha\epsilon}, g_{\alpha\epsilon}) = J(\overline{U_\alpha}, \overline{g_\alpha}).$$

From (30) and (31) we obtain that if $\epsilon \to 0^+$ we have

$$\frac{1}{2\epsilon}\int_\Omega (\Delta U_{\alpha\epsilon} + g_{\alpha\epsilon} + \Delta v_0)^2 dx \longrightarrow 0 \quad , \quad \frac{1}{2\epsilon}\int_{\Gamma_2} (\frac{\partial U_{\alpha\epsilon}}{\partial n} + q + \frac{\partial v_0}{\partial n})^2 d\gamma \longrightarrow 0$$

$$\frac{1}{2\epsilon}\int_{\Gamma_{1,1}} (\frac{\partial U_{\alpha\epsilon}}{\partial n} + \alpha U_{\alpha\epsilon} + \frac{\partial v_0}{\partial n})^2 d\gamma \longrightarrow 0$$

then

$$\lim_{\epsilon\to 0} \|U_{\alpha\epsilon} - z_d\|_H^2 + M\|g_{\alpha\epsilon}\|_H^2 = \|\overline{U_\alpha} - z_d\|_H^2 + M\|\overline{g_\alpha}\|_H^2.$$

Next of the weak convergence and the convergence in norm in $H \times H$ we have the strong convergence, that is

$$U_{\alpha\epsilon} \longrightarrow \overline{U_\alpha} \text{ in } H \text{ and } g_{\alpha\epsilon} \longrightarrow \overline{g_\alpha} \text{ in } H \text{ when } \epsilon \to 0^+.$$

On the other hand, $\lim_{\epsilon\to 0} \Delta U_{\alpha\epsilon} + g_{\alpha\epsilon} + \Delta v_0 = 0$ i.e. $\lim_{\epsilon\to 0} -\Delta U_{\alpha\epsilon} = \overline{g_\alpha} + \Delta v_0 = -\Delta\overline{U_\alpha}$, and taking into account that $-\Delta : H^2(\Omega)\cap V_{01} \to H$ is a isomorphism, the thesis holds. □

Now we define $\overline{p_\alpha}$ as the solution of the following elliptic problem

$$\begin{cases} -\Delta p_\alpha = \overline{U_\alpha} - z_d \text{ in } \Omega \\ p_\alpha = 0 \text{ on } \Gamma_{1,0}; \quad -\frac{\partial p}{\partial \eta} = \alpha\, p_\alpha \text{ on } \Gamma_{1,1} \\ \frac{\partial p}{\partial \eta} = 0 \text{ on } \Gamma_2; \quad \frac{\partial p}{\partial \eta} = 0 \text{ on } \Gamma_3 \end{cases}$$

and we prove the following corollaries

**Corollary 3.** *For fixed $\alpha > 0$, when $\epsilon \to 0^+$, we get the following limits:*

*a) $p_{\alpha\epsilon} \to \overline{p_\alpha}$ in $H^2(\Omega) \cap V_{01}$.*

*b) $\frac{\partial U_{\alpha\epsilon}}{\partial n} \to \frac{\partial \overline{U_\alpha}}{\partial n}$ in $L^2(\Gamma_2)$ and there exists a constant $d'_\alpha > 0$ such that $\|\frac{\partial U_{\alpha\epsilon}}{\partial n}\|_{L^2(\Gamma_2)} \leq d'_\alpha$.*

*c) $p_{\alpha\epsilon} \to \overline{p_\alpha}$ in $L^2(\Gamma_2)$ and there exists a constant $d''_\alpha > 0$ such that $\|p_{\alpha\epsilon}\|_{L^2(\Gamma_2)} \leq d''_\alpha$.*

*d)$U_{\alpha\epsilon} \to \overline{U_\alpha}$ in $L^2(\Gamma_{1,1})$.*

*e)$\frac{\partial U_{\alpha\epsilon}}{\partial n} \to \frac{\partial \overline{U_\alpha}}{\partial n}$ in $L^2(\Gamma_{1,1})$.*

*f) $p_{\alpha\epsilon} \to \overline{p_\alpha}$ in $L^2(\Gamma_{1,1})$; therefore there exists a constant $d'''_\alpha > 0$ such that $\|p_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})} \leq d'''_\alpha$.*

*Proof.* (a)-(c) This follows as in Corollary 1.

d) By the Trace Theorem we have that there exists a constant $D_\alpha > 0$ such that

$$\|\frac{\partial U_{\alpha\epsilon}}{\partial n} - \frac{\partial \overline{U_\alpha}}{\partial n}\|_{L^2(\Gamma_{1,1})} \leq D_\alpha \|U_{\alpha\epsilon} - \overline{U_\alpha}\|_{H^2(\Omega)}.$$

