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A REVIEW OF OPTIMAL CONTROL PROBLEMS FOR ELLIPTIC VARIATIONAL AND HEMIVARIATIONAL INEQUALITIES AND THEIR ASYMPTOTIC BEHAVIORS

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Abstract. We consider a d-dimensional bounded domain Ω which regular boundary consists of the union of three disjoint portions. We study different optimal control problems (distributed, boundary and simultaneous distributed-boundary) for systems governed by elliptic variational inequalities or elliptic hemivariational inequalities. For both cases, we also consider a parameter, like a heat transfer coefficient on a portion of the boundary, which tends to infinity. We prove an existence result for three different optimal control problems, and we show the asymptotic behavior results for the corresponding optimal controls and system states.

1. Introduction. In this paper, we review several previous works of our authorship and some of them in collaboration with other authors. We consider elliptic mixed problems defined in a d-dimensional domain Ω, whose regular boundary Γ consists of the union of three (or possibly two) disjoint portions. These problems are governed by the Poisson

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equation in Ω and by mixed boundary conditions on Γ. More precisely, we consider Dirichlet, Neumann and Robin boundary conditions. We remark that, under additional hypotheses on the data, these problems can be considered as steady-state two phase Stefan problems, which have been extensively studied in several papers such as [10, 34, 35, 36, 37, 38]. In [12, 13], related to these mixed elliptic problems, we formulate distributed optimal control problems on the internal energy, which are dependent of a parameter (heat transfer coefficient). We study existence, uniqueness and asymptotic behaviour of the optimal solutions when this parameter goes to infinity. In [14], we consider boundary optimal control problems on the heat flux and we obtain similar existence, uniqueness and convergence results when heat transfer coefficient goes to infinity. In [15], simultaneous distributed-boundary optimal control problems have been formulated and similar results to [12, 13, 14] have been obtained.

More recently, in [11], a non-monotone multivalued subdifferential boundary condition on a portion of the boundary described by the Clarke generalized gradient of a locally Lipschitz function has been considered. Such multivalued relation is met in certain types of steady-state heat conduction problems as well as in several boundary semipermeability models, see [24, 27, 28, 29, 40, 41], which are motivated by problems arising in hydraulics, fluid flow problems through porous media, and electrostatics, where the solution represents the pressure and the electric potentials. The weak formulations of these problems are given by boundary hemivariational inequalities. In [11], existence result for a class of boundary hemivariational inequality has been proved. In [16], distributed optimal control problems on the internal energy has been formulated for this kind of boundary hemivariational inequality and existence and asymptotic behavior of optimal controls and system states has been obtained. In [4], boundary and simultaneous distributed-boundary optimal control problems related to the same class of boundary hemivariational inequality has been studied and similar results to [16] has been proved.

The paper is structured as follows. In Section 2, we consider mixed elliptic problems and we give their variational and hemivariational formulations. We consider preliminaries concept and we give some existence results and properties of monotonicity, convergence and continuous dependence of data. Furthermore, we present three examples which satisfy the hypotheses considered. In Section 3, we formulate distributed, boundary and simultaneous distributed-boundary optimal control problems related with the mixed elliptic problems governed by variational equalities. We prove existence and uniqueness of the optimal solutions and we obtain convergence results of the optimal controls and the optimal direct and adjoint states, when the heat transfer coefficient goes to infinity. Finally, in Section 4, we consider distributed, boundary and simultaneous distributed-boundary optimal control problems related with the mixed elliptic problems governed by hemivariational inequalities. We prove existence of the optimal solutions and we obtain convergence results of the optimal controls and the optimal system states, when the heat transfer coefficient goes to infinity.

2. Mixed elliptic problems. In this section, we consider elliptic mixed problems defined in a d-dimensional domain, which are governed by the Poisson equation with mixed conditions on the regular boundary of the domain. That is, we consider Dirichlet, Neumann and Robin boundary conditions and a multivalued condition on a portion of boundary. The weak formulations of these problems are given by variational equalities or hemivariational inequalities depending on the boundary conditions we impose. We will give some necessary definitions and we will prove some important properties.

2.1. Problems with variational equalities. We consider a bounded domain Ω in $\mathbb{R}^d$ which regular boundary Γ consists of the union of three disjoint portions Γ_i, $i = 1, 2, 3$ with $|\Gamma_i| > 0$, where $|\Gamma_i|$ denotes the $(d-1)$-dimensional Hausdorff measure of the portion Γ_i on Γ. The outward normal vector on the boundary is denoted by n. We formulate the following two steady-state heat conduction problems with mixed boundary conditions:

$$-\Delta u = g \ \text{ in } \ \Omega, \quad u\big|_{\Gamma_1} = 0, \quad -\frac{\partial u}{\partial n}\big|_{\Gamma_2} = q, \quad u\big|_{\Gamma_3} = b \tag{1}$$

$$-\Delta u = g \ \text{ in } \ \Omega, \quad u\big|_{\Gamma_1} = 0, \quad -\frac{\partial u}{\partial n}\big|_{\Gamma_2} = q, \quad -\frac{\partial u}{\partial n}\big|_{\Gamma_3} = \alpha(u-b) \tag{2}$$

where u is the temperature in Ω, g is the internal energy in Ω, b is the temperature on Γ_3 for (1) and the temperature of the external neighborhood of Γ_3 for (2), q is the heat flux on Γ_2 and $\alpha > 0$ is the heat transfer coefficient on Γ_3, which satisfy the hypothesis: $g \in H = L^2(\Omega)$, $q \in Q = L^2(\Gamma_2)$ and $b \in H^{\frac{1}{2}}(\Gamma_3)$.

We denote

$$V = H^1(\Omega), \quad V_0 = \{v \in V \mid v = 0 \ \text{ on } \ \Gamma_1\},$$

$$K = \{v \in V \mid v = 0 \ \text{ on } \ \Gamma_1, \ v = b \ \text{ on } \ \Gamma_3\},$$

$$K_0 = \{v \in V \mid v = 0 \ \text{ on } \ \Gamma_1 \cup \Gamma_3\},$$

$$(g,h) = \int_\Omega gh\,dx, \quad (q,\eta)_Q = \int_{\Gamma_2} q\eta\,d\gamma,$$

$$a(u,v) = \int_\Omega \nabla u\,\nabla v\,dx, \quad b_\alpha(u,v) = a(u,v) + \alpha\int_{\Gamma_3}\gamma(u)\gamma(v)d\gamma,$$

$$L(v) = \int_\Omega gv\,dx - \int_{\Gamma_2} q\gamma(v)\,d\gamma, \quad L_\alpha(v) = L(v) + \alpha\int_{\Gamma_3} b\gamma(v)\,d\gamma,$$

where $\gamma\colon V \to L^2(\Gamma)$ denotes the trace operator on Γ. In what follows, we write u for the trace of a function $u \in V$ on the boundary. In a standard way, we obtain the following variational formulations to problems (1) and (2), respectively:

$$\text{find } \ u_\infty \in K \ \text{ such that } \ a(u_\infty, v) = L(v) \ \text{ for all } \ v \in K_0, \tag{3}$$

$$\text{find } \ u_\alpha \in V_0 \ \text{ such that } \ b_\alpha(u_\alpha, v) = L_\alpha(v) \ \text{ for all } \ v \in V_0. \tag{4}$$

The standard norms on V and V_0 are denoted by

$$\|v\|_V = \Big(\|v\|_H^2 + \|\nabla v\|^2_{L^2(\Omega;\mathbb{R}^d)}\Big)^{1/2} \ \text{ for } \ v \in V,$$

$$\|v\|_{V_0} = \|\nabla v\|_{L^2(\Omega;\mathbb{R}^d)} \ \text{ for } \ v \in V_0.$$

It is well known by the Poincaré inequality, see [6, 20], that on V_0 the above two norms are equivalent. Note that the form a is bilinear, symmetric, continuous and coercive with constant $m_a > 0$, i.e.

$$a(v,v) = \|v\|_{V_0}^2 \geq m_a \|v\|_V^2 \quad \text{for all} \quad v \in V_0. \tag{5}$$

Note also that the form b_α is bilinear, symetric, continuous and coercive in V, i.e.

$$b_\alpha(v,v) \geq \lambda_\alpha ||v||_V^2, \forall v \in V \tag{6}$$

where $\lambda_\alpha = \lambda_1 \min\{1, \alpha\}$ and λ_1 is the coerciveness constant for the bilinear form a_1 [36].

It is well known that the regularity of solution to the mixed elliptic problems (1) and (2) are problematic in the neighborhood of a part of the boundary, see for example the monograph [19]. A regularity results for elliptic problems with mixed boundary conditions can be found in [1, 2, 21]. Moreover, sufficient hypotheses on the data in order to have H^2 regularity for elliptic variational inequalities are given in [30]. We remark that, under additional hypotheses on the data g, q and b, problems (1) and (2) can be considered as steady-state two phase Stefan problems, see, for example, [10, 34, 36, 38].

The problems (3) and (4) have been extensively studied in several papers such as [10, 34, 35, 36, 37]. Some properties of monotonicity and convergence, when the parameter α goes to infinity, obtained in the aforementioned works, are recalled in the following result.

Theorem 2.1. *If the data satisfy $b = const. > 0$, $g \in H$ and $q \in Q$ with the properties $q \geq 0$ on Γ_2 and $g \leq 0$ in Ω, then*

(i) $u_\infty \leq b$ in Ω,
(ii) $u_\alpha \leq b$ in Ω,
(iii) $u_\alpha \leq u_\infty$ in Ω,
(iv) if $\alpha_1 \leq \alpha_2$, then $u_{\alpha_1} \leq u_{\alpha_2}$ in Ω,
(v) $u_\alpha \to u_\infty$ in V, as $\alpha \to +\infty$.

Proof. See [10, 34, 36, 37]. ■

2.2. Problems with hemivariational inequalities. We consider the mixed nonlinear boundary value problem studied in [11]. We begin by giving some definitions and properties necessary for the development of these topics.

Let $(X, \|\cdot\|_X)$ be a reflexive Banach space, X^* be its dual, and $\langle \cdot, \cdot \rangle$ denote the duality between X^* and X. For a real valued function defined on X, we have the following definitions [5, Section 2.1] and [7, 8, 25].

Definition 2.2. A function $\varphi \colon X \to \mathbb{R}$ is said to be locally Lipschitz if for every $x \in X$ there exist U_x a neighborhood of x and a constant $L_x > 0$ such that

$$|\varphi(y) - \varphi(z)| \leq L_x \|y - z\|_X \quad \text{for all} \quad y, z \in U_x.$$

For such a function the generalized (Clarke) directional derivative of j at the point $x \in X$ in the direction $v \in X$ is defined by

$$\varphi^0(x; v) = \limsup_{y \to x,\, \lambda \to 0^+} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}.$$

The generalized gradient (subdifferential) of φ at x is a subset of the dual space X^* given by

$$\partial\varphi(x) = \{\zeta \in X^* \mid \varphi^0(x; v) \geq \langle \zeta, v \rangle \quad \text{for all} \quad v \in X\}.$$

We shall use the following properties of the generalized directional derivative and the generalized gradient, see [25, Proposition 3.23].

PROPOSITION 2.3. *Assume that $\varphi\colon X \to \mathbb{R}$ is a locally Lipschitz function. Then the following hold:*

(i) *for every $x \in X$, the function $X \ni v \mapsto \varphi^0(x;v) \in \mathbb{R}$ is positively homogeneous, and subadditive, i.e.,*

$$\varphi^0(x;\lambda v) = \lambda\varphi^0(x;v) \quad \text{for all} \quad \lambda \geq 0,\ v \in X,$$

$$\varphi^0(x;v_1+v_2) \leq \varphi^0(x;v_1) + \varphi^0(x;v_2) \quad \text{for all} \quad v_1, v_2 \in X,$$

respectively.

