SUFFICIENT CONDITIONS TO OBTAIN A STEADY STATE-STEFAN PROBLEM WITH INTERNAL ENERGY AND DIRICHLET AND ROBIN BOUNDARY CONDITIONS.

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ABSTRACT: We consider the problem of the steady-state temperature distribution of a material submitted to an internal energy g. We assume the material is contained in a regular bounded domain $\Omega \subset \mathbb{R}^n$, $\partial\Omega = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$, with a fixed positive temperature on Γ_1 and a Robin condition for the heat flux on Γ_2 (or Newton's type law). We obtain monotonicity properties for the temperature and we state two different sufficient conditions in order it is of non constant sign in Ω, that is, we have steady state two-phase Stefan problem.We also show an example where the necessary and sufficient conditions are given.

RESUMEN: Consideramos el problema de la distribución estacionaria de temperatura de un material sometido a una energía interna g. Asumimos que el material está contenido en un dominio regular acotado $\Omega \subset \mathbb{R}^n$, $\partial\Omega = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$, con temperatura positiva fija sobre Γ_1 y tal que el flujo satisface sobre Γ_2 una condición de Robin (o ley de tipo Newton). Demostramos propiedades de monotonía para la temperatura y establecemos dos tipos diferentes de condiciones suficientes para que la temperatura no sea de signo constante en Ω, es decir, para tener un problema estacionario de Stefan a dos fases.También mostramos un ejemplo donde se dan condiciones necesarias y suficientes.

KEY WORDS : steady-state Stefan problem, variational inequalities, mixed boundary conditions.

AMS SUBJECT CLASSIFICATION : 35R35, 35J05, 35J85.

1. INTRODUCTION

We consider the problem of the steady-state temperature distribution of a body or a container with a material which is submitted to an internal energy g. We assume the body to be a regular bounded domain $\Omega \subset \mathbb{R}^n$, with $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and Γ_i are sufficiently regular disjoint portions of $\partial\Omega$ for $i = 1, 2, 3$ and Γ_1 and Γ_2 have positive $(n-1)$ dimensional measure. For simplicity we state the phase change temperature of the material occupying Ω to be 0^oC.

We fix the temperature on Γ_1 to be positive and the heat flux on Γ_2 to verify a Newton's type law (or a Robin condition) and to be null on Γ_3.

In this paper we study necessary or sufficient conditions for data such that we have two phases of the material in $\overline{\Omega}$ (i.e. the temperature is of non constant sign in $\overline{\Omega}$). ([3], [5], [8], [9]).

The weak formulation of our problem is given by

$$(1)\qquad \begin{cases} -\Delta u = g \quad \text{in } \mathcal{D}'(\Omega) \\ u|_{\Gamma_1} = B > 0 \\ -\dfrac{\partial u}{\partial n}\Big|_{\Gamma_2} = \alpha\,(u - T) \\ \dfrac{\partial u}{\partial n}\Big|_{\Gamma_3} = 0 \end{cases}$$

where $u = k_2\theta^+ - k_1\theta^-$, θ is the temperature, k_1 and k_2 are respectively the thermal conductivities of solid and liquid phase, $b = \dfrac{B}{k_2}$ and T are fixed temperatures on Γ_1 and Γ_2 respectively, α is a heat transfer coefficient on Γ_2 and n denotes the exterior normal to Γ_2. ([4])

It is well known ([7], [10]), that the variational formulation of (1) is given by

$$(2)\qquad \begin{cases} a_\alpha\,(u,v) = L_{\alpha Tg}\,(v)\,, \qquad \forall v \in V_1 \\ u \in K_B \end{cases}$$

where

$$(3)\qquad a_\alpha\,(u,v) = \int_\Omega \nabla u \nabla v\, dx + \alpha \int_{\Gamma_2} \gamma_2\,(u)\,\gamma_2\,(v)\; ds$$

$$(4)\qquad L_{\alpha Tg}\,(v) = \int_\Omega g v\, dx + \alpha \int_{\Gamma_2} T\gamma_2\,(v)\; ds$$

with

$$(5)\quad V = H^1\,(\Omega)\,, \quad V_1 = \{v \in V : \gamma_1\,(v) = 0\}\,, \quad K_B = \{v \in V : \gamma_1\,(v) = B\}$$

and γ_i are the 0-order trace operator on Γ_i for $i = 1, 2$.

