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CONVERGENCE OF BOUNDARY OPTIMAL CONTROL PROBLEMS WITH RESTRICTIONS IN MIXED ELLIPTIC STEFAN-LIKE PROBLEMS

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Abstract

We consider a steady-state heat conduction problem P_α with mixed boundary conditions for the Poisson equation in a bounded multidimensional domain Ω depending on a positive parameter α which represents the heat transfer coefficient on a portion Γ_1 of the boundary of Ω. We consider, for each $\alpha > 0$, a cost function J_α and we formulate boundary optimal control problems with restrictions over the heat flux q on a complementary portion Γ_2 of the boundary of Ω. We obtain that the optimality conditions are given by a complementary free boundary problem in Γ_2 in terms of the adjoint state. We prove that the optimal control q_{op_α} and its corresponding system state $u_{q_{op_\alpha}\alpha}$ and adjoint state $p_{q_{op_\alpha}\alpha}$ for each α are strongly convergent to q_{op}, $u_{q_{op}}$ and $p_{q_{op}}$ in $L^2(\Gamma_2)$, $H^1(\Omega)$, and $H^1(\Omega)$ respectively when $\alpha \to \infty$. We also

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prove that these limit functions are respectively the optimal control, the system state and the adjoint state corresponding to another boundary optimal control problem with restrictions for the same Poisson equation with a different boundary condition on the portion Γ_1. We use the elliptic variational inequality theory in order to prove all the strong convergences. In this paper, we generalize the convergence result obtained in Belgacem et al. [3] by considering boundary optimal control problems with restrictions on the heat flux q defined on Γ_2 and the parameter α (which goes to infinity) is defined on Γ_1.

1. Introduction

We consider a bounded domain Ω in $\mathbb{R}^n$ whose regular boundary Γ consists of the union of two disjoint portions Γ_1 y Γ_2 with $\text{meas}(\Gamma_1) > 0$ and $\text{meas}(\Gamma_2) > 0$. We denote with $\text{meas}(\Gamma)$ the $(n-1)$-dimensional measure of Γ. We consider the following two steady-state heat conduction problems P and P_α (for each parameter $\alpha > 0$), respectively with mixed boundary conditions:

$$\Delta u = g \text{ in } \Omega \quad u|_{\Gamma_1} = b, \quad -\frac{\partial u}{\partial n}|_{\Gamma_2} = q \tag{1}$$

and

$$-\Delta u = g \text{ in } \Omega \quad -\frac{\partial u}{\partial n}|_{\Gamma_1} = \alpha(u-b) \quad -\frac{\partial u}{\partial n}|_{\Gamma_2} = q, \tag{2}$$

where g is the internal energy in Ω, b is the temperature on Γ_1 for (1) and the temperature of the external neighborhood of Γ_1 for (2), q is the heat flux on Γ_2 and $\alpha > 0$ is the heat transfer coefficient of Γ_1 (Newton's law or Robin condition on Γ_1). They satisfy the following assumptions:

$$g \in H = L^2(\Omega),\ q \in Q = L^2(\Gamma_2),\ b \in H^{\frac{1}{2}}(\Gamma_1). \tag{3}$$

Problems (1) and (2) can be considered as the steady-state Stefan problem for suitable data q, g and b [8, 17, 18, 20].

Let u_q and $u_{q\alpha}$ be the unique solutions of the mixed elliptic problems (1) and (2), respectively for each $q \in Q$ and $\alpha > 0$ whose variational equalities are given by [9, 14] and [19]:

$$a(u_q, v) = L_q(v),\ \forall v \in V_0,\ u_q \in K \tag{4}$$

and

$$a_\alpha(u_{q\alpha}, v) = L_{q\alpha}(v),\ \forall v \in V,\ u_{q\alpha} \in V, \tag{5}$$

where

$$V = H^1(\Omega);\ V_0 = \{v \in V / v|_{\Gamma_1} = 0\} \text{ and } K = v_0 + V_0$$

for a given $v_0 \in V$, $v_0|_{\Gamma_1} = b$, and

$$(g, h)_H = \int_\Omega ghdx;\ (q, \eta)_Q = \int_{\Gamma_2} q\eta d\gamma,\ (u, v)_{L^2(\Gamma_1)} = \int_{\Gamma_1} uvd\gamma, \tag{6}$$

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx;\ a_\alpha(u, v) = a(u, v) + \alpha(u, v)_{L^2(\Gamma_1)},$$

$$L_q(v) = (g, v)_H - (q, v)_Q;\ L_{q\alpha}(v) = L_q(v) + \alpha(b, v)_{L^2(\Gamma_1)}.$$

We consider q as a control variable for the cost functionals $J : Q \to \mathbb{R}_0^+$ and $J_\alpha : Q \to \mathbb{R}_0^+$ respectively given by

$$J(q) = \frac{1}{2}\|u_q - z_d\|_H^2 + \frac{M}{2}\| q \|_Q^2 \tag{7}$$

and

$$J_\alpha(q) = \frac{1}{2}\|u_{q\alpha} - z_d\|_H^2 + \frac{M}{2}\| q \|_Q^2, \tag{8}$$

where $z_d \in H$ and $M = \text{const.} > 0$ are given.

We can formulate the following boundary optimal control problems with restrictions [5, 8, 12, 15, 16]:

$$\text{Find } q_{op} \in U_{ad} \text{ such that } J(q_{op}) = \min_{q \in U_{ad}} J(q) \tag{9}$$

and

$$\text{find } q_{op_\alpha} \in U_{ad} \text{ such that } J_\alpha(q_{op_\alpha}) = \min_{q \in U_{ad}} J_\alpha(q) \tag{10}$$

respectively, where $U_{ad} = \{q \in Q : q \geq 0 \text{ en } \Gamma_2\}$ is the admissible control set, a nonempty, closed and convex subset of Q.