(ii) *for every $x \in X$, we have $\varphi^0(x;v) = \max\{\langle \zeta, v\rangle \mid \zeta \in \partial\varphi(x)\}$.*

(iii) *the function $X \times X \ni (x,v) \mapsto \varphi^0(x;v) \in \mathbb{R}$ is upper semicontinuous.*

(iv) *for every $x \in X$, the gradient $\partial\varphi(x)$ is a nonempty, convex, and weakly compact subset of X^*.*

(v) *the graph of the generalized gradient $\partial\varphi$ is closed in $X \times$ (weak–X^*)–topology.*

Now, we are in a position to formulate the aforementioned problem. The mixed nonlinear boundary value problem is given by

$$-\Delta u = g \ \text{ in } \ \Omega, \qquad u\big|_{\Gamma_1} = 0, \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q, \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_3} \in \alpha\,\partial j(u). \tag{7}$$

Here, as in the problem (2), α is a positive constant while the function $j\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$, called a superpotential (nonconvex potential), is such that $j(x,\cdot)$ is locally Lipschitz for a.e. $x \in \Gamma_3$ and not necessary differentiable. Since in general $j(x,\cdot)$ is nonconvex, so the multivalued condition on Γ_3 in problem (7) is described by a nonmonotone relation expressed by the generalized gradient of Clarke. Such multivalued relation in problem (7) is met in certain types of steady-state heat conduction problems (the behavior of a semipermeable membrane of finite thickness, a temperature control problems, etc.). Further, problem (7) can be considered as a prototype of several boundary semipermeability models, see [24, 27, 28, 41], which are motivated by problems arising in hydraulics, fluid flow problems through porous media, and electrostatics, where the solution represents the pressure and the electric potentials. Note that the analogous problems with maximal monotone multivalued boundary relations (that is the case when $j(x,\cdot)$ is a convex function) were considered in [3, 9], see also references therein.

Under the above notation, the weak formulation to the elliptic problem (7) becomes the following boundary hemivariational inequality:

$$\text{find } \ \overline{u}_\alpha \in V_0 \ \text{ such that } \ a(\overline{u}_\alpha, v) + \alpha \int_{\Gamma_3} j^0(\overline{u}_\alpha; v)\, d\gamma \geq L(v) \ \text{ for all } \ v \in V_0. \tag{8}$$

Here and in what follows we often omit the variable x and we simply write $j(r)$ instead of $j(x,r)$. Observe that if $j(x,\cdot)$ is a convex function for a.e. $x \in \Gamma_3$, then the problem (8) reduces to the variational inequality of second kind:

$$\text{find} \quad \overline{u}_\alpha \in V_0 \quad \text{such that}$$

$$a(\overline{u}_\alpha, v - \overline{u}_\alpha) + \alpha \int_{\Gamma_3} (j(v) - j(\overline{u}_\alpha))\, d\gamma \geq L(v - \overline{u}_\alpha) \ \text{ for all } \ v \in V_0. \tag{9}$$

Note that when $j(r) = \frac{1}{2}(r-b)^2$, problem (9) reduces to a variational inequality corresponding to problem (2).

The stationary heat conduction models with nonmonotone multivalued subdifferential interior and boundary semipermeability relations cannot be described by convex potentials. They use locally Lipschitz potentials and their weak formulations lead to hemivariational inequalities, see [27, Chapter 5.5.3] and [28].

In [11], for the problem (8), sufficient conditions were studied that guarantee the existence of a solution and the comparison properties and asymptotic behavior, as $\alpha \to +\infty$, stated in Theorem 2.1. Moreover, continuous dependence of solutions was obtained. In order to provide an existence result for the following elliptic hemivariational inequality

$$\text{find } \overline{u} \in V_0 \text{ such that } a(\overline{u}, v) + \alpha \int_{\Gamma_3} j^0(\overline{u}; v)\, d\gamma \geq h(v) \text{ for all } v \in V_0 \tag{10}$$

with $h \in V_0^*$, in [11], the following hypotheses were considered.

<u>$H(j)$</u>: $j\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is such that

(a) $j(\cdot, r)$ is measurable for all $r \in \mathbb{R}$,

(b) $j(x, \cdot)$ is locally Lipschitz for a.e. $x \in \Gamma_3$,

(c) there exist c_0, $c_1 \geq 0$ such that $|\partial j(x,r)| \leq c_0 + c_1|r|$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_3$,

(d) $j^0(x, r; b-r) \leq 0$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_3$ with a constant $b \in \mathbb{R}$.

Note that the existence results for elliptic hemivariational inequalities can be found in several contributions, see [6, 17, 18, 23, 25, 26, 27, 31, 32, 33]. In comparison to other works, the new hypothesis is $H(j)$(d). Under this condition, in [11], both existence of solution to problem (10) and a convergence result when $\alpha \to \infty$ have been proved. Moreover, if the hypothesis $H(j)$(d) is replaced by the relaxed monotonicity condition (see [11] for details)

$$j^0(x, r; s-r) + j^0(x, s; r-s) \leq m_j\, |r-s|^2$$

for all r, $s \in \mathbb{R}$, a.e. $x \in \Gamma_3$ with $m_j \geq 0$, and the smallness condition

$$m_a > \alpha\, m_j \|\gamma\|^2$$

is assumed, then problem (10) is uniquely solvable, see [26, Lemma 20] for the proof. However, this smallness condition is not suitable in the study of problem (10) since for a sufficiently large value of α, it is not satisfied.

Theorem 2.4. *If $H(j)$ holds, $h \in V_0^*$ and $\alpha > 0$, then the hemivariational inequality* (10) *has a solution.*

Proof. This results applying a surjectivity result in [25, Proposition 3.61] and partially follow arguments of [26, Lemma 20]. Here, we will give an idea of the proof, for details see [11, Theorem 4].

i) If we consider $A : V_0 \to V_0^*$ such that $\langle Au, v\rangle = a(u, v)$, $\forall u, v \in V_0$, we prove that the operator A is a linear, bounded ($\|A(u)\|_{V_0^*} \leq \|u\|_{V_0}$) and coercive ($\langle Av, v\rangle = \|v\|_{V_0}^2$). Moreover, A is a pseudomonotone operator.

ii) Next, we define $F : L^2(\Gamma_3) \to \mathbb{R}$ such that

$$F(y) = \int_{\Gamma_3} j(x, y(x))d\gamma, \; y \in L^2(\Gamma_3).$$

The functional F enjoys the following properties (see [25]).

p_1) F is well defined and Lipschitz continuous on bounded subsets of $L^2(\Gamma_3)$, hence also locally Lipschitz,

p_2) $F^0(y, z) \leq \int_{\Gamma_3} j(x, y(x), z(x))d\gamma$, $y, z \in L^2(\Gamma_3)$.

p_3) $\|\partial F(y)\|_{L^2(\Gamma_3)} \leq \overline{c_1} + \overline{c_2}\|y\|_{L^2(\Gamma_3)}$, $y \in L^2(\Gamma_3)$ with $\overline{c_1}, \overline{c_2} \geq 0$.

iii) Now, we define $B : V_0 \to 2^{V_0^*}$ such that

$$B(v) = \alpha\gamma^* \partial F(\gamma v), \; \forall v \in V_0$$

where $\gamma^* : L^2(\Gamma) \to V_0^*$ denotes the adjoint of the trace γ. B is pseudomonotone and bounded multivalued operator.

iv) We prove that $A + B$ is a bounded, pseudomonotone and coercive multivalued operator, hence also surjective.

v) Next, there exists $u \in V_0$ such that $(A + B)u \ni h$.

vi) We obtain that u solves problem (8). ∎

Note that, from Theorem 4.5 it follows that for each $\alpha > 0$, problem (8) has a solution $u_\alpha \in V_0$ while [6, Corollary 2.102] entails that problem (3) has a unique solution $u_\infty \in K$. Moreover, it is easy to observe that problem (3) can be equivalently formulated as follows

$$\text{find} \;\; u_\infty \in K \;\; \text{such that} \;\; a(u_\infty, v - u_\infty) = L(v - u_\infty) \;\; \text{for all} \;\; v \in K. \tag{11}$$

In what follows we need the hypothesis on the data.

$\underline{(H_0)}$: $g \in H$, $g \leq 0$ in Ω, $q \in Q$, $q \geq 0$ on Γ_2.

THEOREM 2.5. *If $H(j)$, (H_0) hold and $b \geq 0$, then*

(a) $\overline{u}_\alpha \leq b$ *in* Ω,

(b) $\overline{u}_\alpha \leq u_\infty$ *in* Ω,

where $\overline{u}_\alpha \in V_0$ is a solution to problem (8) *and $u_\infty \in K$ is the unique solution to problem* (3).

Proof. a) Let $w = \overline{u}_\alpha - b$. Since $w\big|_{\Gamma_1} = -b \leq 0$, then $w^+\big|_{\Gamma_1} = 0$. If we choose $v = -w^+ \in V_0$ in (8), by (H_0) we have $L(w^+) \leq 0$, then

$$a(w^+, w^+) \leq \alpha \int_{\Gamma_3} j^0(\overline{u}_\alpha; -(\overline{u}_\alpha - b)^+) \, d\gamma.$$

Next, by $H(j)$(d) and the coerciveness of a, we deduce $m_a\|w^+\|_V^2 \leq 0$. Hence $w^+ = 0$ in Ω, and $\overline{u}_\alpha \leq b$ in Ω.

b) If we denote $w = \overline{u}_\alpha - u_\infty$, we have that $w\big|_{\Gamma_1} = 0$. If we take $v = -w^+ \in V_0$ in (8), by (a) we have that $w\big|_{\Gamma_3} = (\overline{u}_\alpha - b)\big|_{\Gamma_3} \leq 0$ and consequently $w^+ \in K_0$. Taking $v = w^+ \in K_0$ in (3), we have

$$a(w^+, w^+) \leq \alpha \int_{\Gamma_3} j^0(\overline{u}_\alpha; -w^+) \, d\gamma.$$

Since $u_\infty = b$ on Γ_3, by $H(j)$(d) and the coerciveness of a, we deduce $m_a\|w^+\|_V^2 \leq 0$. Therefore, $w^+ = 0$ in Ω and $u_\alpha \leq u_\infty$ in Ω. ■

In what follows, we comment on the monotonicity property analogous to condition (iv) stated for problem (3) in Theorem 2.1.

Proposition 2.6. *Assume that $H(j)$ and (H_0) hold, and*

$$j^0(x,r;-(r-s)^+) + c\,j^0(x,s;(r-s)^+) \leq 0 \tag{12}$$

for all $c \geq 1$, all r, $s \in \mathbb{R}$, $r \leq b$, $s \leq b$ and a.e. $x \in \Gamma_3$. Let $\overline{u}_{\alpha_i} \in V_0$ denote the unique solution to the inequality (8) *corresponding to $\alpha_i > 0$, $i = 1, 2$. Then the following monotonicity property holds:*

$$\alpha_1 \leq \alpha_2 \quad \Longrightarrow \quad \overline{u}_{\alpha_1} \leq \overline{u}_{\alpha_2} \ \ \text{in} \ \ \Omega.$$

Proof. Let $0 < \alpha_1 \leq \alpha_2$ and $w = \overline{u}_{\alpha_1} - \overline{u}_{\alpha_2}$ in Ω. It is sufficient to prove that $w^+ = 0$ in Ω. Since $w\big|_{\Gamma_1} = 0$, we have $w^+ \in V_0$. We choose $v = -w^+ \in V_0$ in problem (8) for α_1, $v = w^+ \in V_0$ in problem (8) for α_2 and by adding, we have

$$-a(w,w^+) + \alpha_1 \int_{\Gamma_3} j^0(\overline{u}_{\alpha_1};-w^+)\,d\Gamma + \alpha_2 \int_{\Gamma_3} j^0(\overline{u}_{\alpha_2};w^+)\,d\Gamma \geq 0$$

which implies

$$\begin{aligned} a(w^+,w^+) &\leq \int_{\Gamma_3} \Big(\alpha_1\, j^0(\overline{u}_{\alpha_1};-w^+) + \alpha_2\, j^0(\overline{u}_{\alpha_2};w^+)\Big)\,d\Gamma \\ &= \alpha_1 \int_{\Gamma_3} \Big(j^0(\overline{u}_{\alpha_1};-w^+) + \frac{\alpha_2}{\alpha_1}\, j^0(\overline{u}_{\alpha_2};w^+)\Big)\,d\Gamma \leq 0. \end{aligned}$$

Using the coercivity of the form a, we deduce that $w^+ = 0$, which completes the proof. ■

Next, with the aim of studying the asymptotic behavior of solutions to problem (8) when $\alpha \to \infty$, it is necessary to consider the following additional hypothesis on the superpotential j.

$\underline{(H_1)}$: if $j^0(x,r;b-r) = 0$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_3$, then $r = b$.

