Simultaneous Optimal Control Problems for Elliptic Hemivariational Inequalities

Claudia M. Gariboldi and Domingo A. Tarzia

Abstract In this paper, we study simultaneous distributed-boundary optimal control problems on the internal energy and the heat flux for a system governed by a class of elliptic boundary hemivariational inequalities with a parameter. The system has been originated by a steady-state heat conduction problem with non-monotone multivalued subdifferential boundary condition on a portion of the boundary of the domain described by the Clarke generalized gradient of a locally Lipschitz function. We prove existence and asymptotic behavior results for the optimal controls and the system states for the optimal control problems, when the parameter, like a heat transfer coefficient, tends to infinity on a portion of the boundary.

Keywords Control theory · Elliptic inequality · Hemivariational inequalities

1 Introduction

We consider a bounded domain Ω in $\mathbb{R}^d$ whose regular boundary Γ consists of the union of three disjoint portions $\Gamma_i, i = 1, 2, 3$ with $|\Gamma_i| > 0$, where $|\Gamma_i|$ denotes the $(d-1)$-dimensional Hausdorff measure of the portion Γ_i on Γ. The outward normal vector on the boundary is denoted by n. We formulate the following mixed nonlinear boundary value problem, which has been recently studied in [10, 13, 32]:

$$-\Delta u = g \text{ in } \Omega, \quad u\big|_{\Gamma_1} = 0, \quad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q, \quad -\frac{\partial u}{\partial n}\Big|_{\Gamma_3} \in \alpha\, \partial j(u), \tag{1}$$

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where u is the temperature in Ω, g is the internal energy in Ω and q is the heat flux on Γ_2, which satisfy the hypothesis: $g \in H = L^2(\Omega)$ and $q \in Q = L^2(\Gamma_2)$. The parameter α is a positive constant which can be considered as the heat transfer coefficient on the boundary while the function $j\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$, called a superpotential, is such that $j(x, \cdot)$ locally Lipschitz for a.e. $x \in \Gamma_3$ and not necessary differentiable. Since in general $j(x, \cdot)$ is nonconvex, so the multivalued condition on Γ_3 in problem (1) is described by a nonmonotone relation expressed by the generalized gradient of Clarke [6]. The problem (1) can be considered as a prototype of several boundary semipermeability models, see [17, 21, 22, 31]. Analogous problems with maximal monotone multivalued boundary relations, that is the case when $j(x, \cdot)$ is a convex function, were considered in [3, 8].

The weak formulation of the elliptic problem (1) becomes the following elliptic boundary hemivariational inequality [10]:

$$\text{find } u \in V_0 \text{ such that } a(u, v) + \alpha \int_{\Gamma_3} j^0(u; v)\, d\Gamma \geq L(v) \text{ for all } v \in V_0 \tag{2}$$

where

$$V = H^1(\Omega) \quad \text{and} \quad V_0 = \{v \in V \mid v = 0 \text{ on } \Gamma_1\}.$$

Here and in what follows we often omit the variable x and we simply write $j(r)$ instead of $j(x, r)$. The stationary heat conduction models with nonmonotone multivalued subdifferential interior and boundary semipermeability relations can not be described by convex potentials. They use locally Lipschitz potentials and their weak formulations lead to hemivariational inequalities, see [21, Sect. 5.5.3] and [22]. The theory of hemivariational and variational inequalities has been proposed in the 1980s by Panagiotopoulos, see [21, 23, 24], as variational formulations of important classes of inequality problems in mechanics. Most recently, new kinds of variational, hemivariational, and variational-hemivariational inequalities have been investigated, see [5, 19, 25].