It is well known that the solution $u_{q\alpha}$ is strongly convergent to u_q in V for a given heat flux q defined on Γ_2 as $\alpha \to \infty$ [17, 18, 19]. The use of the variational inequality theory in connection with optimization and optimal control problems was done, for example in [1, 2, 4, 6, 9, 13] and [16].

In Section 2, we prove that the functional J is coercive and Gâteaux differentiable on Q and J' is a Lipschitzian and strictly monotone application on Q. We also prove the existence and uniqueness of the boundary optimal control with restriction q_{op} for the problem (9) and we give the corresponding optimality condition as a complementary free boundary problem in terms of the optimal control q_{op} and the optimal adjoint state $p_{q_{op}}$ of the system.

Similarly, in Section 3, we prove that the functional J_α is coercive and Gâteaux differentiable on Q and J'_α is a Lipschitzian and strictly monotone application on Q, for each $\alpha > 0$. We also prove the existence and uniqueness of the boundary optimal controls with restrictions q_{op_α} for the problem (10) for each $\alpha > 0$ and we give the corresponding optimality conditions as a complementary free boundary problem on Γ_2 in terms of the optimal control q_{op_α} and the optimal adjoint state $p_{q_{op_\alpha}\alpha}$ of the system.

In Section 4, we study the convergence when $\alpha \to \infty$ of the boundary optimal control problems with restrictions (10) corresponding to the state system (2). We prove that the optimal state system $u_{q_{op_\alpha}\alpha}$ and the optimal adjoint state $p_{q_{op_\alpha}\alpha}$ of problem (10) are strongly convergent in V to the corresponding optimal state system $u_{q_{op}}$ and optimal adjoint state $p_{q_{op}}$ for a boundary optimal control problem with restriction (9) respectively when $\alpha \to \infty$. Finally, the strong convergence in Q of the optimal controls q_{op_α} of problem (10) to the optimal control q_{op} of problem (9) is also proved when $\alpha \to \infty$.

In [3], it was considered a boundary optimal control problem with $\Gamma = \Gamma_1$ and the Dirichlet control variable is the temperature b which is defined in the same boundary, where the penalization parameter $\varepsilon = \frac{1}{\alpha}$ is given. In this case, the boundary optimal control is proportional to the corresponding adjoint state. In the present paper, we generalize the results obtained in [3] by considering a Neumann boundary optimal control with restrictions on the heat flux q on Γ_2 and the parameter $\alpha\left(= \frac{1}{\varepsilon}\right)$ which goes to infinity is defined on a complementary boundary portion Γ_1. In particular, our optimality conditions for optimal control problems (9) and (10) are given by a free boundary problem for the optimal control and its adjoint state on Γ_2, that is (30) and (49) respectively, which are different to the proportionality between them obtained in [3].

For distributed optimal control problems, the convergence $\alpha \to \infty$ was proved in [9] by using a fixed theorem argument and in [10] by using only the variational inequality theory.

2. Problem P and its Corresponding Boundary Optimal Control Problem

Let $C : Q \to V_0$ be the application such that

$$C(q) = u_q - u_0, \tag{11}$$

where u_0 is the solution of the problem (4) for $q = 0$ whose variational equality is given by

$$a(u_0, v) = L_0(v), \ \forall v \in V_0, \ u_0 \in K \tag{12}$$

with

$$L_0(v) = (g, v)_H.$$

Let $\prod : Q \times Q \to \mathbb{R}$ and $L : Q \to \mathbb{R}$ be defined by the following expressions:

$$\prod(q, \eta) = (C(q), C(\eta))_H + M(q, \eta)_Q, \ \forall q, \eta \in Q \tag{13}$$

$$L(q) = (C(q), z_d - u_0)_H, \ \forall q \in Q.$$

We have that a is a bilinear, continuous and symmetric form on V and coercive on V_0, that is [14]:

$$\exists \lambda > 0 \text{ such that } a(v, v) \geq \lambda \| v \|_V^2, \ \forall v \in V_0. \tag{14}$$

Lemma 1. *We have*

(i) *C is a linear and continuous application.*

(ii) *$\prod$ is a bilinear, continuous, symmetric and coercive form over Q,* *that is,*

$$\Pi(q, q) \geq M\| q \|_Q^2, \ \forall q \in Q. \tag{15}$$

(iii) *L is linear and continuous on Q.*

(iv) *J can be also written as*

$$J(q) = \frac{1}{2}\Pi(q, q) - L(q) + \frac{1}{2}\| u_0 - z_d \|_H^2, \ \forall q \in Q. \tag{16}$$

(v) *J is a coercive functional over Q, that is*

$$\begin{aligned}&(1-t)J(q_2) + tJ(q_1) - J((1-t)q_2 + tq_1)\\ &= \frac{t(1-t)}{2}[\|u_{q_2} - u_{q_1}\|_H^2 + M\| q_2 - q_1 \|_Q^2]\\ &\geq \frac{Mt(1-t)}{2}\| q_2 - q_1 \|_Q^2, \ \forall q_1, q_2 \in Q, \forall t \in [0, 1].\end{aligned} \tag{17}$$

(vi) *There exists a unique optimal control $q_{op} \in Q$ such that*

$$J(q_{op}) = \min_{q \in U_{ad}} J(q). \tag{18}$$

Proof (i)-(iii). It follows as [11] and [15]. In particular, we have $u_q = u_0 + z_q$, where u_0 is the unique solution of the variational equality (12) and z_q is the unique solution of the following variational equality:

$$a(z_q, v) = -(q, v)_Q, \ \forall v \in V_0, \ z_q \in V_0.$$

Moreover, we have

$$u_{c_1q_1+c_2q_2} = c_1u_{q_1} + c_2u_{q_2} + (1 - c_1 - c_2)u_0, \ \forall q_1, q_2 \in Q, \forall c_1, c_2 \in \mathbb{R}.$$

(iv)-(v) It follows from the definition of J, Π and L and a similar way that [9].

(vi) It follows taking into account (i)-(v) [9, 14, 15].