Theorem 2.7. *Assume $H(j)$, (H_0) and (H_1). Let $\{\overline{u}_\alpha\} \subset V_0$ be a sequence of solutions to problem* (8) *and $u_\infty \in K$ be the unique solution to problem* (3)*. Then $\overline{u}_\alpha \to u_\infty$ in V, as $\alpha \to +\infty$.*

Proof. We will give a sketch of the proof, see [11, Theorem 7] for details.

i) We prove that the sequence $\{\overline{u}_\alpha\}$ is bounded in V, $\forall \alpha > 0$.

ii) Next, there exists $c_1 > 0$ (independent of α) such that

$$-\int_{\Gamma_3} j^0(\overline{u}_\alpha, u_\infty - \overline{u}_\alpha)d\gamma \leq \frac{c_1}{\alpha}.$$

iii) We obtain that there exists $u^* \in V_0$ such that $\overline{u}_\alpha \rightharpoonup u^*$ weakly in V, as $\alpha \to \infty$.

iv) Next, we prove that u^* satisfies: $a(u^*, w-u^*) \geq L(w-u^*)$, $\forall w \in K$ and we have that $u^* \in K$.

v) We have that $u^* = u_\infty$.

vi) Finally, $\overline{u}_\alpha \to u_\infty$ strongly in V, as $\alpha \to +\infty$. ■

Now, we present a result on continuous dependence of solution to problem (8) on the internal energy g and the heat flux q for fixed $\alpha > 0$. First, we give a previous result.

LEMMA 2.8. *Let $g_n \in H$, $q_n \in Q$ for $n \in \mathbb{N}$. Define $L_n \in V^*$, $n \in \mathbb{N}$, by*

$$L_n(v) = \int_\Omega g_n v\, dx - \int_{\Gamma_2} q_n v\, d\gamma \quad \text{for} \quad v \in V.$$

If $g_n \rightharpoonup g$ weakly in H, $q_n \rightharpoonup q$ weakly in $L^2(\Gamma_2)$, and $v_n \in V$, $v_n \rightharpoonup v$ weakly in V, then

$$L_n(v_n) \to L(v), \quad \text{as} \quad n \to \infty,$$

and there exists a constant $C > 0$ independent of n such that $\|L_n\|_{V^*} \leq C$ for all $n \in \mathbb{N}$.

Proof. The proof results from the compactness of the embedding V into H and of the trace operator from V into $L^2(\Gamma)$. ∎

The continuous dependence result reads as follows.

THEOREM 2.9. *Assume that $\alpha > 0$ is fixed, L, $L_n \in V^*$, $n \in \mathbb{N}$ and $H(j)$ holds. Let $u_n \in V_0$, $n \in \mathbb{N}$, be a solution to problem* (8) *corresponding to L_n, and*

$$\lim L_n(z_n) = L(z) \quad \text{for any} \quad z_n \rightharpoonup z \text{ weakly in } V, \text{ as } n \to \infty. \tag{13}$$

Then, there exists a subsequence of $\{u_n\}$ which converges weakly in V to a solution of problem (8) *corresponding to L. If, in addition, the following hypotheses hold:*

$$j^0(x, r; s - r) + j^0(x, s; r - s) \leq m_j\, |r - s|^2 \quad \text{for all} \quad r, s \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3, \tag{14}$$

$$m_a > \alpha\, m_j \|\gamma\|^2, \tag{15}$$

where $m_j \geq 0$, then problem (8) *has a unique solution u and $u_n \in V_0$ corresponding to L and L_n, respectively, and the whole sequence $\{u_n\}$ converges to u in V, as $n \to \infty$.*

Proof. See [11, Theorem 9] for details. ∎

Finally, we present three examples of functions which satisfy the hypotheses $H(j)$, (H_1) and (14). Note that the first example is a nonconvex function and the second and third examples are convex fuctions. Moreover, the last example allows us to arrive to the Robin boundary condition.

EXAMPLE 2.10. Let $j\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$j(r) = \begin{cases} (r - b)^2 & \text{if} \quad r < b, \\ 1 - e^{-(r-b)} & \text{if} \quad r \geq b \end{cases}$$

for $r \in \mathbb{R}$ with a constant $b \in \mathbb{R}$. This function is nonconvex, locally Lipschitz and its subdifferential is given by

$$\partial j(r) = \begin{cases} 2(r - b) & \text{if} \quad r < b, \\ [0, 1] & \text{if} \quad r = b, \\ e^{-(r-b)} & \text{if} \quad r > b \end{cases}$$

for all $r \in \mathbb{R}$. Hence, we have $|\partial j(r)| \leq 1 + 2|b| + 2|r|$ for all $r \in \mathbb{R}$. Moreover, using Proposition 2.3(ii), one has

$$j^0(r; b-r) = \max\{\zeta\,(b-r) \mid \zeta \in \partial j(r)\} = \begin{cases} -2(b-r)^2 & \text{if} \quad r < b, \\ 0 & \text{if} \quad r = b, \\ e^{-(r-b)}(b-r) & \text{if} \quad r > b \end{cases}$$

for all $r \in \mathbb{R}$. Thus $H(j)$ is satisfied. By the above formula, we also infer that (H_1) is satisfied and the condition (14) holds with $m_j = 1$.

Example 2.11. We define $j\colon \mathbb{R} \to \mathbb{R}$ by

$$j(r) = |r-b| = \begin{cases} -r+b & \text{if} \quad r \leq b, \\ r-b & \text{if} \quad r > b \end{cases}$$

for $r \in \mathbb{R}$ with a constant $b \in \mathbb{R}$. Then, we have for all $r \in \mathbb{R}$

$$\partial j(r) = \begin{cases} -1 & \text{if} \quad r < b, \\ [-1,1] & \text{if} \quad r = b, \\ 1 & \text{if} \quad r > b \end{cases} \quad \text{and} \quad j^0(r; b-r) = \begin{cases} b-r & \text{if} \quad r > b, \\ 0 & \text{if} \quad r = b, \\ r-b & \text{if} \quad r < b \end{cases}$$

for all $r \in \mathbb{R}$. Thus, $j^0(r; b-r) \leq 0$ for all $r \in \mathbb{R}$. Also, we observe that if $j^0(r; b-r) = 0$ for all $r \in \mathbb{R}$, then $r = b$. In consequence, the properties $H(j)$ and (H_1) are verified. Further, since j is convex, it satisfies (14) with $m_j = 0$.

Example 2.12. Let $j\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$j(r) = \frac{1}{2}(r-b)^2$$

for $r \in \mathbb{R}$ with $b \in \mathbb{R}$. Then

$$j^0(r; s) = (r-b)\,s \quad \text{and} \quad \partial j(r) = r-b$$

for r, $s \in \mathbb{R}$. Moreover, we have $j^0(r; b-r) = (r-b)\,(b-r) = -(b-r)^2 \leq 0$ for all $r \in \mathbb{R}$. Also, for all $r \in \mathbb{R}$, if $j^0(r; b-r) = 0$, then $(r-b)\,(b-r) = -(b-r)^2 = 0$, which implies $r = b$. Hence, we deduce that j satisfies properties $H(j)$, (H_1) and j satisfies (14) with $m_j = 0$.

3. Optimal control problems with variational equalities. In this section, we consider optimal control problems related with mixed elliptic problems of type considered in subsection 2.1. More precisely, we review the optimal control problems studied in [12, 13, 14, 15].

3.1. Optimal control problems on the internal energy. In [12], we consider a bounded domain Ω in $\mathbb{R}^d$ which regular boundary Γ consists of the union of two disjoint portions Γ_i, $i = 1, 2$ with $|\Gamma_i| > 0$, where $|\Gamma_i|$ denotes the $(d-1)$-dimensional Hausdorff measure of the portion Γ_i on Γ. We formulate, in a similar way to problems (1) and (2), the following mixed elliptic problems:

$$-\Delta u = g \quad \text{in } \Omega, \quad u|_{\Gamma_1} = b, \quad -\frac{\partial u}{\partial n}\bigg|_{\Gamma_2} = q, \tag{16}$$

$$-\Delta u = g \quad \text{in } \Omega, \quad -\left.\frac{\partial u}{\partial n}\right|_{\Gamma_1} = \alpha(u-b), \quad -\left.\frac{\partial u}{\partial n}\right|_{\Gamma_2} = q, \tag{17}$$

where g is the internal energy in Ω, b is the temperature on Γ_1 for (16) and the temperature of the external neighborhood of Γ_1 for (17), q is the heat flux on Γ_2 and $\alpha > 0$ is the heat transfer coefficient of Γ_1, that satisfy the following assumptions $g \in H$, $q \in Q$, $b \in H^{\frac{1}{2}}(\Gamma_1)$.

We denote by u_g and $u_{\alpha g}$ the unique solutions of the mixed elliptic problems (16) and (17), respectively, for which variational equalities are given by [20]

$$a\left(u_g, v\right) = L_g(v), \quad \forall v \in V_0, \quad u_g \in K, \tag{18}$$

$$a_\alpha\left(u_{\alpha g}, v\right) = L_{\alpha g}(v), \quad \forall v \in V, \quad u_{\alpha g} \in V, \tag{19}$$

where

$$V = H^1(\Omega), \quad V_0 = \{v \in V : v = 0 \text{ on } \Gamma_1\}, \quad K = v_0 + V_0,$$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad a_\alpha(u,v) = a(u,v) + \alpha \int_{\Gamma_1} uv d\gamma,$$

$$L_g(v) = (g,v) - \int_{\Gamma_2} qv d\gamma, \quad L_{\alpha g}(v) = L_g(v) + \alpha \int_{\Gamma_1} bv d\gamma$$

for a given $v_0 \in V$, $v_0|_{\Gamma_1} = b$.

We consider the following distributed optimal control problems [22, 39] given by:

$$\text{find} \quad g^* \in H \quad \text{such that} \quad J(g^*) = \min_{g \in H} J(g) \tag{20}$$

with

$$J(g) = \frac{1}{2}||u_g - z_d||_H^2 + \frac{M}{2}||g||_H^2 \tag{21}$$

where u_g is the unique solution to the variational equality (18), $z_d \in H$ given and M a positive constant.

For each $\alpha > 0$, we formulate the following distributed optimal control problem:

$$\text{find} \quad g_\alpha^* \in H \quad \text{such that} \quad J_\alpha(g_\alpha^*) = \min_{g \in H} J_\alpha(g) \tag{22}$$

with

$$J_\alpha(g) = \frac{1}{2}||u_{\alpha g} - z_d||_H^2 + \frac{M}{2}||g||_H^2 \tag{23}$$

where $u_{\alpha g}$ is a solution to the problem (19), $z_d \in H$ given and M a positive constant.

In [12], following [22], we prove existence and uniqueness of optimal solution to the problem (20) and (22), for each $\alpha > 0$. For this purpose, we define the following mappings.

Let $C : H \to V_0$ be the mapping such that $C(g) = u_g - u_0$, where u_0 is the solution of problem (18) for $g = 0$. Let $\Pi : H \times H \to \mathbb{R}$ and $L : H \to \mathbb{R}$ be defined by the following expressions:

$$\Pi(g,h) = (C(g), C(h)) + M(g,h), \quad \forall g, h \in H,$$

$$L(g) = (C(g), z_d - u_0), \quad \forall g \in H.$$

For each $\alpha > 0$, we define $C_\alpha : H \to V_0$ such that $C_\alpha(g) = u_{\alpha g} - u_{\alpha 0}$, where $u_{\alpha 0}$ is the solution of problem (19) for $g = 0$. Let $\Pi_\alpha : H \times H \to \mathbb{R}$ and $L_\alpha : H \to \mathbb{R}$ be defined by the following expressions:

$$\Pi_\alpha(g,h) = (C_\alpha(g), C_\alpha(h)) + M(g,h), \quad \forall g,h \in H,$$

$$L_\alpha(g) = (C_\alpha(g), z_d - u_{\alpha 0}), \quad \forall g \in H.$$

We obtain the following results, whose proofs can be seen in [12].

