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Convergence of Distributed Optimal Controls on the Internal Energy in Mixed Elliptic Problems when the Heat Transfer Coefficient Goes to Infinity*

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Abstract. We consider a steady-state heat conduction problem P_α with mixed boundary conditions for the Poisson equation depending on a positive parameter α, which represents the heat transfer coefficient on a portion Γ_1 of the boundary of a given bounded domain in R^n. We formulate distributed optimal control problems over the internal energy g for each α. We prove that the optimal control g_{op_α} and its corresponding system $u_{g_{\mathrm{op}_\alpha}\alpha}$ and adjoint $p_{g_{\mathrm{op}_\alpha}\alpha}$ states for each α are strongly convergent to g_{op}, $u_{g_{\mathrm{op}}}$ and $p_{g_{\mathrm{op}}}$, respectively, in adequate functional spaces. We also prove that these limit functions are respectively the optimal control, and the system and adjoint states corresponding to another distributed optimal control problem for the same Poisson equation with a different boundary condition on the portion Γ_1. We use the fixed point and elliptic variational inequality theories.

Key Words. Variational inequality, Distributed optimal control, Mixed elliptic problem, Adjoint state, Steady-state Stefan problem, Optimality condition, Fixed point.

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1. Introduction

We consider a bounded domain Ω in $\mathbb{R}^n$ whose regular boundary Γ consists of the union of two disjoint portions Γ_1 and Γ_2 with $\text{meas}(\Gamma_1) > 0$ and $\text{meas}(\Gamma_2) > 0$. We denote with $\text{meas}(\Gamma)$ the $(n-1)$-dimensional Lebesgue measure of Γ.

We consider the following two steady-state heat conduction problems P and P_α (for each parameter $\alpha > 0$), respectively, with mixed boundary conditions:

$$-\Delta u = g \quad \text{in } \Omega, \qquad u|_{\Gamma_1} = b, \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q, \tag{1}$$

and

$$-\Delta u = g \quad \text{in } \Omega, \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_1} = \alpha(u-b), \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q, \tag{2}$$

where g is the internal energy in Ω, b is the temperature on Γ_1 for (1) and the temperature of the external neighborhood of Γ_1 for (2), q is the heat flux on Γ_2 and $\alpha > 0$ is the heat transfer coefficient of Γ_1 (Newton's law on Γ_1), that satisfy the following assumptions:

$$g \in H = L^2(\Omega), \qquad q \in L^2(\Gamma_2), \qquad b \in H^{1/2}(\Gamma_1). \tag{3}$$

Problems (1) and (2) can be considered as the steady-state Stefan problem for suitable data q, g and b [5], [8], [11], [17], [18], [20].

Let u_g and $u_{g\alpha}$ be the unique solutions of the mixed elliptic problems (1) and (2), respectively, whose variational equalities are given by [14]

$$a(u_g, v) = L_g(v), \qquad \forall v \in V_0, \quad u_g \in K, \tag{4}$$

and

$$a_\alpha(u_{g\alpha}, v) = L_{g\alpha}(v), \qquad \forall v \in V, \quad u_{g\alpha} \in V, \tag{5}$$

where

$$\begin{gathered} V = H^1(\Omega), \qquad V_0 = \{v \in V / v|_{\Gamma_1} = 0\}, \\ K = v_0 + V_0, \qquad (g,h) = (g,h)_H = \int_\Omega gh\,dx, \\ a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx, \qquad a_\alpha(u,v) = a(u,v) + \alpha\int_{\Gamma_1} bv\,d\gamma, \\ L_g(v) = (g,v)_H - \int_{\Gamma_2} qv\,d\gamma, \qquad L_{g\alpha}(v) = L_g(v) + \alpha\int_{\Gamma_1} bv\,d\gamma \end{gathered} \tag{6}$$

for a given $v_0 \in V$, $v_0|_{\Gamma_1} = b$.

We consider g as a control variable for the cost functionals $J: H \to \mathbb{R}_0^+$ and $J_\alpha: H \to \mathbb{R}_0^+$ respectively given by

$$J(g) = \tfrac{1}{2}\|u_g - z_d\|_H^2 + \frac{M}{2}\|g\|_H^2 \tag{7}$$

and

$$J_\alpha(g) = \tfrac{1}{2}\|u_{g\alpha} - z_d\|_H^2 + \frac{M}{2}\|g\|_H^2, \tag{8}$$

where $z_d \in H$ is given and $M = \text{const.} > 0$.

Then we can formulate the following distributed optimal control problems [7], [9], [10], [15]:

$$\text{Find} \quad g_{\text{op}} \in H \quad \text{such that} \quad J(g_{\text{op}}) = \min_{g \in H} J(g) \tag{9}$$

and

$$\text{Find} \quad g_{\text{op}_\alpha} \in H \quad \text{such that} \quad J_\alpha(g_{\text{op}_\alpha}) = \min_{g \in H} J_\alpha(g), \tag{10}$$

respectively.

The use of variational inequality theory in connection with optimal control problems was done, for example, in [1]–[4], [6], [13] and [16]. In [12] an optimization problem corresponding to (1) is studied in order to avoid a change phase process.

In Section 2 we prove that the functional J is coercive and Gâteaux differentiable on H, and J' is a Lipschitzian and strictly monotone application on H. We also prove the existence and uniqueness of the distributed optimal control problem (9) and we characterize this optimal energy g_{op} as a fixed point on H of a suitable operator W over its adjoint state p_g for a large parameter M.

Similary, in Section 3 we prove that the functional J_α is coercive and Gâteaux differentiable on H, and J'_α is a Lipschitzian and strictly monotone application on H for all $\alpha > 0$. We also prove the existence and uniqueness of the distributed optimal control problem (10) and we characterize this optimal energy g_{op_α} as a fixed point on H of a suitable operator W_α over its adjoint state $p_{g\alpha}$ for a large parameter M.