We define the adjoint state p_q corresponding to (1) for each $q \in Q$, as the unique solution of the following mixed elliptic problem.

$$-\Delta p_q = u_q - z_d \text{ in } \Omega; \ p_q |_{\Gamma_1} = 0; \ \frac{\partial p_q}{\partial n} |_{\Gamma_2} = 0 \tag{19}$$

whose variational formulation is given by

$$a(p_q, v) = (u_q - z_d, v)_H, \ \forall v \in V_0, \ p_q \in V_0. \tag{20}$$

Lemma 2. *We have*

(i) *The adjoint state* p_q *satisfies the following equalities*:

$$(C(\eta), u_q - z_d)_H = a(p_q, C(\eta)) = -(p_q, \eta)_Q. \tag{21}$$

(ii) *J is a Gâteaux differentiable functional and* J' *is given by*

$$\begin{aligned}\langle J'(q), \eta - q\rangle &= (u_\eta - u_q, u_q - z_d)_H + M(q, \eta - q)_Q \\ &= \Pi(q, \eta - q) - L(\eta - q), \ \forall q, \eta \in Q.\end{aligned} \tag{22}$$

(iii) *The Gâteaux derivative of J can be written as*

$$J'(q) = Mq - p_q, \ \forall q \in Q. \tag{23}$$

(iv) *The optimality condition for the problem* (9) *is given by*

$$(Mq_{op} - p_{qop}, \eta - q_{op})_Q \geq 0, \ \forall \eta \in U_{ad}, \ q_{op} \in U_{ad}. \tag{24}$$

Proof (i). It follows from the definition of p_q and taking into account that

$$a(p_q, C(\eta)) = a(p_q, u_\eta - u_0) = a(p_q, u_\eta) - a(p_q, u_0) = -(p_q, \eta)_Q.$$

(ii) For $t > 0$, we have

$$\begin{aligned}\frac{1}{t}[J(q + t(\eta - q)) - J(q)] &= \frac{t}{2}(u_\eta - u_q, u_\eta - u_q)_H + (u_q - z_d, u_\eta - u_q)_H \\ &\quad + M(q, \eta - q)_Q + \frac{Mt}{2}(\eta - q, \eta - q)_Q\end{aligned}$$

and passing to the limit $t \to 0^+$, we obtain (22).

(iii) From (i) and (ii), we have that $\forall \eta \in Q$:

$$\begin{aligned}\langle J'(q), \eta\rangle &= \Pi(q, \eta) - L(\eta) \\ &= M(q, \eta)_Q + (C(\eta), u_q - z_d)_H = (Mq - p_q, \eta)_Q,\end{aligned}$$

therefore $J'(q) = Mq - p_q$.

(iv) It follows from (ii), [14] and [15].

Now, we obtain some useful estimations.

Lemma 3 (i). *The application* $q \in Q \to u_q \in V$ *is Lipschitzian, i.e.*,

$$\|u_{q_2} - u_{q_1}\|_V \leq \frac{\|\gamma_0\|}{\lambda}\|q_2 - q_1\|_Q, \ \forall q_1, q_2 \in Q, \tag{25}$$

where γ_0 *is the trace operator.*

(ii) *For all* $q_1, q_2 \in Q$, *we have*

$$\|p_{q_2} - p_{q_1}\|_V \leq \frac{1}{\lambda}\|u_{q_2} - u_{q_1}\|_H. \tag{26}$$

(iii) *The application* $q \in Q \to p_q \in V_0$ *is strictly monotone. Moreover, we have*

$$\|u_{q_2} - u_{q_1}\|_H^2 = -(p_{q_2} - p_{q_1}, q_2 - q_1)_Q, \ \forall q_1, q_2 \in Q. \tag{27}$$

(iv) J' *is a Lipschitzian and strictly monotone application, that is*

$$\| J'(q_2) - J'(q_1)\|_Q \leq \left(M + \frac{\|\gamma_0\|^2}{\lambda^2}\right)\| q_1 - q_2\|_Q, \ \forall q_1, q_2 \in Q \tag{28}$$

and

$$\begin{aligned}\langle J'(q_2) - J'(q_1), q_2 - q_1\rangle &= \|u_{q_2} - u_{q_1}\|_H^2 + M\| q_2 - q_1\|_Q^2 \\ &\geq M\| q_2 - q_1\|_Q^2, \ \forall q_1, q_2 \in Q. \end{aligned}\tag{29}$$

Proof (i). This results from the following inequalities:

$$\begin{aligned}\lambda\|u_{q_2} - u_{q_1}\|_V^2 &\leq a(u_{q_2} - u_{q_1}, u_{q_2} - u_{q_1}) = -(u_{q_2} - u_{q_1}, q_2 - q_1)_Q \\ &\leq \| q_2 - q_1\|_Q\|u_{q_2} - u_{q_1}\|_Q \leq \| q_2 - q_1\|_Q\|\gamma_0\|\|u_{q_2} - u_{q_1}\|_V,\end{aligned}$$

where γ_0 is the trace operator.

(ii) This follows as [9].

(iii) If we take $v = p_{q_1} - p_{q_2} \in V_0$ in the variational equality (4) for u_{q_1} and u_{q_2} respectively, then we obtain

$$\begin{aligned}-(p_{q_2} - p_{q_1}, q_2 - q_1)_Q &= a(p_{q_2} - p_{q_1}, u_{q_2} - u_{q_1}) \\ &= a(p_{q_2}, u_{q_2} - u_{q_1}) - a(p_{q_1}, u_{q_2} - u_{q_1}) \\ &= (u_{q_2} - z_d, u_{q_2} - u_{q_1})_H - (u_{q_1} - z_d, u_{q_2} - u_{q_1})_H \\ &= \|u_{q_2} - u_{q_1}\|_H^2, \ \forall q_1, q_2 \in Q.\end{aligned}$$