Lemma 3.1. *a) C is a linear and continuous mapping, Π is a bilinear, continuous, symmetric and coercive form on $H \times H$ and L is linear and continuous on H.*

b) The functional J can be also written as

$$J(g) = \frac{1}{2}\Pi(g,h) - L(g) + \frac{1}{2}\left\|u_0 - z_d\right\|_H^2, \quad \forall g \in H.$$

c) There exists a unique optimal control $g^* \in H$ such that

$$J\left(g^*\right)) = \min_{g\in H} J(g).$$

Lemma 3.2. *For each $\alpha > 0$, we have:*

a) C_α is a linear and continuous mapping, Π_α is a bilinear, continuous, symmetric and coercive form on $H \times H$ and L_α is linear and continuous on H.

b) The functional J_α can be also written as

$$J_\alpha(g) = \frac{1}{2}\Pi_\alpha(g,h) - L_\alpha(g) + \frac{1}{2}\left\|u_{\alpha 0} - z_d\right\|_H^2, \quad \forall g \in H.$$

c) There exists a unique optimal control $g_\alpha^* \in H$ such that

$$J_\alpha\left(g_\alpha^*\right) = \min_{g\in H} J_\alpha(g).$$

We define the adjoint state p_g corresponding to (16) or (18), for each $g \in H$, as the unique solution of the following mixed elliptic problem

$$-\Delta p_g = u_g - z_d \text{ in } \Omega, \quad p_g|_{\Gamma_1} = 0, \quad \left.\frac{\partial p_g}{\partial n}\right|_{\Gamma_2} = 0,$$

whose variational formulation is given by

$$a\left(p_g, v\right) = \left(u_g - z_d, v\right), \quad \forall v \in V_0, \quad p_g \in V_0. \tag{24}$$

For each $\alpha > 0$, we define the adjoint state $p_{\alpha g}$ as the unique solution of the following mixed elliptic problem corresponding to (17) or (19), for each $g \in H$

$$-\Delta p_{\alpha g} = u_{\alpha g} - z_d \text{ in } \Omega, \quad -\left.\frac{\partial p_{\alpha g}}{\partial n}\right|_{\Gamma_1} = \alpha p_{\alpha g}, \quad \left.\frac{\partial p_{\alpha g}}{\partial n}\right|_{\Gamma_2} = 0,$$

which variational formulation is given by

$$a_\alpha\left(p_{\alpha g}, v\right) = \left(u_{\alpha g} - z_d, v\right), \quad \forall v \in V, \quad p_{\alpha g} \in V. \tag{25}$$

Next, we give the optimality conditions to the problems (20) and (22).

Lemma 3.3. *a) The optimality condition for problem (20) is given by $J'\left(g^*\right) = 0$ in H, that is,*

$$p_{g^*} + Mg^* = 0 \textit{ in } H.$$

b) For each $\alpha > 0$, the optimality condition for problem (22) is given by $J'_\alpha(g^*_\alpha) = 0$ in H, that is,

$$p_{\alpha g^*_\alpha} + M g^*_\alpha = 0 \text{ in } H.$$

Proof. a) This results taking into account that $\forall g, h \in H$

$$(J'(g), h) = (u_g - z_d, C(h)) + M(g, h) = \Pi(g, h) - L(g)$$

and

$$(u_g - z_d, C(h)) = a(p_g, C(h)) = (p_g, h).$$

b) For each $\alpha > 0$, we have that $\forall g, h \in H$

$$\langle J'_\alpha(g), h\rangle = (u_{\alpha g} - z_d, C_\alpha(h)) + M(g, h) = \Pi_\alpha(g, h) - L_\alpha(g),$$

and

$$(u_{\alpha g} - z_\alpha, C_\alpha(h)) = a_\alpha(p_{\alpha g}, C_\alpha(h)) = (p_{\alpha g}, h). \;\blacksquare$$

Now, we consider the operator $W : H \to V_0 \subset H$ defined by

$$W(g) = -\frac{1}{M} p_g, \quad g \in H$$

and for each $\alpha > 0$, the operator $W_\alpha : H \to V_0 \subset H$ defined by

$$W_\alpha(g) = -\frac{1}{M} p_{\alpha g}, \quad g \in H.$$

We prove the following property.

LEMMA 3.4. *a) W is a Lipschitz operator over H, i.e.*

$$\|W(g_2) - W(g_1)\|_H \leq \frac{1}{\lambda^2 M} \|g_1 - g_2\|_H, \quad \forall g_1, g_2 \in H,$$

and it is a contraction for all $M > 1/\lambda^2$, where λ is the coerciveness constant of the bilinear form a.

b) W_α is a Lipschitz operator over H, i.e.

$$\|W_\alpha(g_2) - W_\alpha(g_1)\|_H \leq \frac{1}{\lambda_\alpha^2 M} \|g_1 - g_2\|_H, \quad \forall g_1, g_2 \in H,$$

and it is a contraction for all $M > 1/\lambda_\alpha^2$, where λ_α is the coerciveness constant of the bilinear form a_α.

Proof. a) By using the coerciveness of the bilinear form a we have

$$\lambda \|p_{g_2} - p_{g_1}\|_V^2 \leq a(p_{g_2} - p_{g_1}, p_{g_2} - p_{g_1}) \leq \|u_{g_2} - u_{g_1}\|_H \|p_{g_2} - p_{g_1}\|_H$$

therefore

$$\|p_{g_2} - p_{g_1}\|_v \leq \frac{1}{\lambda} \|u_{g_2} - u_{g_1}\|_H$$

and taking into account that the mapping $g \in H \to u_g \in V$ is Lipschitzian, that is,

$$\|u_{g_2} - u_{g_1}\|_V \leq \frac{1}{\lambda} \|g_2 - g_1\|_H, \quad \forall g_1, g_2 \in H$$

we obtain

$$\|W(g_2) - W(g_1)\|_H \leq \frac{1}{\lambda^2 M} \|g_1 - g_2\|_H.$$

b) In a similar way that (a), by using the coerciveness of the bilinear form a_α, we obtain that

$$\|p_{\alpha g_2} - p_{\alpha g_1}\|_v \leq \frac{1}{\lambda}\|u_{\alpha g_2} - u_{\alpha g_1}\|_H$$

and taking into account that $g \in H \to u_{\alpha g} \in V$ is a Lipschitzian application, we have

$$\|W_\alpha(g_2) - W_\alpha(g_1)\|_H \leq \frac{1}{\lambda_\alpha^2 M}\|g_1 - g_2\|_H . \blacksquare$$

We have a convergence result for fixed data, when α goes to infinity.

LEMMA 3.5. *For all $\alpha > 0$, $q \in Q$ and $b \in H^{\frac{1}{2}}(\Gamma_1)$, we have that:*

a) $u_{\alpha g} \to u_g$ strongly in V as $\alpha \to +\infty$, $\forall g \in H$.
b) $p_{\alpha g} \to p_g$ strongly in V as $\alpha \to +\infty$, $\forall g \in H$.

Proof. An idea of the proof is as follows, for details see [12, Lemma 3.5].

a) We prove that:

i) The sequence $\{u_{\alpha g}\}$ is bounded in V, $\forall \alpha > 0$.
ii) There exists $c_1 > 0$ (independent of α) such that

$$\int_{\Gamma_1}(u_{\alpha g} - b)^2 d\gamma \leq \frac{(c_1)^2}{\lambda_1(\alpha - 1)}.$$

iii) There exists $w_g \in V$ such that $u_{\alpha g} \rightharpoonup w_g$ weakly in V, as $\alpha \to \infty$.
iv) $w_g \in K$ satisfies $a(w_g, v) = L(v)$, $\forall v \in V_0$.
v) By uniqueness, we have that $w_g = u_g$.
vi) $u_{\alpha g} \to u_g$ strongly in V, as $\alpha \to +\infty$.

b) We obtain that:

i) The sequence $\{p_{\alpha g}\}$ is bounded in V, $\forall \alpha > 0$.
ii) There exists $c_2 > 0$ (independent of α) such that

$$\int_{\Gamma_1}(p_{\alpha g} - p_g)^2 d\gamma \leq \frac{(c_2)^2}{\lambda_1(\alpha - 1)}.$$

iii) There exists $\xi_g \in V$ such that $u_{\alpha g} \rightharpoonup \xi_g$ weakly in V, as $\alpha \to +\infty$.
iv) $\xi_g \in V_0$ satisfies $a(\xi_g, v) = (u_g - z_d, v)$, $\forall v \in V_0$.
v) By uniqueness, $\xi_g = p_g$.
vi) $p_{\alpha g} \to p_g$ strongly in V, as $\alpha \to +\infty$. $\blacksquare$

In [12], we obtain the following convergence result for the optimal solutions g_α^*, $u_{\alpha g_\alpha^*}$ and $p_{\alpha g_\alpha^*}$ of the optimal control problems (22) to the optimal solutions g^*, u_{g^*} and p_{g^*} of the problem (20), when the parameter α goes to infinity. This result is presented as follows.

THEOREM 3.6. *If $M > \frac{1}{\lambda_1}$, with λ_1 the coerciveness constant of a_1, we have that, when $\alpha \to +\infty$:*

a) If g^* and g_α^* are the unique solutions of the optimal control problems (20) and (22), respectively, then $g_\alpha^* \to g^*$ strongly in H.

b) If u_{g^*} and $u_{\alpha g^*_\alpha}$ are the system states corresponding to problems (18) and (19), respectively, then $u_{\alpha g^*_\alpha} \to u_{g^*}$ strongly in V.

c) If p_{g^*} and $p_{\alpha g^*_\alpha}$ are the adjoint states corresponding to problems (18) and (19), respectively, then $p_{\alpha g^*_\alpha} \to p_{g^*}$ strongly in V.

Proof. We will give a scheme of the proof in three steps. For details see [12, Theorem 4.1].

STEP 1. By using that g^*_α is the unique solution of problem (22), we obtain that there exist positive constants c_1, c_2 and c_3 such that

$$||g^*_\alpha||_H \le c_1; \quad ||u_{\alpha g^*_\alpha}||_V \le c_2; \quad \int_{\Gamma_1} (u_{\alpha g^*_\alpha} - u_{g^*})^2 d\gamma \le \frac{c_3}{\lambda_1(\alpha - 1)}.$$

Therefore, we deduce that there exist $f \in H$ and $\eta \in K$ such that $g^*_\alpha \rightharpoonup f$ weakly in H and $u_{\alpha g^*_\alpha} \rightharpoonup \eta$ weakly in V, as $\alpha \to +\infty$. Next, taking $v = p_{\alpha g^*_\alpha} - p_{g^*} \in V$ in (25), we prove that there exist positive constants c_4 and c_5 such that

$$||p_{\alpha g^*_\alpha}||_V \le c_4; \quad \int_{\Gamma_1} (p_{\alpha g^*_\alpha} - p_{g^*})^2 d\gamma \le \frac{c_5}{\lambda_1(\alpha - 1)}$$

and there exists $\xi \in V_0$ such that $p_{\alpha g^*_\alpha} \rightharpoonup \xi$ weakly in V, as $\alpha \to +\infty$.

STEP 2. Taking $v \in V_0$ in (25) and (19), respectively, and by passing to the limits, we obtain

$$a(\xi . v) = (\eta - z_d, v), \qquad \forall v \in V_0, \quad \xi \in V_0. \tag{26}$$

and

$$a(\eta . v) = (f, v) - \int_{\Gamma_2} qv \, d\gamma, \qquad \forall v \in V_0, \quad \eta \in K. \tag{27}$$

Now, by using Lemma 3.4, we have $f = -\frac{1}{M}\xi$ in H. From the uniqueness of fixed point we have $g^* = -\frac{1}{M}p_{g^*}$ in H and therefore, $f = g^*$, $\eta = u_{g^*}$ and $\xi = p_{g^*}$.

STEP 3. The strong convergence are obtained by the previous weak convergence and the following inequalities:

$$\lambda_1 ||p_{\alpha g^*_\alpha} - p_{g^*}||^2_V \le (u_{\alpha g^*_\alpha} - z_d, p_{\alpha g^*_\alpha} - p_{g^*}) - a(p_{g^*}, p_{\alpha g^*_\alpha} - p_{g^*}),$$

$$||g^*_\alpha - g^*||_H \le \frac{1}{M} ||p_{\alpha g^*_\alpha} - p_{g^*}||_V,$$

$$\lambda_1 ||u_{\alpha g^*_\alpha} - u_{g^*}||^2_V \le a(u_{\alpha g^*} - u_{g^*}, u_{\alpha g^*_\alpha} - u_{g^*}). \ \blacksquare$$

In [13], we obtain a new proof of the convergence results obtained in [12] for the optimal solutions of the optimal control problems (22) to the optimal solutions of the problem (20), when $\alpha \to \infty$. This result is given as follows.

THEOREM 3.7. *We have that, when $\alpha \to +\infty$:*

a) If g^* and g^*_α are the unique solutions of the optimal control problems (20) and (22), respectively, then $g^*_\alpha \to g^*$ strongly in H.

b) If u_{g^*} and $u_{\alpha g^*_\alpha}$ are the system states corresponding to problems (18) and (19), respectively, then $u_{\alpha g^*_\alpha} \to u_{g^*}$ strongly in V.

c) If p_{g^*} and $p_{\alpha g^*_\alpha}$ are the adjoint states corresponding to problems (18) and (19), respectively, then $p_{\alpha g^*_\alpha} \to p_{g^*}$ strongly in V.