In Section 4 we study the convergence when $\alpha \to \infty$ of the optimal control problem (10) corresponding to the state system (2). We prove that the optimal state system $u_{g_{\text{op}_\alpha}\alpha}$ and the optimal adjoint system $p_{g_{\text{op}_\alpha}\alpha}$ of problem (10) are strongly convergent in V to the corresponding $u_{g_{\text{op}}}$ and $p_{g_{\text{op}}}$ for problem (9), respectively, when $\alpha \to \infty$. Finally, the strong convergence in H of the optimal control g_{op_α} of problem (10) to the optimal control g_{op} of problem (9) is also proved when $\alpha \to \infty$.

2. Problem P and Its Corresponding Optimal Control Problem

Let $C\colon\ H \to V_0$ be the application such that

$$C(g) = u_g - u_0, \tag{11}$$

where u_0 is the solution of problem (4) for $g = 0$ whose variational equality is given by

$$a(u_0, v) = L_0(v), \qquad \forall v \in V_0, \quad u_0 \in K, \tag{12}$$

with

$$L_0(v) = -\int_{\Gamma_2} qv\, d\gamma.$$

Let Π: $H \times H \to \mathbb{R}$ and L: $H \to \mathbb{R}$ be defined by the following expressions:

$$\Pi(g,h) = (C(g), C(h)) + M(g,h), \qquad \forall g, h \in H, \tag{13}$$

$$L(g) = (C(g), z_d - u_0), \qquad \forall g \in H.$$

We have that a is a bilinear, continuous and symmetric form on V and coercive on V_0, that is [14],

$$\exists \lambda > 0 \quad \text{such that} \quad a(v,v) \geq \lambda \|v\|_V^2, \qquad \forall v \in V_0. \tag{14}$$

Lemma 2.1.

(i) *C is a linear and continuous application.*
(ii) *Π is linear, continuous, symmetric and coercive form on H, that is,*

$$\Pi(g,g) \geq M\|g\|_H^2, \qquad \forall g \in H. \tag{15}$$

(iii) *L is linear and continuous on H.*
(iv) *J can be also written as*

$$J(g) = \tfrac{1}{2}\Pi(g,h) - L(g) + \tfrac{1}{2}\|u_0 - z_d\|_H^2, \qquad \forall g \in H. \tag{16}$$

(v) *There exists a unique optimal control $g_{op} \in H$ such that*

$$J(g_{op}) = \min_{g \in H} J(g). \tag{17}$$

(vi) *The application $g \in H \to u_g \in V$ is Lipschitzian, that is,*

$$\|u_{g_2} - u_{g_1}\|_V \leq \frac{1}{\lambda}\|g_2 - g_1\|_H, \qquad \forall g_1, g_2 \in H. \tag{18}$$

Proof. (i)–(iii) This follows as in [12] and [15].
(iv) From the definitions of J, Π and L, we have

$$\begin{aligned} J(g) &= \tfrac{1}{2}\|u_g + u_0 - u_0 - z_d\|_H^2 + \frac{M}{2}\|g\|_H^2 \\ &= \tfrac{1}{2}\|u_g - u_0\|_H^2 - (u_g - u_0, z_d - u_0) + \tfrac{1}{2}\|u_0 - z_d\|_H^2 + \frac{M}{2}\|g\|_H^2 \\ &= \tfrac{1}{2}\Pi(g,h) - L(g) + \tfrac{1}{2}\|u_0 - z_d\|_H^2. \end{aligned}$$

(v) This is a result of (ii)–(iv) [14], [15].
(vi) If we take $v = u_{g_1} - u_{g_2} \in V_0$ in the variational equality (4) for u_{g_2}, that is,

$$a(u_{g_2}, u_{g_1} - u_{g_2}) = (g_2, u_{g_1} - u_{g_2}) - \int_{\Gamma_2} q(u_{g_1} - u_{g_2})\, d\gamma,$$

and if we take $v = u_{g_2} - u_{g_1} \in V_0$ in the variational equality (4) for u_{g_1}, that is,

$$a(u_{g_1}, u_{g_2} - u_{g_1}) = (g_1, u_{g_2} - u_{g_1}) - \int_{\Gamma_2} q(u_{g_2} - u_{g_1})\, d\gamma,$$

then we obtain

$$a(u_{g_2} - u_{g_1}, u_{g_2} - u_{g_1}) = (g_2 - g_1, u_{g_2} - u_{g_1})$$

and taking into account that a is a coercive form we get

$$\lambda \|u_{g_2} - u_{g_1}\|_V^2 \leq a(u_{g_2} - u_{g_1}, u_{g_2} - u_{g_1}) \leq \|g_2 - g_1\|_H \|u_{g_2} - u_{g_1}\|_H,$$

and therefore (18). □

We define the adjoint state p_g corresponding to (1) or (4), for each $g \in H$, as the unique solution of the following mixed elliptic problem:

$$-\Delta p_g = u_g - z_d \quad \text{in } \Omega, \qquad p_g|_{\Gamma_1} = 0, \qquad \left.\frac{\partial p_g}{\partial n}\right|_{\Gamma_2} = 0, \tag{19}$$

whose variational formulation is given by

$$a(p_g, v) = (u_g - z_d, v), \qquad \forall v \in V_0, \quad p_g \in V_0. \tag{20}$$

Now we will obtain some useful properties of the functional J.

Lemma 2.2.

(i) *J is a Gâteaux differentiable functional and J' is given by*

$$\langle J'(g), h\rangle = (u_g - z_d, C(h)) + M(g, h) = \Pi(g, h) - L(g), \quad \forall g, h \in H. \tag{21}$$

(ii) *The adjoint state p_g satisfy the following equalities:*

$$(p_g, h) = (u_g - z_d, C(h)) = a(p_g, C(h)). \tag{22}$$

(iii) *The Gâteaux derivative of J can be written as*

$$J'(g) = p_g + Mg, \qquad \forall g \in H. \tag{23}$$

(iv) *The optimality condition for problem* (9) *is given by $J'(g_{op}) = 0$ in H, that is,*

$$p_{g_{op}} + Mg_{op} = 0 \quad \text{in } H. \tag{24}$$

Proof. (i) For $t > 0$, we have

$$\frac{1}{t}[J(g + t(f - g)) - J(g)] = \frac{t}{2}(u_f - u_g, u_f - u_g) + (u_g - z_d, u_f - u_g) + M(g, f - g) + \frac{Mt}{2}(f - g, f - g),$$

and passing to the limit $t \to 0$, we obtain (21).