(e) and (f) It follows as (b) and (c) respectively. □

**Corollary 4.** *i) For fixed $\alpha > 0$, there exists $\epsilon_{\alpha 0} > 0, k_{\alpha 0} > 0$ such that:*

$$\sup_{0<\epsilon<\epsilon_{\alpha 0}} \|g_{\alpha\epsilon}\|_H \leq k_{\alpha 0}; \quad \sup_{0<\epsilon<\epsilon_{\alpha 0}} \|U_{\alpha\epsilon}\|_{H^2(\Omega)} \leq k_{\alpha 0}; \quad \sup_{0<\epsilon<\epsilon_{\alpha 0}} \|p_{\alpha\epsilon}\|_{H^2(\Omega)} \leq k_{\alpha 0}$$

$$\sup_{0<\epsilon<\epsilon_{\alpha 0}} \|\sqrt{\epsilon}\lambda_{\alpha\epsilon}\|_H \leq k_{\alpha 0}; \sup_{0<\epsilon<\epsilon_{\alpha 0}} \|\sqrt{\epsilon}\delta_{\alpha\epsilon}\|_{L^2(\Gamma_2)} \leq k_{\alpha 0}; \sup_{0<\epsilon<\epsilon_{\alpha 0}} \|\sqrt{\epsilon}\gamma_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})} \leq k_{\alpha 0}$$

*ii) There exists a constant $D'_\alpha > 0$, such that $\sup_{0<\epsilon<\epsilon_{\alpha 0}} \|\lambda_{\alpha\epsilon}\|_H \leq D'_\alpha$.*

*iii) There exists a constant $D''_\alpha > 0$ such that $\sup_{0<\epsilon<\epsilon_{\alpha 0}} \|\delta_{\alpha\epsilon}\|_{L^2(\Gamma_2)} \leq D''_\alpha$.*

*iv) There exists a constant $D'''_\alpha > 0$ such that $\sup_{0<\epsilon<\epsilon_{\alpha 0}} \|\gamma_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})} \leq D'''_\alpha$.*

*Proof.* i) It results in similar way that Corollary 2 and taking into account that if $\epsilon \to 0^+$ we have

$$\begin{aligned} \|\sqrt{\epsilon}\gamma_{\alpha\epsilon}\|^2_{L^2(\Gamma_{1,1})} &= \int_{L^2(\Gamma_{1,1})} \epsilon\gamma^2_{\alpha\epsilon} = \frac{\epsilon}{\epsilon^2} \int_{L^2(\Gamma_{1,1})} (\frac{\partial U_{\alpha\epsilon}}{\partial n} + \alpha U_{\alpha\epsilon} + \frac{\partial v_0}{\partial n})^2 dx \\ &= \frac{1}{\epsilon} \int_{L^2(\Gamma_{1,1})} (\frac{\partial U_{\alpha\epsilon}}{\partial n} + \alpha U_{\alpha\epsilon} + \frac{\partial v_0}{\partial n})^2 dx \longrightarrow 0 \text{ in } L^2(\Gamma_{1,1}). \end{aligned}$$

ii) Let an arbitrary $s \in B^2(\Omega)$, with $B^2(\Omega)$ the unit ball in $H$, by adding the inequalities (25), $\forall v \in K_1, \forall g \in U_{ad}$ we have

$$(32) \qquad (p_{\alpha\epsilon} - \lambda_{\alpha\epsilon}, -\Delta(v - U_{\alpha\epsilon}))_H + (p_{\alpha\epsilon} + \delta_{\alpha\epsilon}, \tfrac{\partial(v-U_{\alpha\epsilon})}{\partial n})_{L^2(\Gamma_2)}$$
$$+(p_{\alpha\epsilon} + \gamma_{\alpha\epsilon}, \tfrac{\partial(v-U_{\alpha\epsilon})}{\partial n} + \alpha(v - U_{\alpha\epsilon}))_{L^2(\Gamma_{1,1})} + (Mg_{\alpha\epsilon} + \lambda_{\alpha\epsilon}, g - g_{\alpha\epsilon})_H \geq 0.$$

If we take $g = g_{\alpha 1} \in U_{ad}$ and $v = U_{\alpha s} \in K_1$ such that $T(g_{\alpha 1} + s) = U_{\alpha s}$, we have

$$\begin{aligned}
&(p_{\alpha\epsilon}, g_{\alpha 1} + s + \Delta v_0 + g_{\alpha\epsilon} - g_{\alpha\epsilon} + \Delta U_{\alpha\epsilon})_H \\
&-(\lambda_{\alpha\epsilon}, g_{\alpha 1} + s + \Delta v_0 + g_{\alpha\epsilon} - g_{\alpha\epsilon} + \Delta U_{\alpha\epsilon})_H \\
&+(p_{\alpha\epsilon} + \delta_{\alpha\epsilon}, -q - \frac{\partial v_0}{\partial n} - \frac{\partial U_{\alpha\epsilon}}{\partial n})_{L^2(\Gamma_2)} \\
&+(p_{\alpha\epsilon} + \gamma_{\alpha\epsilon}, -\frac{\partial v_0}{\partial n} - \frac{\partial U_{\alpha\epsilon}}{\partial n} - \alpha U_{\alpha\epsilon})_{L^2(\Gamma_{1,1})} \\
&+(M g_{\alpha\epsilon} + \lambda_{\alpha\epsilon}, g_{\alpha 1} - g_{\alpha\epsilon})_H \geq 0.
\end{aligned}$$