Proof. This proof is different from the previous theorem in step 2, for details see [13, Theorem 4.1]. That is, by variational equalities (26) and (27), from uniqueness of solution of the variational equalities (19) and (24), we have $\eta = u_f$ and $\xi = p_f$, respectively. Now, taking into account that $\forall h \in H$

$$J(f) = J_\alpha(f) \leq \liminf_{\alpha\to\infty} J_\alpha(g^*_\alpha) \leq \liminf_{\alpha\to\infty} J_\alpha(h) = \lim_{\alpha\to\infty} J_\alpha(h) = J(h)$$

and from the uniqueness of the optimal control, we obtain that $f = g^*$. Therefore $\eta = u_f = u_{g^*}$ and $\xi = p_f = p_{g^*}$. ■

3.2. Optimal control problems on the heat flux. In [14], we consider the mixed elliptic problems (16) and (17) and we denote by u_q and $u_{\alpha q}$ the unique solutions of the following variational equalities:

$$a\left(u_q, v\right) = L_q(v), \quad \forall v \in V_0, \quad u_q \in K, \tag{28}$$

$$a_\alpha\left(u_{\alpha q}, v\right) = L_{q\alpha}(v), \quad \forall v \in V, \quad u_{\alpha q} \in V, \tag{29}$$

where V, V_0, K, a and a_α are given as in the previous subsection and

$$L_q(v) = (g, v) - \int_{\Gamma_2} qv d\gamma, \quad L_{\alpha q}(v) = L_q(v) + \alpha \int_{\Gamma_1} bv d\gamma.$$

We consider $U_{ad} = \{q \in Q : q \geq 0 \text{ on } \Gamma_2\}$ and we formulate the following distributed optimal control problems [22, 39]:

$$\text{find} \quad q^* \in U_{ad} \quad \text{such that} \quad J_2(q^*) = \min_{q\in U_{ad}} J_2(q) \tag{30}$$

with

$$J_2(q) = \frac{1}{2}||u_q - z_d||^2_H + \frac{M}{2}||q||^2_Q \tag{31}$$

where u_q is the unique solution to the variational equality (28), $z_d \in H$ is given and M is a positive constant. For each $\alpha > 0$, we formulate the following distributed optimal control problem:

$$\text{find} \quad q^*_\alpha \in U_{ad} \quad \text{such that} \quad J_{2\alpha}(q^*_\alpha) = \min_{q\in U_{ad}} J_{2\alpha}(q) \tag{32}$$

with

$$J_{2\alpha}(q) = \frac{1}{2}||u_{\alpha q} - z_d||^2_H + \frac{M}{2}||q||^2_Q \tag{33}$$

where $u_{\alpha q}$ is a solution to the problem (29), $z_d \in H$ given and M a positive constant.

In [14], in a similar way to [12], we prove existence and uniqueness of optimal solutions to the problems (30) and (32).

LEMMA 3.8. *a) There exists a unique optimal control $q^* \in U_{ad}$ to the problem (30).*

b) For each $\alpha > 0$, there exists a unique optimal control $q^*_\alpha \in U_{ad}$ to the problem (32).

Proof. This results in a similar way to Lemma 3.1 and Lemma 3.2. For details see [14, Lemma 1 and Lemma 6]. ■

LEMMA 3.9. *a) The optimality condition for the optimal control problem (30) is given by*

$$(Mq^* - p_{q^*}, \eta - q^*)_Q \geq 0, \quad \forall \eta \in U_{ad}, \quad q^* \in U_{ad}. \tag{34}$$

b) For each $\alpha > 0$, the optimality condition for the optimal control problem (32) is given by

$$(Mq^*_\alpha - p_{\alpha q^*_\alpha}, \eta - q^*)_Q \geq 0, \quad \forall \eta \in U_{ad}, \quad q^*_\alpha \in U_{ad}. \tag{35}$$

Proof. The inequalities (34) and (35) results following [20, 22] and taking into account that, the Gateaux derivative for J_2 is given by

$$\begin{aligned}(J_2'(q), \eta - q) &= (u_\eta - u_q, u_q - z_d) + M(q, \eta - q)_Q \\ &= -(p_q, \eta - q)_Q + M(q, \eta - q)_Q, \qquad \forall \eta, q \in Q\end{aligned}$$

and for each $\alpha > 0$, the Gateaux derivative for $J_{2\alpha}$ is given by

$$\begin{aligned}(J_{2\alpha}'(q), \eta - q) &= (u_{\alpha\eta} - u_{\alpha q}, u_{\alpha q} - z_d) + M(q, \eta - q)_Q \\ &= -(p_{\alpha q}, \eta - q)_Q + M(q, \eta - q)_Q, \qquad \forall \eta, q \in Q. \ \blacksquare\end{aligned}$$

Now, we give the following characterization of the optimal controls.

THEOREM 3.10. *a) Let $q^* \in U_{ad}$ be, q^* is optimal control in Q if and only if $q^* \in Q$ satisfies the complementary conditions*

$$q^* \geq 0 \text{ on } \Gamma_2, \quad Mq^* - p_{q^*} \geq 0 \text{ on } \Gamma_2, \quad q^*(Mq^* - p_{q^*}) = 0 \text{ on } \Gamma_2.$$

b) For each $\alpha > 0$, let $q^*_\alpha \in U_{ad}$ be, q^*_α is optimal control in Q if and only if $q^*_\alpha \in Q$ satisfies the complementary conditions

$$q^*_\alpha \geq 0 \text{ on } \Gamma_2, \quad Mq^*_\alpha - p_{\alpha q^*_\alpha} \geq 0 \text{ on } \Gamma_2, \quad q^*_\alpha(Mq^*_\alpha - p_{\alpha q^*_\alpha}) = 0 \text{ on } \Gamma_2.$$

Proof. We present an idea of the proof, for more details see [14, Theorems 4 and 9].

a) If we take $\eta = 0 \in U_{ad}$ and $\eta = 2q^* \in U_{ad}$ in (34), we obtain

$$(Mq^* - p_{q^*}, q^*) = 0$$

next

$$(Mq^* - p_{q^*}, \eta) \geq (Mq^* - p_{q^*}, q^*) = 0, \quad \forall \eta \in U_{ad}$$

therefore $Mq^* - p_{q^*} \geq 0$ on Γ_2 and since $q^* \geq 0$ on Γ_2, we have that

$$(Mq^* - p_{q^*})q^* = 0.$$

Conversely, $\forall \eta \in U_{ad}$ we have

$$(Mq^* - p_{q^*}, \eta - q^*) = (Mq^* - p_{q^*}, \eta) \geq 0$$

therefore q^* is the optimal control in Q.

b) By taking $\eta = 0 \in U_{ad}$ and $\eta = 2q^*_\alpha \in U_{ad}$ in (35) and following a similar way as in (a), we have (b). $\blacksquare$

COROLLARY 3.11. *If we consider the boundary optimal control problems (30) and (32) without restrictions (i.e., $U_{ad} = Q$), we obtain that $q^* = \frac{1}{M} p_{q^*}$ and $q^*_\alpha = \frac{1}{M} p_{\alpha q^*_\alpha}$, respectively, similar to [12].*

In a similar way to the previous subsection, we can prove the following convergence results.

LEMMA 3.12. *For all $\alpha > 0$, $g \in H$ and $b \in H^{\frac{1}{2}}(\Gamma_1)$, we have that:*

a) $u_{\alpha q} \to u_q$ strongly in V as $\alpha \to +\infty$, $\forall q \in Q$.

b) $p_{\alpha q} \to p_q$ *strongly in* V *as* $\alpha \to +\infty$, $\forall q \in Q$.

Proof. An idea of the proof is as follows, for details see [14, Theorem 11].

a) We prove that:

   i) If we take $v = u_{\alpha q} - u_q$ in (29) with $\alpha > 1$, then there exists $c_1 > 0$ (independent of α) such that

$$\lambda_1 ||u_{\alpha q} - u_q||_V^2 + (\alpha - 1) \int_{\Gamma_1} (u_{\alpha q} - u_q)^2 d\gamma \leq c_1 ||u_{\alpha q} - u_q||_V,$$

   where λ_1 is the coerciveness constant of a_1.

   ii) Then, we deduce that there exists $w_q \in V$ such that $u_{\alpha q} \rightharpoonup w_q$ weakly in V, as $\alpha \to \infty$ and

$$\int_{\Gamma_1} (u_{\alpha q} - b)^2 d\gamma \leq \frac{(c_1)^2}{\lambda_1(\alpha - 1)};$$

   iii) Moreover, $w_q \in K$ satisfies $a(w_q, v) = L(v)$, $\forall v \in V_0$ and by uniqueness, we have that $w_q = u_q$;

   iv) Finally, from the inequality

$$\lambda_1 ||u_{\alpha q} - u_q||_V^2 \leq L_q(u_{\alpha q} - u_q) - a(u_q, u_{\alpha q} - u_q)$$

   we obtain that $u_{\alpha q} \to u_q$ strongly in V, as $\alpha \to +\infty$.

b) This results in a similar way to (a). ∎

THEOREM 3.13. *We have that, when* $\alpha \to +\infty$*:*

a) If q^* *and* q^*_α *are the unique solutions of the optimal control problems (30) and (32), respectively, then* $q^*_\alpha \to q^*$ *strongly in* Q.

b) If u_{q^*} *and* $u_{\alpha q^*_\alpha}$ *are the system states corresponding to problems (18) and (19), respectively, then* $u_{\alpha q^*_\alpha} \to u_{q^*}$ *strongly in* V.

c) If p_{q^*} *and* $p_{\alpha q^*_\alpha}$ *are the adjoint states corresponding to problems (18) and (19), respectively, then* $p_{\alpha q^*_\alpha} \to p_{q^*}$ *strongly in* V.

Proof. We will give a scheme of the proof in three steps. For details see [14, Theorem 12].

STEP 1. By using that q^*_α is the unique solution of problem (32), we obtain that there exist positive constants c_1, c_2 and c_3 such that

$$||q^*_\alpha||_Q \leq c_1; \quad ||u_{\alpha q^*_\alpha}||_V \leq c_2; \quad \int_{\Gamma_1} (u_{\alpha q^*_\alpha} - u_{q^*})^2 d\gamma \leq \frac{c_3}{\lambda_1(\alpha - 1)}.$$

Therefore, we deduce that there exist $f \in Q$ and $\eta \in K$ such that $q^*_\alpha \rightharpoonup f$ weakly in Q and $u_{\alpha q^*_\alpha} \rightharpoonup \eta$ weakly in V, as $\alpha \to +\infty$. Next, taking $v = p_{\alpha q^*_\alpha} - p_{q^*} \in V$ in (25), we prove that there exist positive constants c_4 and c_5 such that

$$||p_{\alpha q^*_\alpha}||_V \leq c_4; \quad \int_{\Gamma_1} (p_{\alpha q^*_\alpha} - p_{q^*})^2 d\gamma \leq \frac{c_5}{\lambda_1(\alpha - 1)}$$

and there exists $\xi \in V_0$ such that $p_{\alpha q^*_\alpha} \rightharpoonup \xi$ weakly in V, as $\alpha \to +\infty$.

STEP 2. Taking $v \in V_0$ in (25) and (4), respectively, and by passing to the limits, we obtain

$$a(\xi.v) = (\eta - z_d, v), \quad \forall v \in V_0, \quad \xi \in V_0. \tag{36}$$

and

$$a(\eta.v) = (f, v) - \int_{\Gamma_2} qv \, d\gamma, \quad \forall v \in V_0, \quad \eta \in K. \tag{37}$$

Next, from the uniqueness of solution of the variational equality (19) and (24), we have $\eta = u_f$ and $\xi = p_f$, respectively. Now, taking into account that $\forall h \in Q$

$$J_2(f) = J_{2\alpha}(f) \leq \liminf_{\alpha\to\infty} J_{2\alpha}(q^*_\alpha) \leq \liminf_{\alpha\to\infty} J_{2\alpha}(h) = \lim_{\alpha\to\infty} J_{2\alpha}(h) = J_2(h)$$

and from the uniqueness of the optimal control, we obtain that $f = q^*$. Therefore $\eta = u_f = u_{q^*}$ and $\xi = p_f = p_{q^*}$.