The bilinear form a_1 is coercive on V, i.e.

$$(6)\qquad \exists \lambda_1 > 0/ \quad a_1\,(v,v) \geq \lambda_1 \,\|v\|_V^2\,, \quad \forall v \in V$$

and therefore a_α is also coercive on V with a constant of coerciveness given by $\lambda_\alpha = \lambda_1 . \min\,(1, \alpha)$. ([7], [10])

Moreover, if $\alpha > 0$, $g \in L^2\,(\Omega)$, $B \in H^{\frac{1}{2}}\,(\Gamma_1)$ and $T \in H^{\frac{1}{2}}\,(\Gamma_2)$, problem (2) has a unique solution $u_{\alpha gTB} \in K_B$ ([4], [7], [10]). In the sequel we will study the solution operator

$$\Lambda : \mathbb{R}^+ \times L^2\,(\Omega) \times H^{\frac{1}{2}}\,(\Gamma_2) \times H^{\frac{1}{2}}\,(\Gamma_1) \longrightarrow H^1\,(\Omega)$$

such that

$$(7)\qquad (\alpha, g, T, B) \longmapsto \Lambda\,(\alpha, g, T, B) = u_{\alpha gTB}.$$

It is well known that some difficulties due to the mixed boundary conditions do arrives to prove regularity of $u_{\alpha gTB}$ ([6]). Sufficient conditions to obtain a H^2 regularity for elliptic mixed boundary problems are given, among others, in [1] and, recently in [2], but our interest is to state hypothesis in order to have a solution of non-constant sign.

We obtain monotonicity properties of $\Lambda(\alpha, g, T, B)$ and we state two different sufficient conditions for it to be of non constant sign in Ω (see Theorems 8 and 10).

Finally we check everything in an example for which we obtain the necessary and sufficient conditions in order to have a steady-state two-phase Stefan problem in Ω.

Throughout this paper, the statement $f \geq 0$ where f belongs to any subspace of $L^2(A)$ will mean $f(x) \geq 0$ a.e. in A, and in order to simplify notation, on Γ_1 and Γ_2, u will mean $\gamma_1(u)$ and $\gamma_2(u)$ respectively.

2. ASYMPTOTIC BEHAVIOUR OF $\Lambda(\alpha, g, T, B)$

It is physically reasonable to expect that if $\alpha \longrightarrow +\infty$ then $u_{\alpha gTB}|_{\Gamma_2} \longrightarrow T$.

This fact leads us to consider the function $u = u_{gTB}$, which is the unique solution of the mixed partial differential problem

$$(8) \qquad \begin{cases} -\Delta u = g & \text{in } \mathcal{D}'(\Omega) \\ u|_{\Gamma_1} = B \\ u|_{\Gamma_2} = T \\ \left.\dfrac{\partial u}{\partial n}\right|_{\Gamma_3} = 0 \end{cases}$$

whose equivalent variational formulation is given by:

$$(9) \qquad \begin{cases} a(u,v) = L_g(v), & \forall v \in V_{12} \\ u \in K_{BT} \end{cases}$$

where

$$(10) \qquad a(u,v) = \int_\Omega \nabla u \nabla v \, dx, \qquad L_g(v) = \int_\Omega gv \, dx$$

with

$$(11) \qquad V_{12} = \{v \in V / \ v = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}$$

$$(12) \qquad K_{BT} = \{v \in V / \ v = B \text{ on } \Gamma_1 \text{ and } v = T \text{ on } \Gamma_2\}$$

Theorem 1. *If $\alpha > 1$, $g \in L^2(\Omega)$, $B \in H^{\frac{1}{2}}(\Gamma_1)$ and $T \in H^{\frac{1}{2}}(\Gamma_2)$, $u_{\alpha gTB}$ and u_{gTB} are respectively the unique solutions of problems (2) and (9) then: 1)* $\|u_{\alpha gTB} - u_{gTB}\|_{H^1(\Omega)} \leq \dfrac{k}{\lambda_1}$

2) $(\alpha - 1) \int_{\Gamma_2} (u_{\alpha gTB} - u_{gTB})^2 \, ds \leq \dfrac{k^2}{\lambda_1}$

where k is a constant which depends on u_{gTB} and λ_1 depends only on Ω and Γ_2.