i.e

$$\begin{aligned}
(\lambda_{\alpha\epsilon}, s)_H \quad \leq \quad & (p_{\alpha\epsilon}, g_{\alpha 1} - g_{\alpha\epsilon} + s)_H + (p_{\alpha\epsilon}, \epsilon\lambda_{\alpha\epsilon})_H + (M g_{\alpha\epsilon}, g_{\alpha 1} - g_{\alpha\epsilon})_H \\
& -(p_{\alpha\epsilon}, \epsilon\delta_{\alpha\epsilon})_{L^2(\Gamma_2)} - (p_{\alpha\epsilon}, \epsilon\gamma_{\alpha\epsilon})_{L^2(\Gamma_{1,1})}.
\end{aligned}$$

Then $\forall s \in B^2(\Omega)$, $\forall \epsilon \in (0, \epsilon_{\alpha 0}]$ we have

$$\begin{aligned}
|(\lambda_{\alpha\epsilon}, s)_H| \quad \leq \quad & \|p_{\alpha\epsilon}\|_H \|g_{\alpha 1} - g_{\alpha\epsilon} + s\|_H + \sqrt{\epsilon}\|p_{\alpha\epsilon}\|_H \|\sqrt{\epsilon}\lambda_{\alpha\epsilon}\|_H \\
& + M\|g_{\alpha\epsilon}\|_H \|g_{\alpha 1} - g_{\alpha\epsilon}\|_H + \sqrt{\epsilon}\|p_{\alpha\epsilon}\|_{L^2(\Gamma_2)} \|\sqrt{\epsilon}\delta_{\alpha\epsilon}\|_{L^2(\Gamma_2)} \\
& + \sqrt{\epsilon}\|p_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})} \|\sqrt{\epsilon}\gamma_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})} \\
\leq \quad & k_{\alpha 0}(\|g_{\alpha 1}\|_H + k_{\alpha 0}) + \sqrt{\epsilon_{\alpha 0}}\, k_{\alpha 0}^2 + M\, k_{\alpha 0}(\|g_{\alpha 1}\|_H + k_{\alpha 0}) \\
& + \sqrt{\epsilon_{\alpha 0}} d''_\alpha k_{\alpha 0} + \sqrt{\epsilon_{\alpha 0}} d'''_\alpha k_{\alpha 0} \equiv D'_\alpha
\end{aligned}$$

and therefore $\sup_{0<\epsilon<\epsilon_{\alpha 0}} \|\lambda_{\alpha\epsilon}\|_H \leq D'_\alpha$.

iii) Let an arbitrary $\tau \in B^2(\Gamma_2)$, with $B^2(\Gamma_2)$ the unit ball of $L^2(\Gamma_2)$. If we make $g = g_{\alpha 2} \in U_{ad}$ and $v = U_{\alpha\tau} \in K_1$ in (32) such that $U_{\alpha\tau}$ satisfies

$$(33) \qquad \begin{cases} -\Delta U_{\alpha\tau} = g_{\alpha 2} + \Delta v_0 \text{ in } \Omega \\ -\frac{\partial U_{\alpha\tau}}{\partial n} = q + \frac{\partial v_0}{\partial n} + \tau \text{ on } \Gamma_2, \quad -\frac{\partial U_{\alpha\tau}}{\partial n} = \alpha U_{\alpha\tau} + \frac{\partial v_0}{\partial n} \text{ on } \Gamma_{1,1} \end{cases}$$

we obtain

$$\begin{aligned}
&(p_{\alpha\epsilon} - \lambda_{\alpha\epsilon}, g_{\alpha 2} - g_{\alpha\epsilon} + \epsilon\lambda_{\alpha\epsilon})_H + (p_{\alpha\epsilon} + \delta_{\alpha\epsilon}, -\tau - \epsilon\delta_{\alpha\epsilon})_{L^2(\Gamma_2)} \\
&+(p_{\alpha\epsilon} + \gamma_{\alpha\epsilon}, -\epsilon\gamma_{\alpha\epsilon})_{L^2(\Gamma_{1,1})} + (M g_\epsilon + \lambda_\epsilon, g_{\alpha 2} - g_\epsilon)_H \geq 0
\end{aligned}$$

or equivalently

$$\begin{aligned}
(\delta_{\alpha\epsilon}, \tau)_{L^2(\Gamma_2)} \quad \leq \quad & (p_{\alpha\epsilon}, g_{\alpha 2} - g_{\alpha\epsilon})_H + (p_{\alpha\epsilon}, \epsilon\lambda_{\alpha\epsilon})_H + (M g_{\alpha\epsilon}, g_{\alpha 2} - g_{\alpha\epsilon})_H \\
& -(p_{\alpha\epsilon}, \tau)_{L^2(\Gamma_2)} - (p_{\alpha\epsilon}, \epsilon\delta_{\alpha\epsilon})_{L^2(\Gamma_2)} - (p_{\alpha\epsilon}, \epsilon\gamma_{\alpha\epsilon})_{L^2(\Gamma_{1,1})}.
\end{aligned}$$