STEP 3. The strong convergence is obtained by the previous weak convergence and the following inequalities

$$\lambda_1 ||p_{\alpha q^*_\alpha} - p_{q^*}||^2_V \leq (u_{\alpha q^*_\alpha} - z_d, p_{\alpha q^*_\alpha} - p_{q^*}) - a(p_{q^*}, p_{\alpha q^*_\alpha} - p_{q^*}),$$

$$||q^*_\alpha - q^*||_Q \leq \frac{1}{M} ||p_{\alpha q^*_\alpha} - p_{q^*}||_V,$$

$$||u_{\alpha q^*_\alpha} - u_{q^*}||_V \leq \frac{||\gamma||}{\lambda} ||q^*_\alpha - q^*||_Q$$

where γ denote the trace operator. ■

3.3. Simultaneous optimal control problems on the internal energy and the heat flux. In [15], we consider the mixed elliptic problems (16) and (17) and we denote by u_{gq} and $u_{\alpha gq}$ the unique solutions of the following variational equalities:

$$a(u_{gq}, v) = L_{gq}(v), \quad \forall v \in V_0, \quad u_{gq} \in K, \tag{38}$$

$$a_\alpha(u_{\alpha gq}, v) = L_{\alpha gq}(v), \quad \forall v \in V, \quad u_{\alpha gq} \in V, \tag{39}$$

where V, V_0, K, a and a_α are defined as in previous subsections and

$$L_{gq}(v) = (g, v) - \int_{\Gamma_2} qv d\gamma, \quad L_{\alpha gq}(v) = L_{gq}(v) + \alpha \int_{\Gamma_1} bv d\gamma.$$

We consider $U_{ad} = \{q \in Q : q \geq 0 \text{ on } \Gamma_2\}$ and we formulate the following simultaneous distributed-boundary optimal control problems [39]:

$$\text{find} \quad (\overline{g}, \overline{q}) \in H \times U_{ad} \quad \text{such that} \quad J_3(\overline{g}, \overline{q}) = \min_{(g,q)\in H\times U_{ad}} J_3(g, q) \tag{40}$$

with

$$J_3(g, q) = \frac{1}{2}||u_{gq} - z_d||^2_H + \frac{M_1}{2}||g||^2_H + \frac{M_2}{2}||q||^2_Q \tag{41}$$

and, for each $\alpha > 0$

$$\text{find} \quad (\overline{g}_\alpha, \overline{q}_\alpha) \in H \times U_{ad} \text{ such that } J_{3\alpha}(\overline{g}_\alpha, \overline{q}_\alpha) = \min_{(g,q)\in H\times U_{ad}} J_{3\alpha}(g, q) \tag{42}$$

with

$$J_{3\alpha}(g, q) = \frac{1}{2}||u_{\alpha gq} - z_d||^2_H + \frac{M_1}{2}||g||^2_H + \frac{M_2}{2}||q||^2_Q \tag{43}$$

where u_{gq} is the unique solution to the variational equality (38), $u_{\alpha gq}$ is a solution to the problem (39), $z_d \in H$ is given and M_1 and M_2 are positive constants.

In [15], in a similar way to [12, 14], we prove existence and uniqueness results of optimal solutions to the problem (40) and (42).

LEMMA 3.14. *a) There exists a unique optimal control* $(\overline{g}, \overline{q}) \in H \times U_{ad}$ *to the problem (40) and the optimality condition is given by*

$$(h - \overline{g}, p_{\overline{g}\,\overline{q}} + M_1\overline{g}) + (\eta - \overline{q}, M_2\overline{q} - p_{\overline{g}\,\overline{q}})_Q \geq 0, \quad \forall (h, \eta) \in H \times U_{ad}. \tag{44}$$

b) For each $\alpha > 0$, *there exists a unique optimal control* $(\overline{g}_\alpha, \overline{q}_\alpha) \in H \times U_{ad}$ *to the problem (42) and the optimality condition is given by* $\forall (h, \eta) \in H \times U_{ad}$

$$(h - \overline{g}_\alpha, p_{\alpha\overline{g}_\alpha\overline{q}_\alpha} + M_1\overline{g}_\alpha) + (\eta - \overline{q}_\alpha, M_2\overline{q}_\alpha - p_{\alpha\overline{g}_\alpha\overline{q}_\alpha})_Q \geq 0. \tag{45}$$

Proof. The proof results in a similar way to Lemma 3.1, Lemma 3.2, Lemma 3.8 and Lemma 3.9. For details see [15, Theorem 1 and Theorem 2]. ■

If we consider the simultaneous distributed and boundary optimal control problems (40) and (42) without restrictions, i.e. $U_{ad} = Q$, we can characterize their solutions by using the fixed point theory.

We consider the norm in $H \times Q$ defined by

$$||(g, q)||^2_{H\times Q} = ||g||^2_H + ||q||^2_Q \qquad \forall (g, q) \in H \times Q.$$

We define the operator $W : H \times Q \to H \times Q$ by

$$W(g, q) = \left(-\frac{1}{M_1} p_{gq}, \frac{1}{M_2} p_{gq} \right) \tag{46}$$

and for each $\alpha > 0$, the operator $W_\alpha : H \times Q \to H \times Q$ by the expression

$$W_\alpha(g, q) = \left(-\frac{1}{M_1} p_{\alpha gq}, \frac{1}{M_2} p_{\alpha gq} \right) \tag{47}$$

and we can prove the following result.

THEOREM 3.15. *a)* W *is a Lipschitz operator over* $H \times Q$, *that is, there exists a positive constant* $C_0 = C_0(\lambda, \gamma, M_1, M_2)$ *such that,* $\forall (g_1, q_1), (g_2, q_2) \in H \times Q$

$$\|W(g_2, q_2) - W(g_1, q_1)\|_{H\times Q} \leq C_0 \|(g_2, q_2) - (g_1, q_1)\|_{H\times Q} \tag{48}$$

and W *is a contraction operator if and only if data satisfy that*

$$C_0 = \frac{\sqrt{2}}{\lambda^2} \sqrt{\frac{1}{M_1^2} + \frac{\|\gamma\|^2}{M_2^2}} (1 + \|\gamma\|) < 1. \tag{49}$$

b) W_α *is a Lipschitz operator over* $H \times Q$, *that is, there exists a positive constant* $C_{0\alpha} = C_{0\alpha}(\lambda_\alpha, \gamma, M_1, M_2)$, *such that*

$$\|W_\alpha(g_2, q_2) - W_\alpha(g_1, q_1)\|_{H\times Q} \leq C_{0\alpha} \|(g_2 - g_1, q_2 - q_1)\|_{H\times Q} \tag{50}$$

and W_α *is a contraction operator if and only if data satisfy that*

$$C_{0\alpha} = \frac{\sqrt{2}}{\lambda_\alpha^2} \sqrt{\frac{1}{M_1^2} + \frac{\|\gamma\|^2}{M_2^2}} (1 + \|\gamma\|) < 1. \tag{51}$$

Proof. This results by estimates between the direct and adjoint states and the vector control variable. For details see [15, Theorem 4 and Theorem 6]. ■

COROLLARY 3.16. *a) If data satisfy inequality (49) then the unique solution* $(\overline{g},\overline{q}) \in H\times Q$ *of optimal control problem (40) can be obtained as the unique fixed point of the operator* W*, that is*

$$W(\overline{g},\overline{q}) = (-\frac{1}{M_1}p_{\overline{g}\,\overline{q}}, \frac{1}{M_2}p_{\overline{g}\,\overline{q}}) = (\overline{g},\overline{q}).$$

b) If data satisfy inequality $C_{0\alpha} < 1$*, then the unique solution* $(\overline{g}_\alpha,\overline{q}_\alpha) \in H \times Q$ *of the vectorial optimal control problem (42) can be obtained as the unique fixed point of the operator* W_α*, that is:*

$$W_\alpha(\overline{g}_\alpha,\overline{q}_\alpha) = (-\frac{1}{M_1}p_{\alpha\overline{g}_\alpha\overline{q}_\alpha}, \frac{1}{M_2}p_{\alpha\overline{g}_\alpha\overline{q}_\alpha}) = (\overline{g}_\alpha,\overline{q}_\alpha).$$

Now, we present the convergence results for the simultaneous distributed-boundary optimal control problems (40) and (42).

LEMMA 3.17. *For each* $\alpha > 0$, $(g,q) \in H \times Q$, $b \in H^{1/2}(\Gamma_1)$*, we have:*

a) $u_{\alpha gq} \to u_{gq}$ *strongly in* V *as* $\alpha \to +\infty$.
b) $p_{\alpha gq} \to p_{gq}$ *strongly in* V *as* $\alpha \to +\infty$.

Proof. The proof is similar to that of Lemma 3.5 and Lemma 3.12. An idea of the proof is as follows, for details see [15, Lemma 1].

a) We prove that:

i) If we take $v = u_{\alpha gq} - u_{gq}$ in (39) with $\alpha > 1$, then there exists $c_1 > 0$ (independent of α) such that

$$\lambda_1||u_{\alpha gq} - u_{gq}||_V^2 + (\alpha - 1)\int_{\Gamma_1}(u_{\alpha gq} - u_q)^2 d\gamma \le c_1||u_{\alpha gq} - u_{gq}||_V,$$

where λ_1 is the coerciveness constant of a_1;

ii) Then, we deduce that there exists $w_q \in V$ such that $u_{\alpha gq} \rightharpoonup w_{gq}$ weakly in V, as $\alpha \to \infty$ and

$$\int_{\Gamma_1}(u_{\alpha gq} - b)^2 d\gamma \le \frac{(c_1)^2}{\lambda_1(\alpha - 1)};$$

iii) Moreover, $w_{gq} \in K$ satisfies $a(w_{gq}, v) = L(v)$, $\forall v \in V_0$ and by uniqueness, we have that $w_{gq} = u_{gq}$;

iv) Finally, from the inequality

$$\lambda_1||u_{\alpha gq} - u_{gq}||_V^2 \le L_{gq}(u_{\alpha gq} - u_{gq}) - a(u_{gq}, u_{\alpha gq} - u_{gq})$$

we obtain that $u_{\alpha gq} \to u_{gq}$ strongly in V, as $\alpha \to +\infty$.

b) This results in a similar way to (a). ■

THEOREM 3.18. *We have that, when* $\alpha \to +\infty$*:*

a) If $(\overline{g},\overline{q})$ *and* $(\overline{g}_\alpha,\overline{q}_\alpha)$ *are the unique solutions of the optimal control problems (40) and (42), respectively, then* $(\overline{g}_\alpha,\overline{q}_\alpha) \to (\overline{g},\overline{q})$ *strongly in* $H \times Q$.

b) If $u_{\overline{g}\,\overline{q}}$ *and* $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}$ *are the system states corresponding to problems (18) and (19), respectively, then* $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \to u_{\overline{g}\,\overline{q}}$ *strongly in* V.

c) *If* $p_{\overline{g}\,\overline{q}}$ *and* $p_{\alpha\overline{g}_\alpha\overline{q}_\alpha}$ *are the adjoint states corresponding to problems (18) and (19), respectively, then* $p_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \to p_{\overline{g}\,\overline{q}}$ *strongly in* V.

Proof. We will give a scheme of the proof in three steps. For details see [15, Theorem 7].

STEP 1. By using that $(\overline{g}_\alpha, \overline{q}_\alpha)$ is the unique solution of problem (42), we obtain that there exist positive constants c_1, c_2, c_3 and c_4 such that

$$||\overline{g}_\alpha||_H \le c_1; \quad ||\overline{q}_\alpha||_Q \le c_2; \quad ||u_{\alpha\overline{g}_\alpha\overline{q}_\alpha}||_V \le c_3; \quad ||p_{\alpha\overline{g}_\alpha\overline{q}_\alpha}||_V \le c_4.$$

Therefore, we deduce that there exist $h \in H$, $f \in Q$, $\eta \in K$ and $\xi \in V_0$ such that $\overline{g}_\alpha \rightharpoonup h$ weakly in H, $\overline{q}_\alpha \rightharpoonup f$ weakly in Q, $u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \rightharpoonup \eta$ weakly in V and $p_{\alpha\overline{g}_\alpha\overline{q}_\alpha} \rightharpoonup \xi$ weakly in V, as $\alpha \to +\infty$.

STEP 2. Taking $v \in V_0$ in (19) and passing to the limits, we obtain

$$a(\eta.v) = (h, v) - \int_{\Gamma_2} fv\,d\gamma, \quad \forall v \in V_0, \quad \eta \in K. \tag{52}$$

Next, by uniqueness of solution of the variational equality (18), we have $\eta = u_{hf}$. For $v \in V_0$ in (25) and passing to the limits, we have

$$a(\xi.v) = (u_{hf} - z_d, v), \quad \forall v \in V_0, \quad \xi \in V_0. \tag{53}$$

and by the uniqueness of solution of the variational equality (24), we have $\xi = p_{hf}$. Now, taking into account that $\forall (h', f') \in H \times Q$

$$\begin{aligned} J_3(h, f) &\le \liminf_{\alpha\to\infty} J_{3\alpha}(\overline{g}_\alpha, \overline{q}_\alpha) \le \liminf_{\alpha\to\infty} J_{3\alpha}(h', f') \\ &= \lim_{\alpha\to\infty} J_{3\alpha}(h', f') = J_3(h', f') \end{aligned}$$

and from the uniqueness of the optimal control, we obtain that $h = \overline{g}$ and $f = \overline{q}$. Therefore $u_{hf} = u_{\overline{g}\,\overline{q}}$ and $p_{hf} = p_{\overline{g}\,\overline{q}}$.