3) $\lim\limits_{\alpha \to +\infty} u_{\alpha gTB} = u_{gTB}$ *strongly in V.*

Proof. We name $u_\alpha = \Lambda(\alpha, g, T, B)$ and $u = u_{gTB}$, then, as $u \in K_{BT} \subset K_B$, we can choose $v = u$ in (2) and (7) and we obtain for $w_\alpha = u_\alpha - u$

$$(13) \qquad \lambda_1 \|w_\alpha\|^2_{H^1} + (\alpha - 1) \int_{\Gamma_2} w_\alpha^2 \, ds \leq a_\alpha(w_\alpha, w_\alpha) = -a(u, w_\alpha) + L_g(w_\alpha)$$

$$\leq \left(\|u\|_V + \|g\|_{L_2}\right) \|w_\alpha\|_V$$

Then, if we call $k = \|u\|_V + \|g\|_{L_2}$ we have

(14) $$\forall \alpha > 1 \qquad \|w_\alpha\|_V \leq \frac{k}{\lambda_1} \qquad \text{and} \qquad (\alpha - 1) \int_{\Gamma_2} w_\alpha^2 \, ds \leq \frac{k^2}{\lambda_1}.$$

These a priori estimates ensure the existence of $u^* \in V$ and a convergent subsucesion of u_α, which for simplicity we will also name u_α such that $u_\alpha \rightharpoonup u^*$ weakly in V and, moreover,

$$\liminf_{\alpha \longrightarrow +\infty} \int_{\Gamma_2} (u_\alpha - T)^2 \, ds = 0.$$

The functional $v \longrightarrow \int_{\Gamma_2} v^2 \, ds$ is weakly lower semicontinuous in V, then

$$0 \leq \int_{\Gamma_2} (u^* - T)^2 \, ds \leq \liminf_{\alpha \longrightarrow +\infty} \int_{\Gamma_2} (u_\alpha - T)^2 \, ds = 0$$

and therefore $u^* \in K_B$.The zero order trace operator γ_1 on Γ_1 is continuous, then $u^* \in K_{BT}$, and $u = u^*$ by uniqueness of the solution of the variational equation (9). Thus, we have proved $\lim_{\alpha \to +\infty} u_{\alpha g T B} = u_{gTB}$ weakly in V.Besides, $w_\alpha \rightharpoonup 0$ weakly in V and (13) implies $w_\alpha \to 0$ strongly in V, i.e. $\lim_{\alpha \to +\infty} u_{\alpha g T B} = u_{gTB}$ strongly in V. □

3. MONOTONICITY PROPERTIES OF $\Lambda(\alpha, g, T, B)$

We will prove the operator $\Lambda(\alpha, g, T, B)$ satisfies some monotonicity properties and order bound.

Proposition 1. *If* $u_i = \Lambda(\alpha, g_i, T_i, B_i), \quad i = 1, 2,\ g_1 \leq g_2$ *in* $L^2(\Omega)$, $T_1 \leq T_2$ *in* $H^{\frac{1}{2}}(\Gamma_2)$, $B_1 \leq B_2$ *in* $H^{\frac{1}{2}}(\Gamma_1)$, *then* $u_1 \leq u_2$ *a.e. in* Ω.

Proof. i) We first consider the case $g_1 = g_2 = g$ and $T_1 = T_2 = T.B_1 \leq B_2 \Longrightarrow w = (u_1 - u_2)^+ \in V_1$ and $0 \leq a_\alpha(u_1 - u_2, w) = a(w, w) + \alpha \int_{\Gamma_2} w^2 \, ds = 0$. Then, $w = 0$, that is $u_1 \leq u_2$ a.e. in Ω.

ii) Now we state $B_1 = B_2 = B$, $g_1 \leq g_2$ and $T_1 \leq T_2$. As $w = (u_1 - u_2)^+ \in V_1, 0 \leq a_\alpha(u_1 - u_2, w) = a_\alpha(w, w) = \int_\Omega (g_1 - g_2) w \, dx + \alpha \int_{\Gamma_2} (T_1 - T_2) w \, ds \leq 0$ and $u_1 \leq u_2$ a.e. in Ω.The general case easily follows from (i) and (ii). □

Proposition 2. *Let* $T = T_0$ *be constant on* Γ_2, $\alpha \in \mathbb{R}^+$.

a) If $g \leq 0$ *in* $L^2(\Omega)$, $B \leq T_0$ *in* $H^{\frac{1}{2}}(\Gamma_1)$, *then* $u_{\alpha g T_0 B} = \Lambda(\alpha, g, T_0, B) \leq T_0$ *a.e. in* Ω.

b) If $g \geq 0$ *in* $L^2(\Omega)$, $B \geq T_0$ *in* $H^{\frac{1}{2}}(\Gamma_1)$, *then* $u_{\alpha g T B} \geq T_0$ *a.e. in* Ω.

c) If $g = 0$, $B = T_0$, *then* $u_{\alpha g T B} = B = T_0$.

Proof. a) $a_\alpha(u_{\alpha g T_0 B} - T_0, v) = \int_\Omega g v \, dx, \quad \forall v \in V_1$.