(iv) By using (23), (25) and (26) for all $q_1, q_2 \in Q$, we have

$$\begin{aligned}\| J'(q_2) - J'(q_1)\|_Q &\leq \|p_{q_2} - p_{q_1}\|_Q + M\| q_2 - q_1\|_Q \\ &\leq \left(M + \frac{\|\gamma_0\|^2}{\lambda^2}\right)\| q_2 - q_1\|_Q\end{aligned}$$

and

$$\langle J'(q_2) - J'(q_1), q_2 - q_1 \rangle = (-p_{q_2} + Mq_2 - (-p_{q_1} + Mq_1), q_2 - q_1)_Q$$

$$= (p_{q_1} - p_{q_2}, q_2 - q_1)_Q + M(q_2 - q_1, q_2 - q_1)_Q$$

$$= \|u_{q_2} - u_{q_1}\|_H^2 + M\| q_2 - q_1\|_Q^2 \geq M\| q_2 - q_1\|_Q^2,$$

that is (28) and (29) respectively.

Theorem 4. *Let $q_{op} \in U_{ad}$ be q_{op} is optimal control in Q if and only if $q_{op} \in Q$ satisfies the complementary conditions*

$$q_{op} \geq 0 \text{ on } \Gamma_2, \; Mq_{op} - p_{q_{op}} \geq 0 \text{ on } \Gamma_2, \; q_{op}(Mq_{op} - p_{q_{op}}) = 0 \text{ on } \Gamma_2. \tag{30}$$

Proof. From the optimality condition (24), taking $\eta = 0 \in U_{ad}$ and $\eta = 2q_{op} \in U_{ad}$, we obtain

$$(Mq_{op} - p_{qop}, q_{op})_Q = 0$$

next

$$(Mq_{op} - p_{qop}, \eta)_Q \geq (Mq_{op} - p_{qop}, q_{op})_Q = 0, \forall \eta \in U_{ad}$$

therefore,

$$Mq_{op} - p_{qop} \geq 0 \text{ on } \Gamma_2$$

and since $q_{op} \geq 0$ on Γ_2, we obtain

$$(Mq_{op} - p_{qop})q_{op} = 0 \text{ on } \Gamma_2$$

next, the thesis holds.

Conversely, $\forall \eta \in U_{ad}$ we have

$$(Mq_{op} - p_{qop}, \eta - q_{op})_Q = (Mq_{op} - p_{qop}, \eta)_Q - (Mq_{op} - p_{qop}, q_{op})_Q$$

$$= (Mq_{op} - p_{qop}, \eta)_Q \geq 0$$

therefore, q_{op} is the optimal control in Q.

Corollary 5. *If we take the boundary optimal control problem* (9) *without restrictions (i.e., $U_{ad} = Q$), then we obtain that $q_{op} = \frac{1}{M} p_{op}$. This relation is of the type obtained in* [3].

3. Problem P_α and its Corresponding Boundary Optimal Control Problem

Let $\Pi_\alpha : Q \times Q \to \mathbb{R}$, $L_\alpha : Q \to \mathbb{R}$ and $C_\alpha : Q \to V$ be defined by

$$\Pi_\alpha(q, \eta) = (C_\alpha(q), C_\alpha(\eta))_H + M(q, \eta)_Q, \ \forall q, \eta \in Q$$

$$L_\alpha(q) = (C_\alpha(q), z_d - u_{0\alpha})_H, \ \forall q \in Q$$

$$C_\alpha(q) = u_{q\alpha} - u_{0\alpha}, \ \forall q \in Q, \tag{31}$$

where $u_{q\alpha}$ is the unique solution of the variational equality (5), $u_{0\alpha}$ is the unique solution of (5) for $q = 0$ whose variational equality is given by

$$a_\alpha(u_{0\alpha}, v) = L_{0\alpha}(v), \ \forall v \in V, \ u_{0\alpha} \in V \tag{32}$$

with

$$L_{0\alpha}(v) = \alpha(b, v)_{L^2(\Gamma_1)} + (g, v)_H \tag{33}$$

and a_α is a bilinear, continuous, symmetric and coercive form on V; that is

$$a_\alpha(v, v) \geq \lambda_\alpha \| v \|_V^2, \ \forall v \in V, \tag{34}$$

where $\lambda_\alpha = \lambda_1 \min(1, \alpha) > 0$ for all $\alpha > 0$ and λ_1 is the coerciveness constant for the bilinear form a_1 [19].

We can obtain analogous properties to Lemma 1, following [9], [14] and [15] which proof is omitted.

Lemma 6. *We have, for each* $\alpha > 0$, *the following properties*:

(i) C_α *is a linear and continuous application.*

(ii) Π_α *is a bilinear, continuous, symmetric and coercive form over* Q, *that is*

$$\Pi_\alpha(q, q) \geq M \| q \|_Q^2, \ \forall q \in Q. \tag{35}$$

(iii) L_α *is linear and continuous on* Q.

(iv) J_α *can be also written as*

$$J_\alpha(q) = \frac{1}{2} \Pi_\alpha(q, q) - L_\alpha(q) + \frac{1}{2} \| u_{0\alpha} - z_d \|_H^2, \ \forall q \in Q. \tag{36}$$

(v) J_α *is a coercive functional over* Q, *that is*

$$(1-t)J_\alpha(q_2)+tJ_\alpha(q_1)-J_\alpha((1-t)q_2+tq_1)$$

$$=\frac{t(1-t)}{2}[\|u_{q_2\alpha}-u_{q_1\alpha}\|_H^2+M\|q_2-q_1\|_Q^2]$$

$$\geq\frac{Mt(1-t)}{2}\|q_2-q_1\|_Q^2,\ \forall q_1,q_2\in Q,\ \forall t\in[0,1]. \tag{37}$$

(vi) *There exists a unique optimal control* $q_{op_\alpha}\in Q$ *such that*

$$J_\alpha(q_{op_\alpha})=\min_{q\in U_{ad}}J_\alpha(q). \tag{38}$$

We define the adjoint state $p_{q\alpha}$ as the unique solution of the following mixed elliptic problem corresponding to (2) or (5) for each $q\in Q$ and $\alpha>0$.

$$-\Delta p_{q\alpha}=u_{q\alpha}-z_d \text{ in } \Omega;\ -\frac{\partial p_{q\alpha}}{\partial n}|_{\Gamma_1}=\alpha p_{q\alpha};\ \frac{\partial p_{q\alpha}}{\partial n}|_{\Gamma_2}=0 \tag{39}$$

whose variational formulation is given by

$$a_\alpha(p_{q\alpha},v)=(u_{q\alpha}-z_d,v),\quad \forall v\in V,\ p_{q\alpha}\in V, \tag{40}$$

where $u_{q\alpha}$ is the unique solution of (5).