(ii) This results from the definition of p_g and taking into account that

$$a(p_g, C(h)) = a(p_g, u_h - u_0) = a(p_g, u_h) - a(p_g, u_0) = (p_g, h).$$

(iii), (iv) They follow from (21), (22) and [14] and [15]. □

Let the operator $W\colon H \to V_0 \subset H$ be defined by

$$W(g) = -\frac{1}{M} p_g, \qquad g \in H. \tag{25}$$

We will prove the following property:

Lemma 2.3. *W is a Lipschitz operator over H, i.e.*

$$\|W(g_2) - W(g_1)\|_H \le \frac{1}{\lambda^2 M}\|g_1 - g_2\|_H, \qquad \forall g_1, g_2 \in H, \tag{26}$$

and it is a contraction for all $M > 1/\lambda^2$.

Proof. If we take $v = p_{g_2} - p_{g_1}$ in the variational equality (20) for g_2 we obtain

$$a(p_{g_2}, p_{g_2} - p_{g_1}) = (u_{g_2} - z_d, p_{g_2} - p_{g_1})$$

and in a similar way we have

$$a(p_{g_1}, p_{g_1} - p_{g_2}) = (u_{g_1} - z_d, p_{g_1} - p_{g_2}).$$

Therefore we obtain

$$a(p_{g_2} - p_{g_1}, p_{g_2} - p_{g_1}) = (u_{g_2} - u_{g_1}, p_{g_2} - p_{g_1})$$

and by using the coercivieness of the bilinear form a we have

$$\lambda\|p_{g_2} - p_{g_1}\|_V^2 \le a(p_{g_2} - p_{g_1}, p_{g_2} - p_{g_1}) \le \|u_{g_2} - u_{g_1}\|_H \|p_{g_2} - p_{g_1}\|_H,$$

therefore

$$\|p_{g_2} - p_{g_1}\|_V \le \frac{1}{\lambda}\|u_{g_2} - u_{g_1}\|_H. \tag{27}$$

Next, taking into account inequalities (18) and (27) we obtain

$$\begin{aligned} \|W(g_2) - W(g_1)\|_H &\le \frac{1}{M}\|p_{g_2} - p_{g_1}\|_H \\ &\le \frac{1}{\lambda M}\|u_{g_2} - u_{g_1}\|_H \\ &\le \frac{1}{\lambda^2 M}\|g_1 - g_2\|_H, \end{aligned}$$

that is, (26). □

Now we are in the condition for proving other properties of the functional J.

Lemma 2.4.

(i) *The application $g \in H \to p_g \in V_0$ is strictly monotone. Moreover, we have*

$$(p_{g_2} - p_{g_1}, g_2 - g_1) = \|u_{g_2} - u_{g_1}\|_H^2 \geq 0, \qquad \forall g_1, g_2 \in H. \tag{28}$$

(ii) *J is coercive or H-elliptic, that is,*

$$\begin{aligned}(1-t)J(g_2) + tJ(g_1) - J((1-t)g_2 + tg_1) \\ = \frac{t(1-t)}{2}[\|u_{g_2} - u_{g_1}\|_H^2 + M\|g_2 - g_1\|_H^2] \\ \geq \frac{Mt(1-t)}{2}\|g_2 - g_1\|_H^2, \qquad \forall g_1, g_2 \in H, \quad \forall t \in [0,1].\end{aligned} \tag{29}$$

(iii) *J' is a Lipschitzian and strictly monotone application, that is,*

$$\|J'(g_2) - J'(g_1)\|_H \leq \left(M + \frac{1}{\lambda^2}\right)\|g_1 - g_2\|_H \tag{30}$$

and

$$\begin{aligned}\langle J'(g_2) - J'(g_1), g_2 - g_1\rangle &= \|u_{g_2} - u_{g_1}\|_H^2 + M\|g_2 - g_1\|_H^2 \\ &\geq M\|g_2 - g_1\|_H^2, \qquad \forall g_1, g_2 \in H.\end{aligned} \tag{31}$$

Proof. (i) We have

$$\begin{aligned}(p_{g_2} - p_{g_1}, g_2 - g_1) &= (p_{g_2}, g_2 - g_1) - (p_{g_1}, g_2 - g_1) \\ &= (u_{g_2} - z_d, C(g_2 - g_1)) - (u_{g_1} - z_d, C(g_2 - g_1)) \\ &= (u_{g_2} - u_{g_1}, C(g_2 - g_1)) \\ &= \|u_{g_2} - u_{g_1}\|_H^2 \geq 0, \qquad \forall g_1, g_2 \in H.\end{aligned}$$

(ii) For all $g_1, g_2 \in H$, $t \in [0,1]$ we get

$$\begin{aligned}&(1-t)J(g_2) + tJ(g_1) - J((1-t)g_2 + tg_1) \\ &= (1-t)\left[\tfrac{1}{2}\|u_{g_2} - z_d\|_H^2 + \frac{M}{2}\|g_2\|_H^2\right] \\ &\quad + t\left[\tfrac{1}{2}\|u_{g_1} - z_d\|_H^2 + \frac{M}{2}\|g_1\|_H^2\right] \\ &\quad - \left[\tfrac{1}{2}\|u_{(1-t)g_2+tg_1} - z_d\|_H^2 + \frac{M}{2}\|(1-t)g_2 + tg_1\|_H^2\right] \\ &= \tfrac{1}{2}[(1-t)\|u_{g_2} - z_d\|_H^2 + t\,\|u_{g_1} - z_d\|_H^2 - \|(1-t)u_{g_1} + tu_{g_2} - z_d\|_H^2 \\ &\quad + (1-t)M\|g_2\|_H^2 + tM\|g_1\|_H^2 - M\|(1-t)g_2 + tg_1\|_H^2]\end{aligned}$$

$$= \frac{t(1-t)}{2}[\,\|u_{g_2} - u_{g_1}\|_H^2 + M\|g_2 - g_1\|_H^2\,]$$
$$\geq \frac{Mt(1-t)}{2}\|g_2 - g_1\|_H^2.$$