In relation to the problem (1), we formulate for each $\alpha > 0$, the following simultaneous distributed and Neumann boundary optimal control problem, [4]:

$$\text{find } (\overline{g}_\alpha, \overline{q}_\alpha) \in H \times Q \text{ such that } J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha) = \min_{(g,q) \in H \times Q} J_\alpha(g, q) \tag{3}$$

with

$$J_\alpha(g, q) = \frac{1}{2} ||u_{\alpha g q} - z_d||_H^2 + \frac{M_1}{2} ||g||_H^2 + \frac{M_2}{2} ||q||_Q^2 \tag{4}$$

where $u_{\alpha g q}$ is a solution to the hemivariational inequality (2), $z_d \in H$ is given and M_1 and M_2 are positive constants.

Now, we consider the following classical steady-state heat conduction problem with mixed boundary conditions [1, 2, 14, 15, 27, 28]:

$$-\Delta u = g \text{ in } \Omega, \qquad u\big|_{\Gamma_1} = 0, \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q, \qquad u\big|_{\Gamma_3} = b, \tag{5}$$

with b a constant on Γ_3 and which variational formulation is given by, [29]:

$$\text{find } u_\infty \in K \text{ such that } a(u_\infty, v) = L(v) \text{ for all } v \in K_0 \tag{6}$$

where

$$K = \{v \in V \mid v = 0 \text{ on } \Gamma_1,\ v = b \text{ on } \Gamma_3\}, \quad K_0 = \{v \in V \mid v = 0 \text{ on } \Gamma_1 \cup \Gamma_3\},$$

$$a(u, v) = \int_\Omega \nabla u\, \nabla v\, dx, \quad L(v) = \int_\Omega g v\, dx - \int_{\Gamma_2} q v\, d\Gamma,$$

Note that the form a is bilinear, symmetric, continuous and coercive with constant $m_a > 0$, i.e.

$$a(v, v) = \|v\|^2_{V_0} \geq m_a \|v\|^2_V \text{ for all } v \in V_0. \tag{7}$$

We remark that, under additional hypotheses on the data g, q and b, problem (5) can be considered as steady-state two-phase Stefan problem, see [9, 26, 28, 29].

In relation to the problem (5), we formulate the following simultaneous distributed and Neumann boundary optimal control problem, given by [4, 10, 12, 16, 30]:

$$\text{find } (\overline{g}, \overline{q}) \in H \times Q \text{ such that } J(\overline{g}, \overline{q}) = \min_{(g,q) \in H \times Q} J(g, q) \tag{8}$$

with

$$J(g, q) = \frac{1}{2}||u_{gq} - z_d||^2_H + \frac{M_1}{2}||g||^2_H + \frac{M_2}{2}||q||^2_Q \tag{9}$$

where u_{gq} is the unique solution to the variational equality (6), $z_d \in H$ given and M_1 and M_2 are positive constants given.

The paper is structured as follows. In Sect. 2, we establish preliminaries concepts of the hemivariational inequalities theory, which are necessary for the development of the following sections. In Sect. 3, for each $\alpha > 0$, we obtain an existence result of solution to the simultaneous distributed-boundary optimal control problem (3). In Sect. 4, the strong convergence of a sequence of optimal controls and the system states to the problems (3)–(4) to the unique optimal control and the system state to the problem (8)–(9), are obtained when the parameter α goes to infinity. Here, we obtain the asymptotic behavior of the optimal system states $u_{\alpha \overline{g}_\alpha \overline{q}_\alpha}$ as $\alpha \to \infty$ without any prior knowledge on the monotonicity on α as it was given in [10, 32].

2 Preliminaries

Let $(X, \|\cdot\|_X)$ be a reflexive Banach space, X^* be its dual, and $\langle \cdot, \cdot \rangle$ denote the duality between X^* and X. For a real valued function defined on X, we have the following definitions [6, Section 2.1] and [7, 19].