Remark 1. We note the double dependence on the parameter α for the optimal state system $u_{q_{op_\alpha}\alpha}$ and the optimal adjoint state $p_{q_{op_\alpha}\alpha}$.

Now, we will obtain some properties of the functional J_α.

Lemma 7. *For each fixed* $\alpha>0$, *we have*

(i) *The adjoint state* $p_{q\alpha}$ *satisfies the following equalities*:

$$(u_{q\alpha}-z_\alpha,C_\alpha(\eta))_H=a_\alpha(p_{q\alpha},C_\alpha(\eta))=-(p_{q\alpha},\eta)_Q,\ \forall q,\eta\in Q.$$

(ii) *The Gâteaux derivative* J'_α *is given by*

$$\langle J'_\alpha(q),\eta\rangle=(u_{q\alpha}-z_d,C_\alpha(\eta))_H+M(q,\eta)_Q$$

$$=\Pi_\alpha(q,\eta)-L_\alpha(q),\ \forall q,\eta\in Q. \tag{41}$$

(iii) *The Gâteaux derivative of* J_α *can be written as*

$$J'_\alpha(q)=Mq-p_{q\alpha},\ \forall q\in Q. \tag{42}$$

(iv) *The optimality condition for problem* (10) *is given by*

$$(Mq_{op_\alpha} - p_{qop_\alpha\alpha}, \eta - q_{op_\alpha})_Q \geq 0, \ \forall \eta \in U_{ad}, \ q_{op_\alpha} \in U_{ad}. \tag{43}$$

Proof. (i) This results from the definition of $p_{q\alpha}$ and the following equalities:

$$a_\alpha(p_{q\alpha}, C_\alpha(\eta)) = a_\alpha(p_{q\alpha}, u_{\eta\alpha} - u_{0\alpha})$$

$$= a_\alpha(p_{q\alpha}, u_{\eta\alpha}) - a_\alpha(p_{q\alpha}, u_{0\alpha}) = -(p_{q\alpha}, \eta)_Q.$$

(ii) We have

$$\frac{1}{t}[J_\alpha(q + t(\eta - q)) - J_\alpha(q)]$$

$$= \frac{t}{2}(u_{\eta\alpha} - u_{q\alpha}, u_{\eta\alpha} - u_{q\alpha})_H + (u_{q\alpha} - z_d, u_{\eta\alpha} - u_{q\alpha})_H$$

$$+ M(q, \eta - q)_Q + \frac{Mt}{2}(\eta - q, \eta - q)_Q$$

and passing to the limit $t \to 0^+$, we obtain (41).

(iii)-(iv) It follows in similar way that Lemma 2.

Lemma 8. *For fixed* $\alpha > 0$, *we have*

(i) *The application* $q \in Q \to u_{q\alpha} \in V$ *is a Lipschitzian operator, that is*

$$\|u_{q_2\alpha} - u_{q_1\alpha}\|_V \leq \frac{\|\gamma_0\|}{\lambda_\alpha}\|q_2 - q_1\|_Q, \ \forall q_1, q_2 \in Q. \tag{44}$$

(ii) *For all* $q_1, q_2 \in Q$, *we have*

$$\|p_{q_2\alpha} - p_{q_1\alpha}\|_V \leq \frac{1}{\lambda_\alpha}\|u_{q_2\alpha} - u_{q_1\alpha}\|_H.$$

(iii) *The operator* $q \in Q \to p_{q\alpha} \in V$ *is strictly monotone, that is*

$$-(p_{q_2\alpha} - p_{q_1\alpha}, q_2 - q_1)_Q = \|u_{q_2\alpha} - u_{q_1\alpha}\|_H^2 \geq 0, \ \forall q_1, q_2 \in Q. \tag{45}$$

(iv) J'_α *is a Lipschitzian and strictly monotone operator, that is*

$$\|J'_\alpha(q_2) - J'_\alpha(q_1)\|_Q \leq \left(M + \frac{\|\gamma_0\|^2}{\lambda_\alpha^2}\right)\|q_2 - q_1\|_Q, \ \forall q_1, q_2 \in Q \tag{46}$$

and

$$\langle J'_\alpha(q_2) - J'_\alpha(q_1),\, q_2 - q_1\rangle = \|u_{q_2\alpha} - u_{q_1\alpha}\|_H^2 + M\| q_2 - q_1 \|_Q^2$$

$$\geq M\| q_2 - q_1 \|_Q^2,\ \forall q_1,\, q_2 \in Q. \tag{47}$$

Proof (i). Its result from the following inequalities:

$$\lambda_\alpha\|u_{q_2\alpha} - u_{q_1\alpha}\|_V^2 \leq a_\alpha(u_{q_2\alpha} - u_{q_1\alpha},\, u_{q_2\alpha} - u_{q_1\alpha})$$

$$\leq \| \gamma_0 \| \| q_2 - q_1 \|_Q \|u_{q_2\alpha} - u_{q_1\alpha}\|_V$$

with γ_0 the trace operator.