Next

$$\begin{aligned}
\left|(\delta_{\alpha\epsilon}, \tau)_{L^2(\Gamma_2)}\right| \quad \leq \quad & \|p_{\alpha\epsilon}\|_H \|g_{\alpha 2} - g_{\alpha\epsilon}\|_H + \epsilon\|p_{\alpha\epsilon}\|_H \|\lambda_{\alpha\epsilon}\|_H \\
& + M\|g_{\alpha\epsilon}\|_H \|g_{\alpha 2} - g_{\alpha\epsilon}\|_H \\
& + \sqrt{\epsilon}\|p_{\alpha\epsilon}\|_{L^2(\Gamma_2)} \|\sqrt{\epsilon}\delta_{\alpha\epsilon}\|_{L^2(\Gamma_2)} + \|p_{\alpha\epsilon}\|_{L^2(\Gamma_2)} \|\tau\|_{L^2(\Gamma_2)} \\
& + \sqrt{\epsilon}\|p_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})} \|\sqrt{\epsilon}\gamma_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})} \\
\leq \quad & k_{\alpha 0}\epsilon_{\alpha 0} D'_\alpha + \sqrt{\epsilon_{\alpha 0}} d'''_\alpha k_{\alpha 0} + \sqrt{\epsilon_{\alpha 0}} d''_\alpha k_{\alpha 0} + d''_\alpha \\
& + M k_{\alpha 0}(\|g_{\alpha 2}\|_H + k_{\alpha 0}) \equiv D''_\alpha
\end{aligned}$$

and therefore $\sup_{0<\epsilon<\epsilon_{\alpha 0}} \|\delta_{\alpha\epsilon}\|_{L^2(\Gamma_2)} \leq D''_\alpha$.

iv) Let an arbitrary $\sigma \in B^2(\Gamma_{1,1})$, with $B^2(\Gamma_{1,1})$ the unit ball of $L^2(\Gamma_{1,1})$. If we make $g = g_{\alpha 3} \in U_{ad}$ and $v = U_{\alpha\sigma} \in K_1$ in (32) such that $U_{\alpha\sigma}$ satisfies

$$\text{(34)} \qquad \begin{cases} -\Delta U_{\alpha\sigma} = g_{\alpha 3} + \Delta v_0 \text{ in } \Omega \\ -\frac{\partial U_{\alpha\sigma}}{\partial n} = q + \frac{\partial v_0}{\partial n} \text{ on } \Gamma_2 \\ -\frac{\partial U_{\alpha\sigma}}{\partial n} = \alpha U_{\alpha\sigma} + \frac{\partial v_0}{\partial n} + \sigma \text{ on } \Gamma_{1,1} \end{cases}$$

we obtain

$$\begin{aligned} &(p_{\alpha\epsilon} - \lambda_{\alpha\epsilon}, g_{\alpha 3} - g_{\alpha\epsilon} + \epsilon\lambda_{\alpha\epsilon})_H + (p_{\alpha\epsilon} + \delta_{\alpha\epsilon}, -\epsilon\delta_{\alpha\epsilon})_{L^2(\Gamma_2)} \\ &+(p_{\alpha\epsilon} + \gamma_{\alpha\epsilon}, -\epsilon\gamma_{\alpha\epsilon} - \sigma)_{L^2(\Gamma_{1,1})} + (Mg_\epsilon + \lambda_\epsilon, g_{\alpha 3} - g_\epsilon)_H \geq 0 \end{aligned}$$

or equivalently

$$\begin{aligned} (\gamma_{\alpha\epsilon}, \sigma)_{L^2(\Gamma_{1,1})} \leq\ & (p_{\alpha\epsilon}, g_{\alpha 3} - g_{\alpha\epsilon})_H + (p_{\alpha\epsilon}, \epsilon\lambda_{\alpha\epsilon})_H + (Mg_{\alpha\epsilon}, g_{\alpha 3} - g_{\alpha\epsilon})_H \\ & -(p_{\alpha\epsilon}, \sigma)_{L^2(\Gamma_{1,1})} - (p_{\alpha\epsilon}, \epsilon\gamma_{\alpha\epsilon})_{L^2(\Gamma_{1,1})} - (p_{\alpha\epsilon}, \epsilon\delta_{\alpha\epsilon})_{L^2(\Gamma_2)}. \end{aligned}$$