Definition 1 A function $\varphi \colon X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ there exist U_x a neighborhood of x and a constant $L_x > 0$ such that

$$|\varphi(y) - \varphi(z)| \leq L_x \|y - z\|_X \text{ for all } y, z \in U_x.$$

For such a function the generalized (Clarke) directional derivative of φ at the point $x \in X$ in the direction $v \in X$ is defined by

$$\varphi^0(x; v) = \limsup_{y \to x,\, \lambda \to 0^+} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}.$$

The generalized gradient (subdifferential) of φ at x is a subset of the dual space X^* given by

$$\partial\varphi(x) = \{\zeta \in X^* \mid \varphi^0(x; v) \geq \langle \zeta, v \rangle \text{ for all } v \in X\}.$$

We consider the following hypothesis.

$\underline{H(j)}$: $j \colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is such that

(a) $j(\cdot, r)$ is measurable for all $r \in \mathbb{R}$,
(b) $j(x, \cdot)$ is locally Lipschitz for a.e. $x \in \Gamma_3$,
(c) there exist $c_0, c_1 \geq 0$ such that $|\partial j(x, r)| \leq c_0 + c_1 |r|$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_3$,
(d) $j^0(x, r; b - r) \leq 0$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_3$ with a constant $b \in \mathbb{R}$.

Note that the existence results for elliptic hemivariational inequalities can be found in several contributions, see [5, 18–21]. In [10, Theorem 4], the hypothesis $H(j)$(d) is considered in order to obtain existence of a solution to problem (2) and under this condition the authors have studied the asymptotic behavior when $\alpha \to \infty$ (see [10, Theorem 7]). Moreover, in [10], we can find several examples of locally Lipschitz (nondifferentiable and nonconvex) functions which satisfy the above hypotheses.

3 Existence of Simultaneous Optimal Controls

We know, by [12], that there exists a unique optimal pair $(\overline{g}, \overline{q}) \in H \times Q$ of the simultaneous distributed-boundary optimal control problem (8). In similar way to [13], we have a result on existence of solution to the simultaneous optimal control problem (3) in which the system is governed by the hemivariational inequality (2).

Theorem 1 *For each* $\alpha > 0$*, if* $H(j)(a) - (d)$ *holds, then the simultaneous distributed-boundary optimal control problem (3) governed by the hemivariational inequality (2) has a solution.*

Proof An idea of the proof is a follows, for details see [4, Theorem 4.1].
For each $\alpha > 0$, the functional J_α is bounded and taking into account that the hemivariational inequality (2) has solution (see [10, Theorem 4]), we have that

$$m = \inf\{J_\alpha(g,q), (g,q) \in H \times Q, u_{\alpha g q} \in T_\alpha(g,q)\} \geq 0, \tag{10}$$

where $T_\alpha(g,q)$ denote, for each $(g,q) \in H \times Q$, the set of solutions of (2). Next, for each $\alpha > 0$, let $(g_n^\alpha, q_n^\alpha) \in H \times Q$ be with $n \in \mathbb{N}$ a minimizing sequence to (10) such that

$$m \leq J_\alpha(g_n^\alpha, q_n^\alpha) \leq m + \frac{1}{n}. \tag{11}$$

We obtain that there exists $C_1 > 0$, independent of α, such that

$$||g_n^\alpha||_H \leq C_1 \quad \text{and} \quad ||q_n^\alpha||_Q \leq C_1. \tag{12}$$

Moreover, we can prove that there exists $C_2 > 0$, independent of α, such that

$$||u_{\alpha g_n^\alpha q_n^\alpha}||_{V_0} \leq C_2. \tag{13}$$

Therefore, for each $\alpha > 0$ there exist $f_\alpha \in H$, $\xi_\alpha \in Q$ and $\eta_\alpha \in V_0$ such that, when $n \to \infty$

$$u_{\alpha g_n^\alpha q_n^\alpha} \rightharpoonup \eta_\alpha \quad \text{in} \quad V_0, \qquad g_n^\alpha \rightharpoonup f_\alpha \quad \text{in} \quad H \qquad \text{and} \qquad q_n^\alpha \rightharpoonup \xi_\alpha \quad \text{in} \quad Q.$$