(ii) Its follows from

$$\lambda_\alpha\|p_{q_2\alpha} - p_{q_1\alpha}\|_V^2 \leq a_\alpha(p_{q_2\alpha} - p_{q_1\alpha},\, p_{q_2\alpha} - p_{q_1\alpha})$$

$$= (u_{q_2\alpha} - u_{q_1\alpha},\, p_{q_2\alpha} - p_{q_1\alpha})_H$$

$$\leq \|u_{q_2\alpha} - u_{q_1\alpha}\|_H \|p_{q_2\alpha} - p_{q_1\alpha}\|_V.$$

(iii) We have that

$$-(p_{q_2\alpha} - p_{q_1\alpha},\, q_2 - q_1)_Q = (u_{q_2\alpha} - u_{q_1\alpha},\, C_\alpha(q_2 - q_1))_H$$

$$= \|u_{q_2\alpha} - u_{q_1\alpha}\|_H^2 \geq 0,\ \forall q_1,\, q_2 \in Q. \tag{48}$$

(iv) By using (i) and (ii), we have

$$\| J'_\alpha(q_2) - J'_\alpha(q_1) \|_Q \leq \|p_{q_2\alpha} - p_{q_1\alpha}\|_Q + M\| q_2 - q_1 \|_Q$$

$$\leq \left(M + \frac{\| \gamma_0 \|^2}{\lambda_\alpha^2} \right) \| q_2 - q_1 \|_Q$$

therefore, J'_α is a Lipschitzian application. On the other hand, we get

$$\langle J'_\alpha(q_2) - J'_\alpha(q_1),\, q_2 - q_1\rangle$$

$$= (p_{q_2\alpha} + Mq_2 - (p_{q_1\alpha} + Mq_1),\, q_2 - q_1)_Q$$

$$= \|u_{q_2\alpha} - u_{q_1\alpha}\|_H^2 + M\| q_2 - q_1 \|_Q^2 \geq M\| q_2 - q_1 \|_Q^2$$

and J'_α is a strictly monotone application.

Theorem 9. *Let* $q_{op_\alpha} \in U_{ad}$, q_{op_α} *is optimal control in* Q *if and only if* $q_{op_\alpha} \in Q$ *satisfies the complementary conditions*

$$q_{op_\alpha} \geq 0 \; on \; \Gamma_2, \; Mq_{op_\alpha} - p_{q_{op_\alpha}\alpha} \geq 0 \; on \; \Gamma_2,$$

$$q_{op_\alpha}(M_{q_{op_\alpha}} - p_{q_{op_\alpha}\alpha}) = 0 \; on \; \Gamma_2. \tag{49}$$

Proof. It follows in similar way to the one given in Theorem 4.

Corollary 10. *If we take the boundary optimal control problem* (10) *without restrictions* (*i.e.*, $U_{ad} = Q$), *then we obtain that* $q_{op_\alpha\alpha} = \frac{1}{M} p_{op_\alpha\alpha}$ *for each* $\alpha > 0$.

4. Convergence of the Problem P_α and its Corresponding Optimal Control as $\alpha \to \infty$

Theorem 11. *For all* $\alpha > 0, q \in Q, b \in H^{\frac{1}{2}}(\Gamma_1)$, *we have the following limits*:

$$\text{(i)} \quad \lim_{\alpha \to \infty} \| u_{q\alpha} - u_q \|_V = 0, \; \forall q \in Q$$

$$\text{(ii)} \quad \lim_{\alpha \to \infty} \| u_{0\alpha} - u_0 \|_V = 0$$

$$\text{(iii)} \quad \lim_{\alpha \to \infty} \| p_{q\alpha} - p_q \|_V = 0, \; \forall q \in Q. \tag{50}$$

Proof. (i) If we take $v = u_{q\alpha} - u_q$ in the variational equality (5), for g, α with $\alpha > 1$, in similar way that [9] and [19], then we obtain

$$\begin{aligned}
&\lambda_1 \| u_{q\alpha} - u_q \|_V^2 + (\alpha - 1)(u_{q\alpha} - u_q, u_{q\alpha} - u_q)_{L^2(\Gamma_1)} \\
&\leq a_\alpha(u_{q\alpha} - u_q, u_{q\alpha} - u_q) \\
&= -a(u_q, u_{q\alpha} - u_q) + (q, u_{q\alpha} - u_q)_H - (q, u_{q\alpha} - u_q)_Q \\
&\leq C_1 \| u_{q\alpha} - u_q \|_V
\end{aligned}$$

with C_1 a constant independent of α. Next, for large α, we have

$$\| u_{q\alpha} - u_q \|_V^2 \leq \frac{C_1}{\lambda_1}$$

and

$$(\alpha - 1)(u_{q\alpha} - u_q, u_{q\alpha} - u_q)_{L^2(\Gamma_1)} \leq \frac{(C_1)^2}{\lambda_1}$$

therefore, we deduce that there exists $w_q \in V$ such that

$$u_{q\alpha} \rightharpoonup w_q \text{ weakly in } V$$

and

$$(u_{q\alpha} - b, u_{q\alpha} - b)_{L^2(\Gamma_1)} \leq \frac{(C_1)^2}{\lambda_1(\alpha - 1)} \to 0 \text{ as } \alpha \to \infty$$

next, $w_q \in K$ and taking the limit in the variational equality (5) when $\alpha \to \infty$, we have

$$a(w_q, v) = L_q(v), \forall v \in V_0, w_q \in K$$

and by uniqueness, we have $w_q = u_q$.

Therefore, $u_{q\alpha} \to u_q$ strongly in V as $\alpha \to \infty$ because of the following inequality:

$$\lambda_1 \| u_{q\alpha} - u_q \|_V^2 \leq L_q(u_{q\alpha} - u_q) - a(u_q, u_{q\alpha} - u_q).$$

For the case (ii), we take $q = 0$ in (i).