Next

$$\begin{aligned} \left|(\gamma_{\alpha\epsilon}, \sigma)_{L^2(\Gamma_{1,1})}\right| \leq\ & \|p_{\alpha\epsilon}\|_H \|g_{\alpha 3} - g_{\alpha\epsilon}\|_H + M\|g_{\alpha\epsilon}\|_H \|g_{\alpha 3} - g_{\alpha\epsilon}\|_H \\ & +\epsilon\|p_{\alpha\epsilon}\|_{L^2(\Gamma_2)}\|\delta_{\alpha\epsilon}\|_{L^2(\Gamma_2)} + \epsilon\|p_{\alpha\epsilon}\|_H\|\lambda_{\alpha\epsilon}\|_H \\ & +\|p_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})}\|\sigma\|_{L^2(\Gamma_{1,1})} + \|p_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})}\|\sqrt{\epsilon}\gamma_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})} \\ \leq\ & k_{\alpha 0}(\|g_{\alpha 3}\|_H + k_{\alpha 0}) + Mk_{\alpha 0}(\|g_{\alpha 3}\|_H + k_{\alpha 0}) + \epsilon_{\alpha 0} d''_\alpha D''_\alpha \\ & +\epsilon_{\alpha 0} k_{\alpha 0}\, D'_\alpha + d'''_\alpha + d'''_\alpha k_{\alpha 0} \equiv D'''_\alpha. \end{aligned}$$

and therefore

$$\sup_{0<\epsilon<\epsilon_{\alpha 0}} \|\gamma_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})} \leq D'''_\alpha.$$

□

**Theorem 4.** $(\overline{U_\alpha}, \overline{g_\alpha}) \in D_\alpha$ *is the optimal solution of* (8) *if and only if there exists* $\overline{\lambda_\alpha} \in H$, $\overline{\delta_\alpha} \in L^2(\Gamma_2)$ *and* $\overline{\gamma_\alpha} \in L^2(\Gamma_{1,1})$ *such that*

$$\text{(35)} \quad \forall v \in K_1 \quad , \quad (\overline{p_\alpha} - \overline{\lambda_\alpha}, -\Delta(v - \overline{U_\alpha}))_H + (\overline{p_\alpha} + \overline{\delta_\alpha}, \frac{\partial(v - \overline{U_\alpha})}{\partial n})_{L^2(\Gamma_2)}$$
$$+(\overline{p_\alpha} + \overline{\gamma_\alpha}, \frac{\partial(v - \overline{U_\alpha})}{\partial n} + \alpha(v - \overline{U_\alpha}))_{L^2(\Gamma_{1,1})} \geq 0$$

$$\text{(36)} \qquad \forall g \in U_{ad} \quad , \quad (M\overline{g_\alpha} + \overline{\lambda_\alpha}, g - \overline{g_\alpha})_H \geq 0.$$

*Proof.* From the Corollary 4, we deduce that there exist a subcollection $\{\lambda_{\alpha\epsilon}\}$ and $\overline{\lambda_\alpha}$ in $H$ such that $\lambda_{\alpha\epsilon} \rightharpoonup \overline{\lambda_\alpha}$ weakly in $H$, a subcollection $\{\delta_{\alpha\epsilon}\}$ and $\overline{\delta_\alpha}$ in $L^2(\Gamma_2)$ such that $\delta_{\alpha\epsilon} \rightharpoonup \overline{\delta_\alpha}$ weakly in $L^2(\Gamma_2)$ and a subcollection $\{\gamma_{\alpha\epsilon}\}$ and $\overline{\gamma_\alpha}$ in $L^2(\Gamma_{1,1})$ such that $\gamma_{\alpha\epsilon} \rightharpoonup \overline{\gamma_\alpha}$ weakly in $L^2(\Gamma_{1,1})$, when $\epsilon \to 0$. Moreover, from the Corollary 3, $\forall\, v \in K_1$ we have that

$$\begin{aligned} \lim_{\epsilon\to 0^+} \quad & (p_{\alpha\epsilon} - \lambda_{\alpha\epsilon}, -\Delta(v - U_{\alpha\epsilon}))_H + (p_{\alpha\epsilon} + \delta_{\alpha\epsilon}, \frac{\partial(v - U_{\alpha\epsilon})}{\partial n})_{L^2(\Gamma_2)} \\ & +(p_{\alpha\epsilon} + \gamma_{\alpha\epsilon}, \frac{\partial(v - U_{\alpha\epsilon})}{\partial n} + \alpha(v - U_{\alpha\epsilon}))_{L^2(\Gamma_{1,1})} \\ = \quad & (\overline{p_\alpha} - \overline{\lambda_\alpha}, -\Delta(v - \overline{U_\alpha}))_H + (\overline{p_\alpha} + \overline{\delta_\alpha}, \frac{\partial(v - \overline{U_\alpha})}{\partial n})_{L^2(\Gamma_2)} \\ & +(\overline{p_\alpha} + \overline{\gamma_\alpha}, \frac{\partial(v - \overline{U_\alpha})}{\partial n} + \alpha(v - \overline{U_\alpha}))_{L^2(\Gamma_{1,1})} \geq 0 \end{aligned}$$