Now, for each $\alpha > 0$, taking the upper limit in (2) for $(g_n^\alpha, q_n^\alpha) \in H \times Q$, we have

$$a(\eta_\alpha, v) + \alpha \limsup_{n\to\infty} \int_{\Gamma_3} j^0(u_{\alpha g_n^\alpha q_n^\alpha}; v)\, d\Gamma \geq \int_\Omega f_\alpha v\, dx - \int_{\Gamma_2} \xi_\alpha v\, d\Gamma \ \ \forall v \in V_0. \tag{14}$$

By the compactness of the trace operator from V into $L^2(\Gamma_3)$, we have $u_{\alpha g_n^\alpha q_n^\alpha}\big|_{\Gamma_3} \to \eta_\alpha\big|_{\Gamma_3}$ in $L^2(\Gamma_3)$, as $n \to +\infty$, and at least for a subsequence, $u_{\alpha g_n^\alpha q_n^\alpha}(x) \to \eta_\alpha(x)$ for a.e. $x \in \Gamma_3$ and $|u_{\alpha g_n^\alpha q_n^\alpha}(x)| \leq h_\alpha(x)$ a.e. $x \in \Gamma_3$, where $h_\alpha \in L^2(\Gamma_3)$. Since the function $\mathbb{R} \times \mathbb{R} \ni (r,s) \mapsto j^0(x, r; s) \in \mathbb{R}$ a.e. is upper semicontinuous on Γ_3, see [10, Proposition 3], we obtain

$$\limsup_{n\to\infty} j^0(x, u_{\alpha g_n^\alpha q_n^\alpha}(x); v(x)) \leq j^0(x, \eta_\alpha(x); v(x)) \ \text{ a.e. } \ x \in \Gamma_3.$$

Next, from $H(j)(c)$, we deduce the estimate

$$|j^0(x, u_{\alpha g_n^\alpha q_n^\alpha}(x); v(x))| \leq (c_0 + c_1|u_{\alpha g_n^\alpha q_n^\alpha}(x)|)\, |v(x)| \leq k_\alpha(x) \ \text{ a.e. } \ x \in \Gamma_3$$

where $k_\alpha \in L^1(\Gamma_3)$, $k_\alpha(x) = (c_0 + c_1 h_\alpha(x))|v(x)|$ and we apply the dominated convergence theorem, see [7] to get

$$\limsup_{n\to\infty} \int_{\Gamma_3} j^0(u_{\alpha g_n^\alpha q_n^\alpha}; v)\, d\Gamma \leq \int_{\Gamma_3} \limsup_{n\to\infty} j^0(u_{\alpha g_n^\alpha q_n^\alpha}; v)\, d\Gamma \leq \int_{\Gamma_3} j^0(\eta_\alpha; v)\, d\Gamma.$$

Using the latter in (14), we obtain that $\eta_\alpha \in V_0$ is a solution to the hemivariational inequality (2). Next, we have proved that $\eta_\alpha = u_{\alpha f_\alpha \xi_\alpha}$, where $u_{\alpha f_\alpha \xi_\alpha}$ is a solution of the hemivariational inequality (2) for data $f_\alpha \in H$ and $\xi_\alpha \in Q$, for each $\alpha > 0$. Finally, from (11) and the weak lower semicontinuity of J_α, we have

$$m \geq \liminf_{n\to\infty} J_\alpha(g_n^\alpha, q_n^\alpha) \geq J_\alpha(f_\alpha, \xi_\alpha),$$

and therefore, for each $\alpha > 0$, (f_α, ξ_α) is an optimal pair to simultaneous distributed-boundary optimal control problem (3). □

4 Asymptotic Behavior of Simultaneous Optimal Controls

In this section we expose a result on the asymptotic behavior of the optimal solutions to problem (3) when $\alpha \to \infty$ and we give a short proof, for more details see [4, Theorem 5.1].