(iii) In this case, we take $v = p_{q\alpha} - p_q$ in the variational equality (40) for g and α, following a similar method as before, we obtain

$$\lambda_1 \| p_{q\alpha} - p_q \|_V^2 + (\alpha - 1)(p_{q\alpha} - p_q, p_{q\alpha} - p_q)_{L^2(\Gamma_1)}$$

$$\leq a_\alpha(p_{q\alpha} - p_q, p_{q\alpha} - p_q) \leq C_2 \| p_{q\alpha} - p_q \|_V$$

with C_2 a constant independent of α. Next, for large α, we have

$$\| p_{q\alpha} - p_q \|_V^2 \leq \frac{C_2}{\lambda_1}$$

and

$$(\alpha - 1)(p_{q\alpha} - p_q, p_{q\alpha} - p_q)_{L^2(\Gamma_1)} \leq \frac{(C_2)^2}{\lambda_1}$$

therefore, we deduce that there exists $\xi_q \in V$ such that

$$p_{q\alpha} \rightharpoonup \xi_q \text{ weakly in } V$$

and

$$(p_{q\alpha} - p_q, p_{q\alpha} - p_q)_{L^2(\Gamma_1)} \leq \frac{(C_2)^2}{\lambda_1(\alpha - 1)} \to 0 \text{ as } \alpha \to \infty$$

that is $\xi_q \in V_0$ and taking the limit on the variational equality (40) for $p_{q\alpha}$, we have

$$a(\xi_q, v) = (u_q - z_d, v), \forall v \in V_0, \xi_q \in V_0$$

next, by uniqueness, we obtain $\xi_q = p_q$. Therefore, taking into account the following inequality

$$\lambda_1 \| p_{q\alpha} - p_q \|_V^2 \leq (u_{q\alpha} - z_d, p_{q\alpha} - p_q)_H - a(p_q, p_{q\alpha} - p_q),$$

we have that $p_{q\alpha} \to p_q$ strongly in V.

Now, we will prove that the optimal control q_{op_α} of problem (10) and its corresponding optimal adjoint states $p_{q_{op_\alpha}\alpha}$ and optimal system states $u_{q_{op_\alpha}\alpha}$ are convergent to the optimal control q_{op} of problem (9) and its corresponding optimal adjoint state $p_{q_{op}}$ and optimal system state $u_{q_{op}}$ respectively, when the parameter α (heat transfer coefficient on Γ_1) goes to infinity.

Theorem 12 (i). *If* $p_{q_{op}}$ *and* $p_{q_{op_\alpha}\alpha}$ *are the corresponding adjoint state of the problems* (9) *and* (10) *respectively, then*

$$\lim_{\alpha \to \infty} \| p_{q_{op_\alpha}\alpha} - p_{q_{op}} \|_V = 0. \tag{51}$$

(ii) *If* q_{op} *and* q_{op_α} *are the solutions of the problems* (9) *and* (10) *respectively, then*

$$\lim_{\alpha \to \infty} \| q_{op_\alpha} - q_{op} \|_Q = 0. \tag{52}$$

(iii) *If* $u_{q_{op}}$ *and* $u_{q_{op_\alpha}\alpha}$ *are the corresponding solutions of the problem* P *and problem* P_α *respectively, then*

$$\lim_{\alpha \to \infty} \| u_{q_{op_\alpha}\alpha} - u_{q_{op}} \|_V = 0. \tag{53}$$

Proof. Since q_{op_α} is the solution of the problem (10), we have the following inequality:

$$\frac{1}{2} \| u_{q_{op_\alpha}\alpha} - z_d \|_H^2 + \frac{M}{2} \| q_{op_\alpha} \|_Q^2 \leq \frac{1}{2} \| u_{q\alpha} - z_d \|_H^2 + \frac{M}{2} \| q \|_Q^2, \forall q \in Q,$$

taking $q = 0$, we have

$$\frac{1}{2}\|u_{q_{op_\alpha}\alpha} - z_d\|_H^2 + \frac{M}{2}\|q_{op_\alpha}\|_Q^2 \le \frac{1}{2}\|u_{0\alpha} - z_d\|_H^2 \le C_3, \ \forall \alpha > 0,$$

where C_3 is a constant independent of parameter α because $u_{0\alpha}$ is convergent when $\alpha \to \infty$. Therefore,

$$\|q_{op_\alpha}\|_Q \le C_4 \ \text{ and } \ \|u_{q_{op_\alpha}\alpha}\|_H \le C_5, \tag{54}$$

where C_4 and C_5 are constants independent of α.

Now, if we take $v = u_{q_{op_\alpha}\alpha} - u_{q_{op}}$ in the variational equality (5), following [9] and [19], we obtain for $\alpha > 1$:

$$\lambda_1\|u_{q_{op_\alpha}\alpha} - u_{q_{op}}\|_V^2 + (\alpha - 1)(u_{q_{op_\alpha}\alpha} - u_{q_{op}}, u_{q_{op_\alpha}\alpha} - u_{q_{op}})_{L^2(\Gamma_1)}$$

$$\le a_\alpha(u_{q_{op_\alpha}\alpha} - u_{q_{op}}, u_{q_{op_\alpha}\alpha} - u_{q_{op}}) \le C_6\|u_{q_{op_\alpha}\alpha} - u_{q_{op}}\|_V,$$

where $C_6 = C_6(q_{op}, g, u_{q_{op}}, \|\gamma_0\|)$ is independent of α. Next, we have

$$\|u_{q_{op_\alpha}\alpha} - u_{q_{op}}\|_V^2 \le \frac{C_6}{\lambda_1} \tag{55}$$

and

$$(\alpha - 1)(u_{q_{op_\alpha}\alpha} - u_{q_{op}}, u_{q_{op_\alpha}\alpha} - u_{q_{op}})_{L^2(\Gamma_1)} \le \frac{(C_6)^2}{\lambda_1}$$

therefore, we deduce that

$$\exists \eta \in V \ \text{ such that } u_{q_{op_\alpha}\alpha} \rightharpoonup \eta \text{ weakly in } V \tag{56}$$

and because the following inequalities:

$$0 \le (\eta - u_{q_{op}}, \eta - u_{q_{op}})_{L^2(\Gamma_1)}$$

$$\le \liminf_{\alpha \to \infty}(u_{q_{op_\alpha}\alpha} - u_{q_{op}}, u_{q_{op_\alpha}\alpha} - u_{q_{op}})_{L^2(\Gamma_1)} = 0,$$

we obtain that $\eta \in K$.