and $\forall\, g \in U_{ad}$ we have

$$\lim_{\epsilon\to 0} \; (Mg_{\alpha\epsilon} + \lambda_{\alpha\epsilon}, g - g_{\alpha\epsilon})_H = (M\overline{g_\alpha} + \overline{\lambda_\alpha}, g - \overline{g_\alpha})_H \geq 0.$$

Conversely, let $(v, g) \in D_\alpha$ be, adding (35) and (36) we have

$$\begin{aligned} & (\overline{p_\alpha} + M\overline{g_\alpha}, g - \overline{g_\alpha})_H + (\overline{p_\alpha} + \overline{\delta_\alpha}, \frac{\partial(v - \overline{U_\alpha})}{\partial n})_{L^2(\Gamma_2)} \\ & +(\overline{p_\alpha} + \overline{\gamma_\alpha}, \frac{\partial(v - \overline{U_\alpha})}{\partial n} + \alpha(v - \overline{U_\alpha}))_{L^2(\Gamma_{1,1})} \geq 0. \end{aligned}$$

Next, from the Green's formula, we obtain

$$\begin{aligned} (\overline{p_\alpha}, -\Delta(v - \overline{U_\alpha}))_H \quad = \quad & (-\Delta\overline{p_\alpha}, v - \overline{U_\alpha})_H - (\overline{p_\alpha}, \frac{\partial(v - \overline{U_\alpha})}{\partial n})_{L^2(\Gamma_2)} \\ & -(\overline{p_\alpha}, \frac{\partial(v - \overline{U_\alpha})}{\partial n} + \alpha(v - \overline{U_\alpha}))_{L^2(\Gamma_{1,1})} \\ = \quad & (\overline{U_\alpha} - z_d, v - \overline{U_\alpha})_H - (\overline{p_\alpha}, \frac{\partial(v - \overline{U_\alpha})}{\partial n})_{L^2(\Gamma_2)} \\ & -(\overline{p_\alpha}, \frac{\partial(v - \overline{U_\alpha})}{\partial n} + \alpha(v - \overline{U_\alpha}))_{L^2(\Gamma_{1,1})} \end{aligned}$$

therefore we have

$$\begin{aligned} & (\overline{U_\alpha} - z_d, v - \overline{U_\alpha})_H + (M\overline{g_\alpha}, g - \overline{g_\alpha})_H \\ & +(\overline{\gamma_\alpha}, \frac{\partial(v - \overline{U_\alpha})}{\partial n} + \alpha(v - \overline{U_\alpha}))_{L^2(\Gamma_{1,1})} + (\overline{\delta_\alpha}, \frac{\partial(v - \overline{U_\alpha})}{\partial n})_{L^2(\Gamma_2)} \geq 0. \end{aligned}$$

Since $(v, g) \in D_\alpha$, the condition of optimality is given by

$$J'(\overline{U}, \overline{g})(v - \overline{U}, g - \overline{g}) = (\overline{U_\alpha} - z_d, v - \overline{U_\alpha})_H + (M\overline{g_\alpha}, g - \overline{g_\alpha})_H \geq 0$$

and therefore $(\overline{U_\alpha}, \overline{g_\alpha})$ is the optimal solution of $(8)$. □

## 4. WEAK CONVERGENCE OF THE SOLUTIONS OF THE PENALIZED PROBLEMS FOR FIXED $\varepsilon > 0$ AND $\alpha \to \infty$

**Theorem 5.** *Let fixed $\epsilon > 0$ be, if $(U_{\alpha\epsilon}, g_{\alpha\epsilon})$ is the unique solution of the problem $(10)$ for each $\alpha > 0$ and $(U_\epsilon, g_\epsilon)$ is the unique solution of the problem $(9)$, then $U_{\alpha\epsilon} \rightharpoonup U_\epsilon$ weakly in $H$ and $g_{\alpha\epsilon} \rightharpoonup g_\epsilon$ weakly in $H$ when $\alpha \to \infty$.*

*Proof.* We have

$$J_{\alpha\epsilon}(U_{\alpha\epsilon}, g_{\alpha\epsilon}) \leq J_{\alpha\epsilon}(\overline{U_\alpha}, \overline{g_\alpha}) = J(\overline{U_\alpha}, \overline{g_\alpha}). \tag{37}$$

As $\overline{U_\alpha}$ and $\overline{g_\alpha}$ are strongly convergent in $H$, by [5], there exists a constant $k_1 > 0$ such that

$$J(\overline{U_\alpha}, \overline{g_\alpha}) = \frac{1}{2}\|\overline{U_\alpha} - z_d\|_H^2 + \frac{M}{2}\|\overline{g_\alpha}\|_H^2 \leq k_1.$$