(iii) By using (18), (23) and (27) we have

$$\|J'(g_2) - J'(g_1)\|_H \leq \|p_{g_2} - p_{g_1}\|_H + M\|g_2 - g_1\|_H$$
$$\leq \left(M + \frac{1}{\lambda^2}\right)\|g_2 - g_1\|_H$$

and

$$\begin{aligned}\langle J'(g_2) - J'(g_1), g_2 - g_1\rangle &= (p_{g_2} + Mg_2 - (p_{g_1} + Mg_1), g_2 - g_1)\\ &= (p_{g_2} - p_{g_1}, g_2 - g_1) + M(g_2 - g_1, g_2 - g_1)\\ &= \|u_{g_2} - u_{g_1}\|_H^2 + M\|g_2 - g_1\|_H^2 \geq M\|g_2 - g_1\|_H^2,\end{aligned}$$

that is, (30) and (31), respectively. □

We present an iterative algorithm in order to obtain g_{op}. For each ρ we define the following sequence (g_n) given by [7], [10]

$$g_0 \in H \quad \text{(given, arbitrarily)}, \quad g_{n+1} = (1 - \rho M)g_n - \rho p_{g_n}, \qquad \forall n \geq 0, \tag{32}$$

which will converge to g_{op} for a suitable ρ.

Lemma 2.5. *If ρ is chosen satisfying the inequalities*

$$0 < \rho < \frac{2M}{(M + 1/\lambda^2)^2}, \tag{33}$$

then the algorithm (32) *is strongly convergent in H to the optimal control g_{op} of* (9) *independently of g_0, that is,*

$$\lim_{n\to\infty} \|g_n - g_{op}\|_H = 0, \qquad \text{for any} \quad g_0 \in H. \tag{34}$$

Proof. The operator $T\colon H \to H$ defined by

$$T(g) = (1 - \rho M)g - \rho p_g \tag{35}$$

is a Lipschitz operator, that is,

$$\|T(g_2) - T(g_1)\|_H \leq \sqrt{\gamma(\rho)}\|g_2 - g_1\|_H, \qquad \forall g_1, g_2 \in H, \tag{36}$$

where $\gamma(\rho)$ is given by

$$\gamma(\rho) = 1 - 2M\rho + \left(M + \frac{1}{\lambda^2}\right)^2 \rho^2, \tag{37}$$

because

$$\begin{aligned}\|T(g_2) - T(g_1)\|_H^2 &= \|g_2 - \rho J'(g_2) - g_1 + J'(g_1)\|_H^2 \\ &= \|g_2 - g_1\|_H^2 - 2\rho\,(g_2 - g_1, J'(g_2) - J'(g_1)) \\ &\quad + \rho^2 \|J'(g_2) - J'(g_1)\|_H^2 \\ &\le \|g_2 - g_1\|_H^2 - 2\rho M \|g_2 - g_1\|_H^2 \\ &\quad + \rho^2 \left(M + \frac{1}{\lambda^2}\right)^2 \|g_2 - g_1\|_H^2 \\ &= \left[1 - 2M\rho + \left(M + \frac{1}{\lambda^2}\right)^2 \rho^2\right] \|g_2 - g_1\|_H^2.\end{aligned}$$

Therefore T will be a contraction if and only if $0 \le \gamma(\rho) < 1$, that is, inequality (33).

Moreover, the ρ and γ optimals are given by

$$\rho_{\text{op}} = \frac{M}{(M + 1/\lambda^2)^2}, \qquad \gamma_{\text{op}} = 1 - \left(\frac{M}{M + 1/\lambda^2}\right)^2. \quad \square \tag{38}$$

3. Problem P_α and Its Corresponding Optimal Control Problem

Let $\Pi_\alpha\colon H \times H \to \mathbb{R}$, $L_\alpha\colon H \to \mathbb{R}$ and $C_\alpha\colon H \to V$ be defined by

$$\begin{aligned}&\Pi_\alpha(g, h) = (C_\alpha(g), C_\alpha(h)) + M(g, h), \qquad \forall g, h \in H, \\ &L_\alpha(g) = (C_\alpha(g), z_d - u_{0\alpha}), \qquad \forall g \in H, \\ &C_\alpha(g) = u_{g\alpha} - u_{0\alpha}, \qquad \forall g \in H,\end{aligned} \tag{39}$$

where $u_{g\alpha}$ is the unique solution of the variational equality (5), $u_{0\alpha}$ is the unique solution of (5) for $g = 0$ whose variational equality is given by

$$a_\alpha(u_{0\alpha}, v) = L_{0\alpha}(v), \qquad \forall v \in V, \quad u_{0\alpha} \in V, \tag{40}$$

with

$$L_{0\alpha}(v) = \alpha \int_{\Gamma_1} bv\, d\gamma - \int_{\Gamma_2} qv\, d\gamma, \tag{41}$$

and a_α is a bilinear, continuous, symmetric and coercive form on V, that is,

$$a_\alpha(v, v) \ge \lambda_\alpha \|v\|_V^2, \qquad \forall v \in V, \tag{42}$$

where $\lambda_\alpha = \lambda_1 \min(1, \alpha) > 0$ for all $\alpha > 0$ and λ_1 is the coerciveness constant for the bilinear form a_1 [19].

We can obtain similar properties to Lemma 2.1, following [12], [14], [15] and [18], the proof of which is omitted.

Lemma 3.1.