STEP 3. The strong convergence is obtained by the previous weak convergence and the following inequalities

$$||\overline{g}_\alpha - \overline{g}||_H \le \frac{1}{M_1}||p_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - p_{\overline{g}\,\overline{q}}||_V, \qquad ||\overline{q}_\alpha - \overline{q}||_Q \le \frac{||\gamma||}{M_2}||p_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - p_{\overline{g}\,\overline{q}}||_V.$$

For $\alpha > 1$

$$\begin{aligned} \lambda_1\|u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - u_{\overline{g}\,\overline{q}}\|_V^2 \le{}& (g, u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - u_{\overline{g}\,\overline{q}})_H - (q, u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - u_{\overline{g}\,\overline{q}})_Q \\ & - a(u_{\overline{g}\,\overline{q}}, u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - u_{\overline{g}\,\overline{q}}) \end{aligned}$$

and

$$\begin{aligned} \lambda_1\|p_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - p_{\overline{g}\,\overline{q}}\|_V^2 \le{}& (u_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - z_d, p_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - p_{\overline{g}\,\overline{q}})_H \\ & - a(p_{\overline{g}\,\overline{q}}, p_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - p_{\overline{g}\,\overline{q}}) - \alpha(p_{\overline{g}\,\overline{q}}, p_{\alpha\overline{g}_\alpha\overline{q}_\alpha} - p_{\overline{g}\,\overline{q}})_{L^2(\Gamma_1)} \end{aligned}$$

where λ_1 is the coerciveness constant of bilinear form a_1. ∎

4. Optimal control problems with hemivariational inequalities. In this section, we consider optimal control problems related with mixed elliptic problems governed by variational and hemivariational inequalities considered in subsection 2.2. More precisely, we will review the optimal control problems studied in [4, 16].

4.1. Optimal control problems on the internal energy. We consider distributed optimal control problems of the type studied in [12, 22, 39] given by:

$$\text{find} \quad g^* \in H \quad \text{such that} \quad I(g^*) = \min_{g \in H} I(g) \tag{54}$$

with

$$I(g) = \frac{1}{2}||u_{\infty g} - z_d||_H^2 + \frac{M}{2}||g||_H^2 \tag{55}$$

where $u_{\infty g}$ is the unique solution to the variational equality (3), $z_d \in H$ given and M a positive constant.

For each $\alpha > 0$, we formulate the following distributed optimal control problem

$$\text{find} \quad g_\alpha^* \in H \quad \text{such that} \quad I_\alpha(g_\alpha^*) = \min_{g \in H} I_\alpha(g) \tag{56}$$

with

$$I_\alpha(g) = \frac{1}{2}||\overline{u}_{\alpha g} - z_d||_H^2 + \frac{M}{2}||g||_H^2 \tag{57}$$

where $\overline{u}_{\alpha g}$ is a solution to the hemivariational inequality (8), $z_d \in H$ given and M a positive constant.

In [16], for each $\alpha > 0$, we obtain an existence result of optimal solutions to the optimal control problem (56). Moreover, asymptotic behavior of optimal controls and system states of the problem (56), when the parameter α goes to infinity, was studied.

Now, we pass to a result on existence of solution to the optimal control problem (56) in which the system is governed by the hemivariational inequality (8).

THEOREM 4.1. *For each $\alpha > 0$, if $H(j)$ holds, then the distributed optimal control problems* (56) *has a solution.*

Proof. We give a sketch of the proof. For details, see [16, Theorem 2].

i) For each $\alpha > 0$ and $g \in H$, we have

$$m = \inf\{I_\alpha(g), g \in H, \overline{u}_{\alpha g} \in T_\alpha^1(g)\} \geq 0$$

with $T_\alpha^1(g)$ the set of solutions of (8).

ii) If $g_n^\alpha \in H$ is a minimizing sequence, then there exist positive constants k_1 and k_2 such that

$$||g_n^\alpha||_H \leq k_1 \quad \text{and} \quad ||\overline{u}_{\alpha g_n^\alpha}||_{V_0} \leq k_2.$$

iii) Therefore, there exist $f \in H$ and $\eta_\alpha \in V_0$ such that

$$\overline{u}_{\alpha g_n^\alpha} \rightharpoonup \eta_\alpha \text{ weakly in } V_0 \qquad \text{and} \qquad g_n^\alpha \rightharpoonup f \text{ weakly in } H.$$

iv) Next, we have that $\eta_\alpha \in V_0$ satisfies

$$a(\eta_\alpha, v) + \alpha \int_{\Gamma_3} j^0(\eta_\alpha; v)\, d\gamma \geq \int_\Omega f v\, dx - \int_{\Gamma_2} q v\, d\gamma \quad \text{for all} \quad v \in V_0$$

and therefore $\eta_\alpha = \overline{u}_{\alpha f}$, where $\overline{u}_{\alpha f}$ is a solution of the problem (8) for data $f \in H$ and $q \in Q$.

v) Finally, we have that $m \geq I_\alpha(f)$ and therefore, $(f, \overline{u}_{\alpha f})$ is an optimal pair to optimal control problem (56). ∎

In what follows, we present the asymptotic behavior of the optimal solutions to problem (56), when $\alpha \to +\infty$.

THEOREM 4.2. *Assume $H(j)$ and (H_1). If $(g_\alpha, \overline{u}_{\alpha g_\alpha})$ is an optimal solution to problem (56) and $(g^*, u_{\infty g^*})$ is the unique solution to problem (54), then $g_\alpha \to g^*$ strongly in H and $\overline{u}_{\alpha g_\alpha} \to u_{\infty g^*}$ strongly in V, when $\alpha \to +\infty$.*

Proof. We will make a sketch of the proof in three steps. For details see [16, Theorem 3].

STEP 1. For all $\alpha > 0$, we prove that the sequence $(g_\alpha, \overline{u}_{\alpha g_\alpha})$ is bounded in $H \times H$, that is

$$||g_\alpha||_H \leq k_1 \qquad ||\overline{u}_{\alpha g_\alpha}||_V \leq k_2$$

for positive constants k_1 and k_2. Next, we have that, there exists $k_3 > 0$ (independent of α) such that

$$-\int_{\Gamma_3} j^0(\overline{u}_{\alpha g_\alpha}, u_{\infty g^*} - \overline{u}_{\alpha g_\alpha}) d\gamma \leq \frac{k_3}{\alpha}.$$

Therefore, we obtain that, there exist $\eta \in V$ and $h \in H$ such that, as $\alpha \to +\infty$

$$\overline{u}_{\alpha g_\alpha} \rightharpoonup \eta \text{ weakly in } V \quad \text{and} \quad g_\alpha \rightharpoonup h \text{ weakly in } H.$$

STEP 2. Since V_0 is sequentially weakly closed in V, $\eta \in V_0$ and

$$\eta \in V_0 \ \text{ satisfies } \ L(w - \eta) \leq a(\eta, w - \eta) \ \text{ for all } \ w \in K.$$

Next, we obtain that $\eta \in K$ and

$$\eta \in K \ \text{ satisfies } \ a(\eta, v) = L(v) \ \text{ for all } \ v \in K_0,$$

i.e., $\eta \in K$ is a solution to problem (3) and by the uniqueness of solution to problem (3), we have $\eta = u_{\infty h}$. From the uniqueness of the optimal control problem (65), we obtain $h = g^*$. Therefore, when $\alpha \to +\infty$

$$g_\alpha \rightharpoonup g^* \ \text{ weakly in } \ H \quad \text{and} \quad \overline{u}_{\alpha g_\alpha} \rightharpoonup u_{\infty g^*} \ \text{ weakly in } \ V.$$

STEP 3. We have that

$$m_a \|u_{\infty g^*} - \overline{u}_{\alpha g_\alpha}\|_V^2 \leq a(u_{\infty g^*}, u_{\infty g^*} - \overline{u}_{\alpha g_\alpha}) + L(\overline{u}_{\alpha g_\alpha} - u_{\infty g^*}).$$

Next, from the weak continuity of $a(u_{\infty g^*}, \cdot)$, the compactness of the trace operator and $\overline{u}_{\alpha g_\alpha} \to u_{\infty g^*}$ strongly in H,

$$\overline{u}_{\alpha g_\alpha} \to u_{\infty g^*} \text{ strongly in } V, \quad \text{when} \quad \alpha \to +\infty.$$

Finally, as $g_\alpha \rightharpoonup g^*$ weakly in H and $||g_\alpha||_H \to ||g^*||_H$, we deduce that

$$g_\alpha \to g^* \text{ strongly in } H \quad \text{when} \quad \alpha \to +\infty. \ \blacksquare$$

4.2. Optimal control problems on the heat flux. We consider the boundary optimal control problems studied in [4], which are given by

$$\text{find} \quad q^* \in Q \quad \text{such that} \quad I_2(q^*) = \min_{q \in Q} I_2(q) \tag{58}$$

with

$$I_2(q) = \frac{1}{2}||u_{\infty q} - z_d||_H^2 + \frac{M}{2}||q||_Q^2 \tag{59}$$

and, for each $\alpha > 0$, the problem

$$\text{find} \quad q_\alpha^* \in Q \quad \text{such that} \quad I_{2\alpha}(q_\alpha^*) = \min_{q \in Q} I_{2\alpha}(q) \tag{60}$$

with

$$I_{2\alpha}(q) = \frac{1}{2}||\overline{u}_{\alpha q} - z_d||_H^2 + \frac{M}{2}||q||_Q^2 \tag{61}$$

where $u_{\infty q}$ is the unique solution to the variational equality (3), $\overline{u}_{\alpha q}$ is a solution to the hemivariational inequality (8), $z_d \in H$ given and M a positive constant.

It is know, by [14], that there exists a unique optimal solution $q^* \in Q$ of the boundary optimal control problem (58). In [4], existence of solution to the optimal control problem (60), which is governed by the hemivariational inequality (8), has been proved. This result is presented as follows.

THEOREM 4.3. *For each $\alpha > 0$, if $H(j)$ holds, then the boundary optimal control problems (60) has a solution.*

Proof. We denote, for each $\alpha > 0$ and each $q \in Q$, by $T_\alpha^2(q)$ the set of solutions of (8) and we have that

$$m = \inf\{I_{2\alpha}(q), q \in Q, \overline{u}_{\alpha q} \in T_\alpha^2(q)\} \geq 0. \tag{62}$$

Next, for each $\alpha > 0$, we consider $q_n^\alpha \in Q$ a minimizing sequence to (62) and we prove that there exist $\xi_\alpha \in Q$ and $\eta_\alpha \in V_0$ such that, when $n \to \infty$

$$\overline{u}_{\alpha q_n^\alpha} \rightharpoonup \eta_\alpha \text{ weakly in } V_0 \qquad \text{and} \qquad q_n^\alpha \rightharpoonup \xi_\alpha \text{ weakly in } Q.$$

After that, we obtain that $\eta_\alpha = \overline{u}_{\alpha\xi_\alpha}$ where $\overline{u}_{\alpha\xi_\alpha}$ is a solution of the hemivariational inequality (8) for data $\xi_\alpha \in Q$ and $g \in H$. Finally, we prove that

$$m \geq I_{2\alpha}(\xi_\alpha)$$

and therefore ξ_α is an optimal solution to optimal control problem (60). ∎

In [4], following [16], has been studied the asymptotic behavior of optimal solutions of the problems (60) when the parameter α goes to infinity. This result is presented as follows.

THEOREM 4.4. *Assume $H(j)$ and (H_1). If q_α^* is an optimal solution to problem (60) and q^* is the unique solution to problem (58), then $q_\alpha^* \to q^*$ strongly in Q and $\overline{u}_{\alpha q_\alpha^*} \to u_{\infty q^*}$ strongly in V, when $\alpha \to +\infty$.*

Proof. We give the scheme of the proof in three steps. For details see [4, Theorem 3.2].