$B \leq T_0 \Rightarrow w = (u_{\alpha g T_0 B} - T_0)^+ \in V_1$ and then $0 \leq a_\alpha(w, w) = \int_\Omega g w \, dx \leq 0$.

Thus $u_{\alpha g T_0 B} \leq T_0$.
b) and c) are obtained similarly. □

An immediate consequence of this property and Theorem 1 is the following:

Proposition 3. *Let $T = T_0$ be constant on Γ_2, $u_{\alpha_i} = \Lambda(\alpha_i, g, T_0, B)$, $i = 1, 2$ and $1 < \alpha_1 \leq \alpha_2$.*

a) If $g \leq 0$ in $L^2(\Omega)$, $B \leq T_0$ in $H^{\frac{1}{2}}(\Gamma_1)$, then $u_{\alpha_1} \leq u_{\alpha_2}$ and $u_{\alpha g T_0 B} \uparrow u_{g T_0 B}$ strongly in V.

b) If $g \geq 0$ in $L^2(\Omega)$, $B \geq T_0$ in $H^{\frac{1}{2}}(\Gamma_1)$, then $u_{\alpha_2} \leq u_{\alpha_1}$ and $u_{\alpha g T_0 B} \downarrow u_{g T_0 B}$ strongly in V.

Proof. We consider $w = (u_1 - u_2)^+ \in V_1$. Then, as $u_1 \leq T_0$ by Prop. 3, we have

$$0 \leq a(w, w) = a(u_1 - u_2, w) = \int_{\Gamma_2} [\alpha_1 (T_0 - u_1) - \alpha_2 (T_0 - u_2)]\, w\, ds =$$

$$(\alpha_1 - \alpha_2) \int_{\Gamma_2} (T_0 - u_1)\, w\, ds - \alpha_2 \int_{\Gamma_2} w^2\, ds \leq 0.$$

Then $u_{\alpha_1} \leq u_{\alpha_2}$.
b) is obtained in a similar way.□

□

The next result easily follows from propositions 2 and 3.

Proposition 4. *Let $T = T_0$ be constant on Γ_2 and $g_1 \leq 0 \leq g_2$ in $L^2(\Omega)$, $T_1 \leq T_0 \leq T_2$ in $H^{\frac{1}{2}}(\Gamma_2)$, $B_1 \leq T_0 \leq B_2$ in $H^{\frac{1}{2}}(\Gamma_1)$, then $u_{\alpha_1 g_1 T_1 B_1} \leq T_0 \leq u_{\alpha_2 g_2 T_2 B_2}$ a.e. in Ω, $\forall \alpha_1, \alpha_2 \in \mathbb{R}^+$.*

4. NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF TWO PHASES IN Ω

From now on we will consider without loss of generality $B \geq 0$ as in the Introduction.

It is physically reasonable to expect that, in the case $g \geq 0$, $T \geq 0$ and $B \geq 0$ the solution $u_{\alpha g T B}$ will be non negative, but this result is a trivial consequence of proposition 3, taking into account that $u_{\alpha 000} = 0$. Thus, denying the above statement we have the following:

Proposition 5. *If $B \geq 0$, a necessary condition for the existence of two phases in Ω ($u_{\alpha g T B}$ is of non constant sign in Ω) is that $g \leq 0$ or $T \leq 0$, non simultaneously null.*

In order to obtain sufficient conditions we will first analyze the case of constant data.

Lemma 1. *Let g, T, B be constant, then*

$$u_{\alpha g T B} = \Lambda(\alpha, g, T, B) = T + (B - T)\, U_{1\alpha} + g U_{2\alpha}$$

where $U_{1\alpha} = \Lambda(\alpha, 0, 0, 1)$ and $U_{2\alpha} = \Lambda(\alpha, 1, 0, 0)$ verify:

i) $0 \leq U_{1\alpha} \leq 1$ *in* $L^2(\Omega)$.
ii) $U_{1\alpha} \downarrow u_{001}$ *strongly in* V *when* $\alpha \longrightarrow +\infty$.
iii) $\forall\ 0 < \varepsilon < 1,\ \exists \Gamma^1_{\alpha,\varepsilon} \subset \Gamma_2$ *with* $\left|\Gamma^1_{\alpha,\varepsilon}\right| = \int_{\Gamma^1_{\alpha,\varepsilon}} ds > 0$ *such that*