(i) *C_α is a linear and continuous application.*
(ii) *Π_α is linear, continuous, symmetric and coercive on H, that is,*

$$\Pi_\alpha(g, g) \geq M\|g\|_H^2, \qquad \forall g \in H. \tag{43}$$

(iii) *L_α is linear and continuous on H.*
(iv) *J_α can be also written as*

$$J_\alpha(g) = \tfrac{1}{2}\Pi_\alpha(g, h) - L_\alpha(g) + \tfrac{1}{2}\|u_{0\alpha} - z_d\|_H^2, \qquad \forall g \in H. \tag{44}$$

(v) *There exists a unique optimal control $g_{op_\alpha} \in H$ such that*

$$J_\alpha(g_{op_\alpha}) = \min_{g \in H} J_\alpha(g). \tag{45}$$

(vi) *The application $g \in H \to u_{g\alpha} \in V$ is Lipschitzian, that is,*

$$\|u_{g_2\alpha} - u_{g_1\alpha}\|_V \leq \frac{1}{\lambda_\alpha}\|g_2 - g_1\|_H, \qquad \forall g_1, g_2 \in H. \tag{46}$$

We define the adjoint state $p_{g\alpha}$ as the unique solution of the following mixed elliptic problem corresponding to (2) or (5), for each $g \in H$ and $\alpha > 0$:

$$-\Delta p_{g\alpha} = u_{g\alpha} - z_d \quad \text{in } \Omega, \qquad -\frac{\partial p_{g\alpha}}{\partial n}\bigg|_{\Gamma_1} = \alpha p_{g\alpha}, \qquad \frac{\partial p_{g\alpha}}{\partial n}\bigg|_{\Gamma_2} = 0, \tag{47}$$

whose variational formulation is given by

$$a_\alpha(p_{g\alpha}, v) = (u_{g\alpha} - z_d, v), \qquad \forall v \in V, \quad p_{g\alpha} \in V, \tag{48}$$

where $u_{g\alpha}$ is the unique solution of (5).

Now we obtain some properties of the functional J_α.

Lemma 3.2. *Let $\alpha > 0$.*

(i) *The Gâteaux derivative J'_α is given by*

$$\langle J'_\alpha(g), h\rangle = (u_{g\alpha} - z_d, C_\alpha(h)) + M(g, h) = \Pi_\alpha(g, h) - L_\alpha(g), \quad \forall g, h \in H. \tag{49}$$

(ii) *The adjoint state $p_{g\alpha}$ satisfies the following equalities:*

$$(p_{g\alpha}, h) = (u_{g\alpha} - z_\alpha, C_\alpha(h)) = a_\alpha(p_{g\alpha}, C_\alpha(h)), \qquad \forall g, h \in H. \tag{50}$$

(iii) *The Gâteaux derivative of J_α can be written as*

$$J'_\alpha(g) = p_{g\alpha} + Mg, \qquad \forall g \in H. \tag{51}$$

(iv) *The optimality condition for problem (10) is given by $J'_\alpha(g_{op_\alpha}) = 0$ in H, that is,*

$$p_{g_{op_\alpha}\alpha} + Mg_{op_\alpha} = 0 \quad \text{in } H. \tag{52}$$

Proof. (i) We have

$$\frac{1}{t}[J_\alpha(g+t(f-g))-J_\alpha(g)]$$
$$=\frac{t}{2}(u_{f\alpha}-u_{g\alpha},u_{f\alpha}-u_{g\alpha})+(u_{g\alpha}-z_d,u_{f\alpha}-u_{g\alpha})$$
$$+M(g,f-g)+\frac{Mt}{2}(f-g,f-g)$$

and passing to the limit $t \to 0$, we obtain (49).

(ii) This results from the definition of $p_{g\alpha}$ and taking into account that

$$a(p_{g\alpha},C_\alpha(h))=a(p_{g\alpha},u_h-u_0)=a(p_{g\alpha},u_h)-a(p_{g\alpha},u_0)=(p_{g\alpha},h). \qquad \square$$

Remark 1. We note the double dependence on the parameter α for the optimal state system $u_{g_{op_\alpha}\alpha}$ and the adjoint state $p_{g_{op_\alpha}\alpha}$.

Let the operator W_α: $H \to V \subset H$ be defined by

$$W_\alpha(g)=-\frac{1}{M}p_{g\alpha}, \qquad \forall g \in H. \tag{53}$$

We have the following property:

Lemma 3.3. *W_α is a Lipschitz operator over H, that is,*

$$\|W_\alpha(g_2)-W_\alpha(g_1)\|_H \le \frac{1}{\lambda_\alpha^2 M}\|g_1-g_2\|_H, \qquad \forall g_1,g_2 \in H, \tag{54}$$

and it is a contraction for all $M > 1/\lambda_\alpha^2$.

Proof. If we take $v = p_{g_2\alpha}-p_{g_1\alpha}$ in variational equality (48) for g_2 and g_1, respectively, by substracting them and by using the coerciveness of a_α we have

$$\lambda_\alpha\|p_{g_2\alpha}-p_{g_1\alpha}\|_H^2 \le a_\alpha(p_{g_2\alpha}-p_{g_1\alpha},p_{g_2\alpha}-p_{g_1\alpha})$$
$$=(u_{g_2\alpha}-u_{g_1\alpha},p_{g_2\alpha}-p_{g_1\alpha})_H$$
$$\le \|u_{g_2\alpha}-u_{g_1\alpha}\|_H\|p_{g_2\alpha}-p_{g_1\alpha}\|_H,$$

therefore

$$\|p_{g_2\alpha}-p_{g_1\alpha}\|_V \le \frac{1}{\lambda_\alpha}\|u_{g_2\alpha}-u_{g_1\alpha}\|_H. \tag{55}$$

Next, taking into account inequalities (46) and (55) we obtain

$$\|W_\alpha(g_2)-W_\alpha(g_1)\|_H=\frac{1}{M}\|p_{g_2\alpha}-p_{g_1\alpha}\|_V$$
$$\le \frac{1}{\lambda_\alpha M}\|u_{g_2\alpha}-u_{g_1\alpha}\|_H$$
$$\le \frac{1}{\lambda_\alpha^2 M}\|g_1-g_2\|_H,$$

that is, (54). $\square$

Now, we prove other properties of the functional J_α.

Lemma 3.4.