Theorem 2 *Assume $H(j)$ and (H_1). If $(\overline{g}_\alpha, \overline{q}_\alpha)$ is a optimal solution to simultaneous distributed and Neumann boundary optimal control problem (3) and $(\overline{g}, \overline{q})$ is the unique solution to simultaneous optimal control problem (8), then $(\overline{g}_\alpha, \overline{q}_\alpha) \to (\overline{g}, \overline{q})$ in $H \times Q$ strongly and $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \to u_{\infty\overline{g}\,\overline{q}}$ in V strongly, when $\alpha \to \infty$.*

Proof For each $\alpha > 0$, $(\overline{g}_\alpha, \overline{q}_\alpha)$ is a optimal solution to problem (3), then there exists a constant $C_1 > 0$ such that

$$\frac{1}{2}||u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - z_d||_H^2 + \frac{M_1}{2}||\overline{g}_\alpha||_H^2 + \frac{M_2}{2}||\overline{q}_\alpha||_Q^2 \leq \frac{1}{2}||u_{\alpha 00} - z_d||_H^2 \leq C_1$$

because $\{u_{\alpha 00}\}$ is convergent when $\alpha \to \infty$, see [10, Theorem 7]. Therefore, there exist positive constants C_2, C_3 and C_4, independent of α, such that

$$||\overline{g}_\alpha||_H \leq C_2, \qquad ||\overline{q}_\alpha||_Q \leq C_3 \quad \text{and} \quad ||u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}||_H \leq C_4. \tag{15}$$

Now, if we choose $v = u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \in V_0$ as a test function in (2), we get

$$\begin{aligned} &a(u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) - \alpha \int_{\Gamma_3} j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}; u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\, d\Gamma \\ &\leq a(u_{\infty\overline{g}\,\overline{q}}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) - L(u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}). \end{aligned}$$

Taking into account that $j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}; u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) = j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}; b - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})$ on Γ_3, and by $H(j)$(d), we have $j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}; u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) \leq 0$ on Γ_3. Next, by the boundedness and coerciveness of a, we infer

$$\|u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\|_V \leq \frac{1}{m_a}(M_a\|u_{\infty\overline{g}\,\overline{q}}\|_V + \|L\|_{V^*}) + \|u_{\infty\overline{g}\,\overline{q}}\|_V =: C_5, \tag{16}$$

with $M_a > 0$.
Now, since $a(u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) \geq 0$, from (4), we have

$$-\int_{\Gamma_3} j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}; u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\, d\Gamma \leq \frac{C_6}{\alpha} \tag{17}$$

where $C_6 > 0$ is independent of α.

It follows from (16) that $\{u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\}$ remains in a bounded subset of V. Thus, there exists $\eta \in V$ such that, by passing to a subsequence if necessary, we have

$$u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \rightharpoonup \eta \ \text{ weakly in } \ V, \ \text{ as } \alpha \to \infty. \tag{18}$$

Moreover. from (15), we have that there exists $h \in H$ and $p \in Q$ such that

$$\overline{g}_\alpha \rightharpoonup h \ \text{ weakly in } \ H, \ \text{ as } \alpha \to \infty. \tag{19}$$

$$\overline{q}_\alpha \rightharpoonup p \ \text{ weakly in } \ Q, \ \text{ as } \alpha \to \infty. \tag{20}$$

Next, we observe that $\eta \in V_0$ because $\{u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\} \subset V_0$ and V_0 is sequentially weakly closed in V. Let $w \in K$ and $v = w - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \in V_0$, from (2), since $w = b$ on Γ_3, by $H(j)$(d), we obtain

$$L(w - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) \leq a(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, w - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) \tag{21}$$

and by using the weak lower semicontinuity of $V \ni v \mapsto a(v, v) \in \mathbb{R}$, we deduce

$$\eta \in V_0 \ \text{ satisfies } \ L(w - \eta) \leq a(\eta, w - \eta) \ \text{ for all } \ w \in K. \tag{22}$$