Next, from (37), $\forall \alpha > 0$, for fixed $\epsilon > 0$ we get $J_{\alpha\epsilon}(U_{\alpha\epsilon}, g_{\alpha\epsilon}) \leq k_1$, i.e. we get

$$J(U_{\alpha\epsilon}, g_{\alpha\epsilon}) + \frac{1}{2\epsilon}\|\Delta U_{\alpha\epsilon} + g_{\alpha\epsilon} + \Delta v_0\|_H^2 + \frac{1}{2\epsilon}\|\frac{\partial U_{\alpha\epsilon}}{\partial n} + q + \frac{\partial v_0}{\partial n}\|_{L^2(\Gamma_2)}^2$$
$$+\frac{1}{2\epsilon}\|\frac{\partial U_{\alpha\epsilon}}{\partial n} + \alpha U_{\alpha\epsilon} + \frac{\partial v_0}{\partial n}\|_{L^2(\Gamma_{1,1})}^2 \leq k_1, \ \forall \alpha > 0, \ \text{fixed } \epsilon > 0.$$

Therefore, we deduce that

$$\begin{array}{rcl} a)\ J(U_{\alpha\epsilon}, g_{\alpha\epsilon}) & \leq & k_1 \\ b)\ \|\Delta U_{\alpha\epsilon} + g_{\alpha\epsilon} + \Delta v_0\|_H^2 & \leq & k_1 \\ c)\ \|\frac{\partial U_{\alpha\epsilon}}{\partial n} + q + \frac{\partial v_0}{\partial n}\|_{L^2(\Gamma_2)}^2 & \leq & k_1 \\ d)\ \|\frac{\partial U_{\alpha\epsilon}}{\partial n} + \alpha U_{\alpha\epsilon} + \frac{\partial v_0}{\partial n}\|_{L^2(\Gamma_{1,1})}^2 & \leq & k_1. \end{array}$$

Now, from (a) and the properties of the functional $J$ we have

$$e)\ \|U_{\alpha\epsilon}\|_H \leq k_1; \quad f)\ \|g_{\alpha\epsilon}\|_H \leq k_1.$$

Next, from (b) and (f) we obtain that there exists a constant $k_2 > 0$ such that

$$g)\ \|\Delta U_{\alpha\epsilon}\|_H \leq k_2$$

and therefore from (e) and (f) we have that there exist $U_\epsilon^*$ and $g_\epsilon^*$ in $H$ such that

$$U_{\alpha\epsilon} \rightharpoonup U_\epsilon^* \text{ weakly in } H\ ,\ g_{\alpha\epsilon} \rightharpoonup g_\epsilon^* \text{ weakly in } H \text{ when } \alpha \to \infty.$$

Now, we will see that $U_\epsilon^* \in K$. Since $U_\epsilon^*$ is the weak limit of the collection $\{U_{\alpha\epsilon}\}$ we get that $U_\epsilon^* \in K_1$ . We will prove that $U_\epsilon^* = 0$ on $\Gamma_{1,1}$. In fact, from (d) we deduce that there exists a constant $k_3 > 0$ such that

$$\|\frac{\partial U_{\alpha\epsilon}}{\partial n} + \alpha U_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})} \leq k_3 \tag{38}$$

and by the Trace Theorem we deduce that there exists a constant $k_4 > 0$ such that $\|\frac{\partial U_{\alpha\epsilon}}{\partial n}\|_{L^2(\Gamma_{1,1})} \leq k_4\|U_{\alpha\epsilon}\|_{H^2(\Omega)}$. Since $U_{\alpha\epsilon} = 0$ on $\Gamma_{1,0}$ and $\frac{\partial U_{\alpha\epsilon}}{\partial n} = 0$ on $\Gamma_3$ we have that there exists a constant $k_5 > 0$ such that $\|U_{\alpha\epsilon}\|_{H^2(\Omega)} \leq k_5\|\Delta U_{\alpha\epsilon}\|_H$ and taking into account (g) we get

$$\|\frac{\partial U_{\alpha\epsilon}}{\partial n}\|_{L^2(\Gamma_{1,1})} \leq k_4\|U_{\alpha\epsilon}\|_{H^2(\Omega)} \leq k_4k_5\|\Delta U_{\alpha\epsilon}\|_H \leq k_2k_4k_5 = k_6$$

i.e. there exists a constant $k_6 > 0$ such that $\|\frac{\partial U_{\alpha\epsilon}}{\partial n}\|_{L^2(\Gamma_{1,1})} \leq k_6$, or, taking into account (38), we have that there exists a constant $k_7 > 0$ such that

$$\alpha\|U_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})} \leq k_7. \tag{39}$$