(i) *The operator* $g \in H \to p_{g\alpha} \in V$ *is strictly monotone, that is,*

$$(p_{g_2\alpha} - p_{g_1\alpha}, g_2 - g_1) = \|u_{g_2\alpha} - u_{g_1\alpha}\|_H^2 \geq 0, \qquad \forall g_1, g_2 \in H. \tag{56}$$

(ii) J_α *is coercive or H-elliptic, that is,*

$$\begin{aligned}(1-t)J_\alpha(g_2) &+ tJ_\alpha(g_1) - J_\alpha((1-t)g_2 + tg_1) \\ &= \frac{t(1-t)}{2}[\|u_{g_2\alpha} - u_{g_1\alpha}\|_H^2 + M\|g_2 - g_1\|_H^2] \\ &\geq \frac{Mt(1-t)}{2}\|g_2 - g_1\|_H^2, \qquad \forall g_1, g_2 \in H, \quad \forall t \in [0,1].\end{aligned} \tag{57}$$

(iii) J'_α *is a Lipschitzian and strictly monotone operator, that is,*

$$\|J'_\alpha(g_2) - J'_\alpha(g_1)\|_H \leq \left(M + \frac{1}{\lambda_\alpha^2}\right)\|g_1 - g_2\|_H, \qquad \forall g_1, g_2 \in H, \tag{58}$$

and

$$\begin{aligned}\langle J'_\alpha(g_2) - J'_\alpha(g_1), g_2 - g_1\rangle &= \|u_{g_2\alpha} - u_{g_1\alpha}\|_H^2 + M\|g_2 - g_1\|_H^2 \\ &\geq M\|g_2 - g_1\|_H^2, \qquad \forall g_1, g_2 \in H.\end{aligned} \tag{59}$$

Proof. (i) We have

$$\begin{aligned}(p_{g_2\alpha} - p_{g_1\alpha}, g_2 - g_1) &= (u_{g_2\alpha} - u_{g_1\alpha}, C_\alpha(g_2 - g_1)) \\ &= \|u_{g_2\alpha} - u_{g_1\alpha}\|_H^2 \geq 0, \qquad \forall g_1, g_2 \in H.\end{aligned}$$

(ii) For all $g_1, g_2 \in H, t \in [0,1]$ we obtain

$$\begin{aligned}&(1-t)J_\alpha(g_2) + tJ_\alpha(g_1) - J_\alpha((1-t)g_2 + tg_1) \\ &= (1-t)\left[\tfrac{1}{2}\|u_{g_2\alpha} - z_d\|_H^2 + \frac{M}{2}\|g_2\|_H^2\right] \\ &\quad + t\left[\tfrac{1}{2}\|u_{g_1\alpha} - z_d\|_H^2 + \frac{M}{2}\|g_1\|_H^2\right] \\ &\quad - \left[\tfrac{1}{2}\|u_{(1-t)g_2+g_1\alpha} - z_d\|_H^2 + \frac{M}{2}\|(1-t)g_2 + tg_1\|_H^2\right] \\ &= \frac{t(1-t)}{2}[\|u_{g_2\alpha} - u_{g_1\alpha}\|_H^2 + M\|g_2 - g_1\|_H^2] \\ &\geq \frac{Mt(1-t)}{2}\|g_2 - g_1\|_H^2.\end{aligned}$$

(iii) By using (46) and (55) we have

$$\|J'_\alpha(g_2) - J'_\alpha(g_1)\|_H \le \|p_{g_2\alpha} - p_{g_1\alpha}\|_H + M\|g_2 - g_1\|_H$$
$$\le \left(M + \frac{1}{\lambda_\alpha^2}\right)\|g_2 - g_1\|_H,$$

then J'_α is a Lipschitzian application. On the other hand we get

$$\langle J'_\alpha(g_2) - J'_\alpha(g_1), g_2 - g_1\rangle = (p_{g_2\alpha} + Mg_2 - (p_{g_1\alpha} + Mg_1), g_2 - g_1)$$
$$= \|u_{g_2\alpha} - u_{g_1\alpha}\|_H^2 + M\|g_2 - g_1\|_H^2$$
$$\ge M\|g_2 - g_1\|_H^2,$$

and J'_α is a strictly monotone application. □

Now, we prove the following result of convergence when $\alpha \to \infty$.

Lemma 3.5. *For all $\alpha > 0$, $q \in L^2(\Gamma_2)$, $b \in H^{1/2}(\Gamma_1)$, we have the following limits:*

$$\text{(i)}\ \lim_{\alpha\to\infty} \|u_{g\alpha} - u_g\|_V = 0, \forall g \in H,$$
$$\text{(ii)}\ \lim_{\alpha\to\infty} \|u_{0\alpha} - u_0\|_V = 0, \tag{60}$$
$$\text{(iii)}\ \lim_{\alpha\to\infty} \|p_{g\alpha} - p_g\|_V = 0, \forall g \in H.$$

Proof. (i) If we take $v = u_{g\alpha} - u_g$ in the variational equality (5), for g, α, with $\alpha > 1$ (because $\alpha \to \infty$), following [18] and [19] we obtain

$$\lambda_1\|u_{g\alpha} - u_g\|_V^2 + (\alpha - 1)\int_{\Gamma_1}(u_{g\alpha} - u_g)^2\, d\gamma \le a_\alpha(u_{g\alpha} - u_g, u_{g\alpha} - u_g)$$
$$\le C_1\|u_{g\alpha} - u_g\|_V, \tag{61}$$

with C_1 a constant independent of α. Next for large α we obtain

$$\text{(a)}\ \|u_{g\alpha} - u_g\|_V^2 \le \frac{C_1}{\lambda_1}, \qquad \text{(b)}\ (\alpha - 1)\int_{\Gamma_1}(u_{g\alpha} - u_g)^2\, d\gamma \le \frac{(C_1)^2}{\lambda_1}, \tag{62}$$

and we deduce that there exists $w_g \in V$ such that

$$\text{(a)}\ u_{g\alpha} \rightharpoonup w_g \quad \text{weakly in } V, \qquad \text{(b)}\ \int_{\Gamma_1}(u_{g\alpha} - b)^2\, d\gamma \le \frac{(C_1)^2}{\lambda_1}\frac{1}{(\alpha - 1)} \to 0,$$
$$\text{as}\quad \alpha \to \infty, \tag{63}$$

that is, $w_g \in K$ and taking the limit of the variational equality (5) as $\alpha \to \infty$ we have

$$a(w_g, v) = L_g(v), \qquad \forall v \in V_0, \quad w_g \in K, \tag{64}$$

and, by uniqueness, we have $w_g = u_g$.