Next, if we take $v = p_{q_{op_\alpha}\alpha} - p_{q_{op}}$ in the variational equality (40), we get

$$\lambda_1\|p_{q_{op_\alpha}\alpha} - p_{q_{op}}\|_V^2 + (\alpha - 1)(p_{q_{op_\alpha}\alpha} - p_{q_{op}}, p_{q_{op_\alpha}\alpha} - p_{q_{op}})_{L^2(\Gamma_1)}$$

$$\le a_\alpha(p_{q_{op_\alpha}\alpha} - p_{q_{op}}, p_{q_{op_\alpha}\alpha} - p_{q_{op}}) \le C_7\|p_{q_{op_\alpha}\alpha} - p_{q_{op}}\|_V$$

with $C_7 = C_7(C_5, p_{q_{op}})$. Next, we obtain

$$\|p_{q_{op_\alpha}\alpha} - p_{q_{op}}\|_V^2 \leq \frac{C_7}{\lambda_1} \tag{57}$$

and

$$(\alpha - 1)(p_{q_{op_\alpha}\alpha} - p_{q_{op}}, p_{q_{op_\alpha}\alpha} - p_{q_{op}})_{L^2(\Gamma_1)} \leq \frac{(C_7)^2}{\lambda_1}$$

therefore, we deduce that

$$\exists \xi \in V \text{ such that } p_{q_{op_\alpha}\alpha} \rightharpoonup \xi \text{ weakly in } V \tag{58}$$

and from the following inequalities:

$$\begin{aligned} 0 &\leq (\xi - p_{q_{op}}, \xi - p_{q_{op}})_{L^2(\Gamma_1)} \\ &\leq \liminf_{\alpha \to \infty}(p_{q_{op_\alpha}\alpha} - p_{q_{op}}, p_{q_{op_\alpha}\alpha} - p_{q_{op}})_{L^2(\Gamma_1)} = 0, \end{aligned}$$

we obtain $\xi \in V_0$.

Now, we consider $v \in V_0$ and taking into account (56) and (58) from the variational equality (40), we have

$$a(\xi, v) = (\eta - z_d, v), \forall v \in V_0, \xi \in V_0. \tag{59}$$

Next from (54), we deduce that there exists $f \in Q$ such that $q_{op_\alpha} \rightharpoonup f$ weakly in Q. Therefore, if we put $v \in V_0$ in the variational equality (5) and we pass to the limit $\alpha \to \infty$, we obtain

$$a(\eta, v) = (g, v)_H - (f, v)_Q, \forall v \in V_0, \eta \in K.$$

Now

$$a(\eta, v) = L_f(v), \forall v \in V_0, \eta \in K \tag{60}$$

and from the uniqueness of the solution of the variational equality (4), we have

$$\eta = u_f. \tag{61}$$

On the other hand, from (59), (61) and the uniqueness of the solution of the variational equality (20), we get

$$\xi = p_f.$$

Now

$$J_\alpha(q_{op_\alpha}) \leq J_\alpha(f^*), \forall f^* \in U_{ad}$$

next

$$J(f) = \frac{1}{2}\|u_f - z_d\|_H^2 + \frac{M}{2}\| f \|_Q^2 = \frac{1}{2}\| \eta - z_d \|_H^2 + \frac{M}{2}\| f \|_Q^2$$

$$\leq \liminf_{\alpha\to\infty} J_\alpha(q_{op_\alpha}) \leq \liminf_{\alpha\to\infty} J_\alpha(f^*) = \lim_{\alpha\to\infty} J_\alpha(f^*)$$

$$= J(f^*),\ \forall f^* \in U_{ad}$$

and from the uniqueness of the optimal control problem (9), we obtain that $f = q_{op}$.

Therefore, $\eta = u_f = u_{q_{op}}$ and $\xi = p_f = p_{q_{op}}$.

Moreover, from (58) and the following computation:

$$\lambda_1\|p_{q_{op_\alpha}\alpha} - p_{q_{op}}\|_V^2 \leq a_\alpha(p_{q_{op_\alpha}\alpha} - p_{q_{op}}, p_{q_{op_\alpha}\alpha} - p_{q_{op}})$$

$$= a_\alpha(p_{q_{op_\alpha}\alpha}, p_{q_{op_\alpha}\alpha} - p_{q_{op}}) - a(p_{q_{op}}, p_{q_{op_\alpha}\alpha} - p_{q_{op}})$$

$$= (u_{q_{op_\alpha}\alpha} - z_d, p_{q_{op_\alpha}\alpha} - p_{q_{op}})_H - a(p_{q_{op}}, p_{q_{op_\alpha}\alpha} - p_{q_{op}}),$$

we have (51).

From the optimality conditions (24) and (43), we deduce that

$$(M(q_{op} - q_{op_\alpha}) + (p_{q_{op_\alpha}\alpha} - p_{q_{op}}), q_{op_\alpha} - q_{op})_Q \geq 0$$

now

$$M\|q_{op_\alpha} - q_{op}\|_Q^2 \leq \|p_{q_{op_\alpha}\alpha} - p_{q_{op}}\|_Q\|q_{op_\alpha} - q_{op}\|_Q$$

next

$$\|q_{op_\alpha} - q_{op}\|_Q \leq \frac{\|\gamma_0\|}{M}\|p_{q_{op_\alpha}\alpha} - p_{q_{op}}\|_V$$

and therefore, (52) holds.

From (25), we have

$$\|u_{q_{op_\alpha}} - u_{q_{op}}\|_V \leq \frac{\|\gamma_0\|}{\lambda}\|q_{op_\alpha} - q_{op}\|_Q$$

and taking into account (52), we get (53).

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