Subsequently, from (18), by the compactness of the trace operator, we have $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\big|_{\Gamma_3} \to \eta\big|_{\Gamma_3}$ in $L^2(\Gamma_3)$, as $\alpha \to \infty$. Passing to a subsequence if necessary, we may suppose that $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x) \to \eta(x)$ for a.e. $x \in \Gamma_3$ and there exists $f \in L^2(\Gamma_3)$ such that $|u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x)| \leq f(x)$ a.e. $x \in \Gamma_3$. Using the upper semicontinuity of the function $\mathbb{R} \times \mathbb{R} \ni (r, s) \mapsto j^0(x, r; s) \in \mathbb{R}$ for a.e. $x \in \Gamma_3$, see [10, Proposition 3 (iii)], we get

$$\limsup_{\alpha\to\infty} j^0(x, u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x); b - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x)) \leq j^0(x, \eta(x); b - \eta(x)) \ \text{ a.e. } \ x \in \Gamma_3.$$

Next, taking into account that

$$|j^0(x, u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x); b - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}(x))| \leq k(x) \ \text{ a.e. } \ x \in \Gamma_3$$

with $k \in L^1(\Gamma_3)$, by the dominated convergence theorem, see [7], we obtain

$$\limsup_{\alpha\to\infty}\int_{\Gamma_3} j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}; b-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\,d\Gamma \leq \int_{\Gamma_3} j^0(\eta; b-\eta)\,d\Gamma.$$

Consequently, from $H(j)$(d) and (17), we have

$$0 \leq -\int_{\Gamma_3} j^0(\eta; b-\eta)\,d\Gamma \leq \liminf_{\alpha\to\infty}\left(-\int_{\Gamma_3} j^0(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}; b-u_{\alpha\overline{g}_\alpha\overline{q}_\alpha})\,d\Gamma\right) \leq 0$$

which gives $\int_{\Gamma_3} j^0(\eta; b-\eta)\,d\Gamma = 0$. Again by $H(j)$(d), we get $j^0(x,\eta(x); b-\eta(x)) = 0$ a.e. $x \in \Gamma_3$. By using now the hypothesis (H_1), we have $\eta(x) = b$ for a.e. $x \in \Gamma_3$, which together with (22) implies

$$\eta \in K \ \text{ satisfies } \ L(w-\eta) \leq a(\eta, w-\eta) \ \text{ for all } \ w \in K.$$

Next,let $v := w - \eta \in K_0$ with arbitrary $w \in K$. Hence, $L(v) \leq a(\eta, v)$ for all $v \in K_0$ and we deduce that

$$\eta \in K \ \text{ satisfies } \ a(\eta, v) = L(v) \ \text{ for all } \ v \in K_0,$$

i.e., $\eta \in K$ is a solution to problem (6). By the uniqueness of solution to problem (6), we have $\eta = u_{\infty hp}$ and hence $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \rightharpoonup u_{\infty hp}$ weakly in V, as $\alpha \to \infty$.
Now, $\forall (g, q) \in H \times Q$, we have

$$J(h,p) \leq \liminf_{\alpha\to\infty} J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha) \leq \liminf_{\alpha\to\infty} J_\alpha(g,q) = \lim_{\alpha\to\infty} J_\alpha(g,q) = J(g,q)$$

and from uniqueness of the optimal control problem (8), see [11], we obtain that $h = \overline{g}$ and $p = \overline{q}$, therefore $u_{\infty hp} = u_{\infty\overline{g}\,\overline{q}}$. Next, we have that, when $\alpha \to \infty$

$$\overline{g}_\alpha \rightharpoonup \overline{g} \ \text{ in } \ H, \quad \overline{q}_\alpha \rightharpoonup \overline{q} \ \text{ in } \ Q \quad \text{and} \quad u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \rightharpoonup u_{\infty\overline{g}\,\overline{q}} \ \text{ in } \ V.$$