Therefore, $u_{g\alpha} \to u_g$ strongly in V as $\alpha \to \infty$ because of the following inequality:

$$\lambda_1 \|u_{g\alpha} - u_g\|_V^2 \le L_g(u_{g\alpha} - u_g) - a(u_{g\alpha}, u_{g\alpha} - u_g).$$

For case (ii) we take $g = 0$ in (i).

(iii) In this case we take $v = p_{g\alpha} - p_g$ in the variational equality (48) for g, α and following a similar method as before we obtain

$$\lambda_1 \|p_{g\alpha} - p_g\|_V^2 + (\alpha - 1) \int_{\Gamma_1} (p_{g\alpha} - p_g)^2 \, d\gamma \le a_\alpha(p_{g\alpha} - p_g, p_{g\alpha} - p_g)$$
$$\le C_2 \|p_{g\alpha} - p_g\|_V,$$

with C_2 a constant independent of α . Next, for large α, we have

$$\text{(a)} \ \|p_{g\alpha} - p_g\|_V^2 \le \frac{C_2}{\lambda_1}, \qquad \text{(b)} \ (\alpha - 1) \int_{\Gamma_1} (p_{g\alpha} - p_g)^2 \, d\gamma \le \frac{(C_2)^2}{\lambda_1}, \tag{65}$$

and we deduce that there exists $\xi_g \in V$ such that

$$\text{(a)} \ p_{g\alpha} \rightharpoonup \xi_g \quad \text{weakly in } V,$$
$$\text{(b)} \ \int_{\Gamma_1} (p_{g\alpha} - p_g)^2 \, d\gamma \le \frac{(C_2)^2}{\lambda_1(\alpha - 1)} \to 0, \qquad \text{as} \quad \alpha \to \infty, \tag{66}$$

that is, $\xi_g \in V_0$ and taking the limit on the variational equality (48) for $p_{g\alpha}$ we have

$$a(\xi_g, v) = (u_g - z_d, v), \qquad \forall v \in V_0, \quad \xi_g \in V_0, \tag{67}$$

and, by uniqueness, we obtain $\xi_g = p_g$. Therefore, taking into account the following inequality,

$$\lambda_1 \|p_{g\alpha} - p_g\|_V^2 \le (u_{g\alpha} - z_d, p_{g\alpha} - p_g) - a(p_g, p_{g\alpha} - p_g),$$

we have that $p_{g\alpha} \to p_g$ strongly in V. □

4. Convergence of Problem P_α and Its Corresponding Optimal Control as $\alpha \to \infty$

In this section we prove that the optimal control g_{op_α} of problem (10) and its corresponding adjoint state $p_{g_{op_\alpha}\alpha}$ (48) are convergent to the optimal control g_{op} of problem (9) and its corresponding adjoint state $p_{g_{op}}$ (20), respectively, when the parameter α (heat transfer coefficient on Γ_1) goes to infinity.

Theorem 4.1. *Let $M > 1/\lambda_1^2$. Then we have*:

(i) *If $p_{g_{op}}$ and $p_{g_{op_\alpha}\alpha}$ are the corresponding adjoint states of problems* (9) *and* (10), *respectively, then*

$$\lim_{\alpha \to \infty} \|p_{g_{op_\alpha}\alpha} - p_{g_{op}}\|_V = 0. \tag{68}$$

(ii) *If g_{op} and g_{op_α} are the solutions of problems* (9) *and* (10), *respectively, then*

$$\lim_{\alpha\to\infty} \|g_{\text{op}_\alpha} - g_{\text{op}}\|_H = 0. \tag{69}$$

(iii) *If $u_{g_{\text{op}}}$ and $u_{g_{\text{op}_\alpha}\alpha}$ are the corresponding solutions of problems* P *and* P_α, *respectively, then*

$$\lim_{\alpha\to\infty} \|u_{g_{\text{op}_\alpha}\alpha} - u_{g_{\text{op}}}\|_V = 0. \tag{70}$$

Proof. We prove some preliminary results for the three cases.

Since g_{op_α} is the solution of problem (10), we have the following inequality:

$$\tfrac{1}{2}\|u_{g_{\text{op}_\alpha}\alpha} - z_d\|_H^2 + \frac{M}{2}\|g_{\text{op}_\alpha}\|_H^2 \le \tfrac{1}{2}\|u_{g\alpha} - z_d\|_H^2 + \frac{M}{2}\|g\|_H^2, \qquad \forall g \in H,$$

then, taking $g = 0$, we have

$$\tfrac{1}{2}\|u_{g_{\text{op}_\alpha}\alpha} - z_d\|_H^2 + \frac{M}{2}\|g_{\text{op}_\alpha}\|_H^2 \le \tfrac{1}{2}\|u_{0\alpha} - z_d\|_H^2 \le C_3, \qquad \forall \alpha > 0,$$

where C_3 is a constant independent of parameter α because $u_{0\alpha}$ is convergent when $\alpha \to \infty$. Therefore

$$\|g_{\text{op}_\alpha}\|_H \le C_4 \quad \text{and} \quad \|u_{g_{\text{op}_\alpha}\alpha}\|_H \le C_5, \tag{71}$$

where C_4 and C_5 are constants independent of α.

Now, if we take $v = u_{g_{\text{op}_\alpha}\alpha} - u_{g_{\text{op}}}$ in the variational equality (5), following [18] we obtain, for $\alpha > 1$,

$$\begin{aligned}
&\lambda_1\|u_{g_{\text{op}_\alpha}\alpha} - u_{g_{\text{op}}}\|_V^2 + (\alpha - 1)\int_{\Gamma_1} (u_{g_{\text{op}_\alpha}\alpha} - u_{g_{\text{op}}})^2\, d\gamma \\
&\quad\le a_\alpha(u_{g_{\text{op}_\alpha}\alpha} - u_{g_{\text{op}}}, u_{g_{\text{op}_\alpha}\alpha} - u_{g_{\text{op}}}) \\
&\quad\le C_6\|u_{g_{\text{op}_\alpha}\alpha} - u_{g_{\text{op}}}\|_V,
\end{aligned}$$