STEP 1. Since q_α^* is an optimal solution to problem (60), we deduce that there exist positive constants k_1 and k_2 such that

$$||q_\alpha^*||_Q \leq k_1, \quad ||\overline{u}_{\alpha q_\alpha^*}||_V \leq k_2.$$

Moreover, there exists a positive constant k_3 such that

$$-\int_{\Gamma_3} j^0(\overline{u}_{\alpha q_\alpha^*}; u_{\infty q^*} - \overline{u}_{\alpha q_\alpha^*})\, d\gamma \leq \frac{k_3}{\alpha}.$$

Therefore, there exist $\eta \in V$ and $\xi \in Q$ such that

$$\overline{u}_{\alpha q_\alpha^*} \rightharpoonup \eta \;\text{ weakly in }\; V, \text{ as } \alpha \to +\infty, \tag{63}$$

$$q_\alpha^* \rightharpoonup \xi \ \text{ weakly in } \ Q, \ \text{as } \alpha \to +\infty. \tag{64}$$

STEP 2. We obtain that

$$\eta \in K \ \text{ satisfies } \ a(\eta, v) = L(v) \ \text{ for all } \ v \in K_0,$$

i.e., $\eta \in K$ is a solution to problem (3) and by the uniqueness of solution to problem (3), we have $\eta = u_{\infty\xi}$ and hence $\overline{u}_{\alpha q_\alpha^*} \rightharpoonup u_{\infty\xi}$ weakly in V, as $\alpha \to +\infty$. Next, $\forall q \in Q$

$$I_2(\xi) \leq \liminf_{\alpha\to+\infty} I_{2\alpha}(q_\alpha^*) \leq \liminf_{\alpha\to\infty} I_{2\alpha}(q) = \lim_{\alpha\to\infty} I_{2\alpha}(q) = I_2(q)$$

and from the uniqueness of the optimal control problem (58), we obtain that $\xi = q^*$, therefore $u_{\infty\xi} = u_{\infty q^*}$. Therefore, when $\alpha \to +\infty$

$$q_\alpha^* \rightharpoonup q^* \ \text{ weakly in } \ Q \quad \text{and} \quad \overline{u}_{\alpha q_\alpha^*} \rightharpoonup u_{\infty q^*} \ \text{ weakly in } \ V.$$

STEP 3. By $H(j)$(d) and the coerciveness of the form a, we obtain

$$m_a \left\| u_{\infty q^*} - \overline{u}_{\alpha q_\alpha^*} \right\|_V^2 \leq a(u_{\infty q^*}, u_{\infty q^*} - \overline{u}_{\alpha q_\alpha^*}) + L(\overline{u}_{\alpha q_\alpha^*} - u_{\infty q^*}).$$

Next, we have that $\overline{u}_{\alpha q_\alpha^*} \to u_{\infty q^*}$ strongly in V as $\alpha \to \infty$. Now, from $\overline{u}_{\alpha q_\alpha^*} \to u_{\infty q^*}$ strongly in H and as $q_\alpha^* \rightharpoonup q^*$ weakly in Q we obtain

$$I_2(q^*) \leq \liminf_{\alpha\to\infty} I_{2\alpha}(q_\alpha^*).$$

On the other hand, from the definition of q_α^* and taking into account that $\overline{u}_{\alpha q^*} \to u_{\infty q^*}$ strongly in H, we obtain

$$\limsup_{\alpha\to\infty} I_{2\alpha}(q_\alpha^*) \leq \limsup_{\alpha\to\infty} I_{2\alpha}(q^*) = I_2(q^*)$$

and therefore

$$\lim_{\alpha\to\infty} \left(\frac{1}{2} ||\overline{u}_{\alpha q_\alpha^*} - z_d||_H^2 + \frac{M}{2} ||q_\alpha^*||_Q^2 \right) = \frac{1}{2} ||u_{\infty q^*} - z_d||_H^2 + \frac{M}{2} ||q^*||_Q^2.$$

Finally, when $\alpha \to +\infty$, we have $||q_\alpha^*||_Q^2 \to ||q^*||_Q^2$ and as $q_\alpha^* \rightharpoonup q^*$ weakly in Q, we deduce that $q_\alpha^* \to q^*$ strongly in Q. ■

4.3. Simultaneous optimal control problems on the internal energy and the heat flux. We consider the simultaneous distributed and Neumann boundary optimal control problems studied in [4]. These problems are given by

$$\text{find} \quad (\overline{g}, \overline{q}) \in H \times Q \quad \text{such that} \quad I_3(\overline{g}, \overline{q}) = \min_{(g,q)\in H\times Q} I_3(g, q) \tag{65}$$

with

$$I_3(g, q) = \frac{1}{2} ||u_{\infty gq} - z_d||_H^2 + \frac{M_1}{2} ||g||_H^2 + \frac{M_2}{2} ||q||_Q^2 \tag{66}$$

where $u_{\infty gq}$ is the unique solution to the variational equality (3), $z_d \in H$ given and M_1 and M_2 are given positive constants. For each $\alpha > 0$, the following simultaneous distributed and Neumann boundary optimal control problem

$$\text{find} \quad (\overline{g}_\alpha, \overline{q}_\alpha) \in H \times Q \quad \text{such that} \quad I_{3\alpha}(\overline{g}_\alpha, \overline{q}_\alpha) = \min_{(g,q)\in H\times Q} I_{3\alpha}(g, q) \tag{67}$$

with

$$I_{3\alpha}(g, q) = \frac{1}{2} ||\overline{u}_{\alpha gq} - z_d||_H^2 + \frac{M_1}{2} ||g||_H^2 + \frac{M_2}{2} ||q||_Q^2 \tag{68}$$

where $\overline{u}_{\alpha gq}$ is a solution to the hemivariational inequality (8), $z_d \in H$ is given and M_1 and M_2 are positive constants.

It is known, by [15], that there exists a unique optimal pair $(\overline{g}, \overline{q}) \in H \times Q$ of the simultaneous distributed-boundary optimal control problem (65). In similar way to [16], in [4] a result on existence of solution to the simultaneous optimal control problem (67) which is governed by the hemivariational inequality (8) has been proved. This result and an idea of its proof are presented as follows.

Theorem 4.5. *For each $\alpha > 0$, if $H(j)$ holds, then the simultaneous distributed-boundary optimal control problem (67) governed by the hemivariational inequality (8) has a solution.*

Proof. i) For each $\alpha > 0$ and $(g, q) \in H \times Q$, we have

$$m = inf\{I_{3\alpha}(g, q), (g, q) \in H \times Q, \overline{u}_{\alpha gq} \in T^3_\alpha(g, q)\} \geq 0$$

with $T^3_\alpha(g, q)$ the set of solutions of (8).

ii) Next, if $(g^\alpha_n, q^\alpha_n) \in H \times Q$ is a minimizing sequence, there exist positive constants k_1, k_2 and k_3 such that, as $n \to \infty$

$$||g^\alpha_n||_H \leq k_1, \quad ||q^\alpha_n||_Q \leq k_2 \quad \text{and} \quad ||\overline{u}_{\alpha g^\alpha_n q^\alpha_n}||_{V_0} \leq k_3.$$

iii) Therefore, there exist $f_\alpha \in H$, $\xi_\alpha \in Q$ and $\eta_\alpha \in V_0$ such that

$$q^\alpha_n \rightharpoonup \xi_\alpha \text{ weakly in } Q, \qquad g^\alpha_n \rightharpoonup f_\alpha \text{ weakly in } H$$

$$\overline{u}_{\alpha g^\alpha_n q^\alpha_n} \rightharpoonup \eta_\alpha \text{ weakly in } V_0.$$

iv) Next, we prove that $\eta_\alpha \in V_0$ satisfies

$$a(\eta_\alpha, v) + \alpha \int_{\Gamma_3} j^0(\eta_\alpha; v)\, d\gamma \geq \int_\Omega f_\alpha v\, dx - \int_{\Gamma_2} \xi_\alpha v\, d\gamma \ \forall v \in V_0$$

and therefore $\eta_\alpha = \overline{u}_{\alpha f_\alpha \xi_\alpha}$, where $\overline{u}_{\alpha f_\alpha \xi_\alpha}$ is a solution of the (8) for data $f_\alpha \in H$ and $\xi_\alpha \in Q$.

v) Finally, we have $m \geq I_{3\alpha}(f_\alpha, \xi_\alpha)$ and therefore, (f_α, ξ_α) is an optimal pair for optimal control problem (67). ■

The asymptotic behavior of the optimal solutions to problem (67) when α goes to infinity, studied in [4], is presented as follows.

Theorem 4.6. *Assume $H(j)$ and (H_1). If $(\overline{g}_\alpha, \overline{q}_\alpha)$ is an optimal solution to simultaneous distributed and Neumann boundary optimal control problem (67) and $(\overline{g}, \overline{q})$ is the unique solution to simultaneous optimal control problem (65), then $(\overline{g}_\alpha, \overline{q}_\alpha) \to (\overline{g}, \overline{q})$ in $H \times Q$ strongly and $\overline{u}_{\alpha \overline{g}_\alpha \overline{q}_\alpha} \to u_{\infty \overline{g}\, \overline{q}}$ in V strongly, when $\alpha \to \infty$.*

Proof. We give a sketch of the proof. For details see [4, Theorem 5.1].

Step 1. For all $\alpha > 0$, the sequence (g_α, q_α) is bounded in $H \times Q$ and $\overline{u}_{\alpha g_\alpha q_\alpha}$ is bounded in H, that is

$$||g_\alpha||_H \leq k_1, \quad ||q_\alpha||_Q \leq k_2, \quad ||\overline{u}_{\alpha g_\alpha q_\alpha}||_V \leq k_3$$

for positive constants k_1, k_2 and k_3. Moreover, there exists $k_4 > 0$ (independent of α) such that

$$-\int_{\Gamma_3} j^0(\overline{u}_{\alpha g_\alpha q_\alpha}, u_{\infty \overline{g}\, \overline{q}} - \overline{u}_{\alpha g_\alpha q_\alpha}) d\gamma \leq \frac{k_4}{\alpha}.$$

Next, we prove that there exist $\eta \in V$, $h \in H$ and $p \in Q$ such that, as $\alpha \to +\infty$

$$\overline{u}_{\alpha g_\alpha q_\alpha} \rightharpoonup \eta \text{ weakly in } V$$

$$g_\alpha \rightharpoonup h \text{ weakly in } H \quad \text{and} \quad q_\alpha \rightharpoonup p \text{ weakly in } Q.$$

STEP 2. Since V_0 is sequentially weakly closed in V, $\eta \in V_0$ satisfies

$$L(w-\eta) \leq a(\eta, w-\eta) \ \text{ for all } \ w \in K.$$

Next, we obtain that $\eta \in K$ and

$$\eta \in K \ \text{ satisfies } \ a(\eta, v) = L(v) \ \text{ for all } \ v \in K_0,$$

i.e., $\eta \in K$ is a solution to problem (3) and by the uniqueness of solution to problem (3), we have that $\eta = u_{hp}$. From the uniqueness of the optimal control problem (65), we obtain

$$h = \overline{g} \quad \text{and} \quad p = \overline{q}.$$

Therefore, when $\alpha \to +\infty$

$$g_\alpha \rightharpoonup \overline{g} \ \text{ weakly in } \ H, \quad q_\alpha \rightharpoonup \overline{q} \ \text{ weakly in } \ Q$$

$$\overline{u}_{\alpha g_\alpha q_\alpha} \rightharpoonup u_{\infty \overline{g}\,\overline{q}} \ \text{ weakly in } \ V.$$

STEP 3. We have

$$m_a \left\| u_{\infty \overline{g}\,\overline{q}} - \overline{u}_{\alpha g_\alpha q_\alpha} \right\|_V^2 \leq a(u_{\infty \overline{g}\,\overline{q}}, u_{\infty \overline{g}\,\overline{q}} - \overline{u}_{\alpha g_\alpha q_\alpha}) + L(\overline{u}_{\alpha g_\alpha q_\alpha} - u_{\infty \overline{g}\,\overline{q}}).$$

Next, from the weak continuity of $a(\overline{u}_{\overline{g}\,\overline{q}}, \cdot)$, the compactness of the trace operator and $\overline{u}_{\alpha g_\alpha q_\alpha} \to u_{\infty \overline{g}\,\overline{q}}$ strongly in H,

$$\overline{u}_{\alpha g_\alpha q_\alpha} \to u_{\infty \overline{g}\,\overline{q}} \text{ strongly in } V, \quad \text{when} \quad \alpha \to +\infty.$$

Finally, as $g_\alpha \rightharpoonup \overline{g}$ weakly in H, $q_\alpha \rightharpoonup \overline{q}$ weakly in Q

$$||g_\alpha||_H \to ||\overline{g}||_H \quad \text{and} \quad ||q_\alpha||_Q \to ||\overline{q}||_Q$$

we deduce that, as $\alpha \to +\infty$

$$g_\alpha \to \overline{g} \text{ strongly in } H \quad \text{and} \quad q_\alpha \to \overline{q} \text{ strongly in } Q. \ \blacksquare$$

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