Therefore, from (39) we deduce that $\|U_{\alpha\epsilon}\|_{L^2(\Gamma_{1,1})} \leq \frac{k_7}{\alpha} \to 0$ when $\alpha \to \infty$ i.e. $\lim_{\alpha\to\infty} \int_{\Gamma_{1,1}} (U_{\alpha\epsilon})^2 d\gamma = 0$. Next, from the weakly lower semicontinuous of the function $v \to \int_{\Gamma_{1,1}} v^2 d\gamma$ we have $\int_{\Gamma_{1,1}} (U_\epsilon^*)^2 d\gamma \leq \lim_{\alpha\to\infty} \int_{\Gamma_{1,1}} (U_{\alpha\epsilon})^2 d\gamma = 0$ then $\int_{\Gamma_{1,1}} (U_\epsilon^*)^2 d\gamma = 0$, and therefore $U_\epsilon^* = 0$ in $\Gamma_{1,1}$, that is $U_\epsilon^* \in K$. On the other hand we know that $g_\epsilon^* \in U_{ad}$.

We will prove that $U_\epsilon^* = U_\epsilon$ and $g_\epsilon^* = g_\epsilon$. In fact, from the following inequalities

$$\begin{aligned} J_\epsilon(U_\epsilon^*, g_\epsilon^*) &\leq J_{\alpha\epsilon}(U_\epsilon^*, g_\epsilon^*) \leq \lim\inf_{\alpha\to\infty} J_{\alpha\epsilon}(U_{\alpha\epsilon}, g_{\alpha\epsilon}) \leq \lim\inf_{\alpha\to\infty} J_{\alpha\epsilon}(\overline{U_\alpha}, \overline{g_\alpha}) \\ &= \lim\inf_{\alpha\to\infty} J(\overline{U_\alpha}, \overline{g_\alpha}) = J(\overline{U}, \overline{g}) \leq J(U_\epsilon, g_\epsilon) \leq J_\epsilon(U_\epsilon, g_\epsilon) \end{aligned}$$

and by the uniqueness of the solution of the control problem we deduce that $U_\epsilon^* = U_\epsilon$ and $g_\epsilon^* = g_\epsilon$. □

**Acknowledgments**: This paper has been partially sponsored by the Project PIP No. 5379 from CONICET - UA (Rosario, Argentina), and ANPCYT PAV No. 120-5 from Agencia (Argentina), and "Fondo de Ayuda a la Investigación" from Univ. Austral (Rosario, Argentina).

## References

[1] A. Azzan - E. Kreyszig, On solutions of elliptic equations satisfying mixed boundary conditions, SIAM J. Math. Anal., 13 (1982), 254-262.

[2] A. Bensoussan, Teoría moderna de control óptimo, Cuadern. Inst. Mat. Beppo Levi, # 7, Rosario, 1974.

[3] M. Bergouniux, A penalization method for optimal control of elliptic problems with state constraints, SIAM J. Control and Optimization, 30 (1992), 305-323.

[4] M. Bergouniux, Optimal control of an obstacle problem, Appl. Math. Optim., 36 (1997), 147-172.

[5] C.M. Gariboldi - D.A. Tarzia, Convergence of distributed optimal controls on the internal energy in mixed elliptic problems when the heat transfer coefficient goes to infinity, Appl. Math. Optim., 47 (2003), 213-230.

[6] C.M. Gariboldi - D.A. Tarzia, A new proof of the convergence of distributed optimal controls on the internal energy in mixed elliptic problems, MAT - Serie A, 7 (2004), 31-42.

[7] D. Kinderlehrer - G. Stampacchia, An introduction to variational inequalities and their applications, Academic Press, New York, 1980.

[8] J.L. Lions, Côntrole optimal des systemes gouvernés par des équations aux dérivées partielles, Dunod-Gauthier Villars, Paris, 1968.

[9] M. Milla Miranda - L.A. Medeiros, On a boundary value problem for wave equations: existence uniqueness-asymptotic behavior, Rev. Mat. Apl., 17 (1996), 47-73.

[10] D.A. Tarzia, Sur le problème de Stefan à deux phases, C.R. Acad. Sc. Paris 288A (1979), 941-944.

[11] D.A. Tarzia, An inequality for the constant heat flux to obtain a steady-state two-phase Stefan problem, Eng. Anal., 5 (1988), 177-181.

CLAUDIA M. GARIBOLDI: Depto. Matemática, FCEFQyN, Univ. Nac. de Rio Cuarto, Ruta 36 Km 601, 5800 Rio Cuarto, Argentina; E-mail: cgariboldi@exa.unrc.edu.ar
DOMINGO A. TARZIA: Depto. Matemática-CONICET, FCE, Univ. Austral, Paraguay 1950, S2000FZF Rosario, Argentina; E-mail: Domingo.Tarzia@fce.austral.edu.ar

Trabajo recibido el 28/4/2008 y aceptado el 5/6/2008.