where $C_6 = C_6(g_{\text{op}}, q, u_{g_{\text{op}}})$ is independent of α. Next, we have

$$\begin{aligned}
&\text{(a)}\ \ \|u_{g_{\text{op}_\alpha}\alpha} - u_{g_{\text{op}}}\|_V^2 \le \frac{C_6}{\lambda_1}, \\
&\text{(b)}\ \ (\alpha - 1)\int_{\Gamma_1} (u_{g_{\text{op}_\alpha}\alpha} - u_{g_{\text{op}}})^2\, d\gamma \le \frac{(C_6)^2}{\lambda_1},
\end{aligned} \tag{72}$$

and therefore we deduce that

$$\exists\, \eta \in V \quad \text{such that} \quad u_{g_{\text{op}_\alpha}\alpha} \rightharpoonup \eta \quad \text{weakly in } V, \tag{73}$$

and because of the following inequalities,

$$0 \le \int_{\Gamma_1} (\eta - u_{g_{op}})^2 \, d\gamma \le \liminf_{\alpha\to\infty} \int_{\Gamma_1} (u_{g_{op_\alpha}\alpha} - u_{g_{op}})^2 \, d\gamma = 0,$$

we obtain that $\eta \in K$.

Next, if we take $v = p_{g_{op_\alpha}\alpha} - p_{g_{op}}$ in the variational equality (48) we get

$$\begin{aligned} \lambda_1 \| p_{g_{op_\alpha}\alpha} - p_{g_{op}} \|_V^2 + (\alpha - 1) \int_{\Gamma_1} (p_{g_{op_\alpha}\alpha} - p_{g_{op}})^2 \, d\gamma \\ \le a_\alpha (p_{g_{op_\alpha}\alpha} - p_{g_{op}}, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) \\ \le C_7 \| p_{g_{op_\alpha}\alpha} - p_{g_{op}} \|_V, \end{aligned}$$

with $C_7 = C_7(C_5, p_{g_{op}})$. Next, we obtain

$$\begin{aligned} &\text{(a)} \quad \| p_{g_{op_\alpha}\alpha} - p_{g_{op}} \|_V^2 \le \frac{C_7}{\lambda_1}, \\ &\text{(b)} \quad (\alpha - 1) \int_{\Gamma_1} (p_{g_{op_\alpha}\alpha} - p_{g_{op}})^2 \, d\gamma \le \frac{(C_7)^2}{\lambda_1}, \end{aligned} \tag{74}$$

and therefore we deduce that

$$\exists \xi \in V \quad \text{such that} \quad p_{g_{op_\alpha}\alpha} \rightharpoonup \xi \quad \text{weakly in } V \tag{75}$$

and by the following inequality,

$$0 \le \int_{\Gamma_1} (\xi - p_{g_{op}})^2 \, d\gamma \le \liminf_{\alpha\to\infty} \int_{\Gamma_1} (p_{g_{op_\alpha}\alpha} - p_{g_{op}})^2 \, d\gamma = 0,$$

we obtain $\xi \in V_0$.

Now we consider $v \in V_0$ and, taking into acount (73) and (75), from the variational equality (48) we have

$$a(\xi, v) = (\eta - z_d, v), \qquad \forall v \in V_0, \quad \xi \in V_0. \tag{76}$$

Next, from (71) we deduce that there exists $f \in H$ such that $g_{op_\alpha} \rightharpoonup f$ weakly in H. Therefore if we put $v \in V_0$ in the variational equality (5) and we pass to the limit $\alpha \to \infty$, we obtain

$$a(\eta, v) = (f, v) - \int_{\Gamma_2} q v \, d\gamma, \qquad \forall v \in V_0, \quad \eta \in K. \tag{77}$$

Now, taking into account Lemma 3.3 and the facts that $g_{op_\alpha} \rightharpoonup f$ weakly in H and $p_{g_{op_\alpha}\alpha} \rightharpoonup \xi$ weakly in V, we have

$$f = -\frac{1}{M}\xi \quad \text{in } H. \tag{78}$$

Therefore from the uniqueness of fixed point we have

$$g_{op} = -\frac{1}{M} p_{g_{op}} \quad \text{in } H, \tag{79}$$

and then we obtain that $f = g_{op}$, $\eta = u_{g_{op}}$ and $\xi = p_{g_{op}}$.

Moreover, from (75) and the following computation,

$$\begin{aligned}
\lambda_1 \| p_{g_{op_\alpha}\alpha} - p_{g_{op}} \|_V^2 &\le a_\alpha(p_{g_{op_\alpha}\alpha} - p_{g_{op}}, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) \\
&= a_\alpha(p_{g_{op_\alpha}\alpha}, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) - a(p_{g_{op}}, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) \\
&= (u_{g_{op_\alpha}\alpha} - z_d, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) - a(p_{g_{op}}, p_{g_{op_\alpha}\alpha} - p_{g_{op}})
\end{aligned}$$

we have (68).

From Lemmas 2.3 and 3.3 it results that

$$\| g_{op_\alpha} - g_{op} \|_H = \frac{1}{M} \| p_{g_{op}} - p_{g_{op_\alpha}\alpha} \|_H \le \frac{1}{M} \| p_{g_{op}} - p_{g_{op_\alpha}\alpha} \|_V$$

and therefore (69) holds.

Now we have

$$\begin{aligned}
\lambda_1 \| u_{g_{op_\alpha}\alpha} - u_{g_{op}} \|_V^2 &\le a_\alpha(u_{g_{op_\alpha}\alpha} - u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) \\
&= a_\alpha(u_{g_{op_\alpha}\alpha}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) - a_\alpha(u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) \\
&= L_{g_{op_\alpha}\alpha}(u_{g_{op_\alpha}\alpha} - u_{g_{op}}) - a(u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) \\
&\quad - \alpha \int_{\Gamma_1} b(u_{g_{op_\alpha}\alpha} - b)\, d\gamma \\
&= a(u_{g_{op_\alpha}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) - a(u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}})
\end{aligned}$$

and taking into account (69) and the fact that $u_{g_{op_\alpha}} \to u_{g_{op}}$ strongly in V when $\alpha \to \infty$ because of (18), we get (70). □

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