Now, choosing $v = u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \in V_0$ in problem (2), since $u_{\infty\overline{g}\,\overline{q}} = b$ on Γ_3, by $H(j)$(d) and the coerciveness of the form a, we have

$$m_a \,\|u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}\|_V^2 \leq a(u_{\infty\overline{g}\,\overline{q}}, u_{\infty\overline{g}\,\overline{q}} - u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) + L(u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - u_{\infty\overline{g}\,\overline{q}}).$$

Employing the weak continuity of $a(u_{\infty\overline{g}\,\overline{q}}, \cdot)$, the compactness of the trace operator and taking into account that $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \to u_{\infty\overline{g}\,\overline{q}}$ strongly in H, we conclude that $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \to u_{\infty\overline{g}\,\overline{q}}$ strongly in V, as $\alpha \to \infty$.
Finally, from $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \to u_{\infty\overline{g}\,\overline{q}}$ strongly in H, we deduce

$$\lim_{\alpha\to\infty} \frac{1}{2}||u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - z_d||_H^2 = \frac{1}{2}||u_{\infty\overline{g}\,\overline{q}} - z_d||_H^2 \tag{23}$$

and as $\overline{g}_\alpha \rightharpoonup \overline{g}$ weakly in H and $\overline{q}_\alpha \rightharpoonup \overline{q}$ weakly in Q, then

$$||\overline{g}||_H^2 \leq \liminf_{\alpha\to\infty} ||\overline{g}_\alpha||_H^2 \quad \text{and} \quad ||\overline{q}||_Q^2 \leq \liminf_{\alpha\to\infty} ||\overline{q}_\alpha||_Q^2. \tag{24}$$

Next, from (23) and (24), we obtain

$$J(\overline{g}, \overline{q}) \leq \liminf_{\alpha\to\infty} J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha).$$

On the other hand, from the definition of $(\overline{g}_\alpha, \overline{q}_\alpha)$, we have

$$J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha) \leq J_\alpha(\overline{g}, \overline{q})$$

then, taking into account that $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \to u_{\infty\overline{g}\,\overline{q}}$ strongly in H, see [10, Theorem 7], we obtain

$$\limsup_{\alpha\to\infty} J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha) \leq \limsup_{\alpha\to\infty} J_\alpha(\overline{g}, \overline{q}) = J(\overline{g}, \overline{q})$$

and therefore

$$\lim_{\alpha\to\infty} J_\alpha(\overline{g}_\alpha, \overline{q}_\alpha) = J(\overline{g}, \overline{q})$$

or equivalently

$$\begin{aligned} &\lim_{\alpha\to\infty}\left(\frac{1}{2}||u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - z_d||_H^2 + \frac{M_1}{2}||\overline{g}_\alpha||_H^2 + \frac{M_2}{2}||\overline{q}_\alpha||_Q^2\right) \\ &= \frac{1}{2}||u_{\infty\overline{g}\,\overline{q}} - z_d||_H^2 + \frac{M_1}{2}||\overline{g}||_H^2 + \frac{M_2}{2}||\overline{q}||_Q^2. \end{aligned} \tag{25}$$

Now, from (23) and (25), when $\alpha \to \infty$, we have

$$||\overline{g}_\alpha||_H^2 \to ||\overline{g}||_H^2 \quad \text{and} \quad ||\overline{q}_\alpha||_Q^2 \to ||\overline{q}||_Q^2$$

and as $\overline{g}_\alpha \rightharpoonup \overline{g}$ weakly in H and $\overline{q}_\alpha \rightharpoonup \overline{q}$ weakly in Q, we deduce that $\overline{g}_\alpha \to \overline{g}$ strongly in H and $\overline{q}_\alpha \to \overline{q}$ strongly in Q. This completes the proof. □

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