$U_{1\alpha} \leq \varepsilon$ *a.e. on* $\Gamma^1_{\alpha,\varepsilon} \forall \alpha \geq \dfrac{C_1}{\varepsilon^2}$ *where* $C_1 = 1 + \dfrac{\|u_{001}\|_V^2}{\lambda_1 |\Gamma_2|}$ *depends only on* Ω, Γ_1 *and* Γ_2.
iv) $0 \leq U_{2\alpha}$ *in* $L^2(\Omega)$.
v) $U_{2\alpha} \downarrow u_{100}$ *in* $L^2(\Omega)$ *when* $\alpha \longrightarrow \infty$.
vi) $\forall\ \varepsilon > 0,\ \exists \Gamma^2_{\alpha,\varepsilon} \subset \Gamma_2$ *with* $\left|\Gamma^2_{\alpha,\varepsilon}\right| = \int_{\Gamma^2_{\alpha,\varepsilon}} ds > 0$ *such that*

$U_{2\alpha} \leq \varepsilon$ *a.e. on* $\Gamma^2_{\alpha,\varepsilon}\ \forall \alpha \geq \dfrac{C_2}{\varepsilon^2}$ *where* $C_2 = 1 + \dfrac{\left(\|u_{001}\|_V + |\Omega|\right)^2}{\lambda_1 |\Gamma_2|}$ *depends only on* Ω, Γ_1 *and* Γ_2.
vii) $U_{2\alpha} \in L^\infty(\Omega)$.
viii) $\forall \alpha > 1,\ \exists \Omega_1 \subset \Omega$ *with* $|\Omega_1| = \int_{\Omega_1} dx > 0$ *such that*

$U_{2\alpha} \geq C_3$ *a.e. in* Ω_1 *where* $C_3 = \dfrac{\lambda_1 \|u_{100}\|_V^2}{|\Omega|}$ *depends only on* Ω *and* Γ_2.

Proof. It is well known from the linearity of Problem I that, if g, T and B are constant

$$u_{\alpha gTB} = B\Lambda(\alpha,0,0,1) + T\Lambda(\alpha,0,1,0) + g\Lambda(\alpha,1,0,0)$$

and moreover,

$$\Lambda(\alpha,0,1,0) = 1 - \Lambda(\alpha,0,0,1).$$

Then, we have

$$u_{\alpha gTB} = T + (B - T) U_{1\alpha} + gU_{2\alpha}$$

where $U_{1\alpha} = \Lambda(\alpha,0,0,1)$ and $U_{2\alpha} = \Lambda(\alpha,1,0,0)$ are solutions of (2).

(i), (ii), (iv) and (v) are obviously consequences of the maximum principle or of propositions 2-4.

Besides, from Theorem 1

$$(\alpha - 1)\int_{\Gamma_2} (U_{1\alpha} - u_{001})^2\, ds = (\alpha - 1)\int_{\Gamma_2} U_{1\alpha}^2\, ds \leq \frac{k^2}{\lambda_1}.$$

Then, for any fixed $0 < \varepsilon < 1$, if $\alpha \geq \dfrac{C_1}{\varepsilon^2} > 1 + \dfrac{k^2}{\lambda_1 \varepsilon^2 |\Gamma_2|}$ where $C_1 = 1 + \dfrac{\|u_{001}\|_V^2}{\lambda_1 |\Gamma_2|}$.

It results

$$\int_{\Gamma_2} U_{1\alpha}^2\, ds \leq \varepsilon^2 |\Gamma_2|.$$

$\therefore \exists \Gamma^1_{\alpha,\varepsilon}$ with $\left|\Gamma^1_{\alpha,\varepsilon}\right| > 0$ such that $U_{1\alpha} \leq \varepsilon$ a.e. on $\Gamma^1_{\alpha,\varepsilon}$.

A similar result (vi) holds for $U_{2\alpha}$, where we the constant C_1 is replaced by $C_2 = 1 + \dfrac{\left(\|u_{001}\|_V + |\Omega|\right)^2}{\lambda_1 |\Gamma_2|}$.

In order to prove vii) we will seek for $F \in L^\infty\left(\overline{\Omega}\right)$ such that $U_{2\alpha} \le F$.

We propose a polynomial

$$F(x) = -\frac{1}{2}(x_1)^2 + \sum_{i=1}^{n} p_i x_i + p_0 + q \in L^\infty\left(\overline{\Omega}\right)$$

with $-\frac{1}{2}(x_1)^2 + \sum\limits_{i=1}^{n} p_i x_i + p_0 \ge 0$, $\forall x \in \overline{\Omega}$.

Then, $\forall q > 0$ it results $F(x) \ge q > 0$ in $\overline{\Omega}$.

Moreover,

$$\left.\left(\frac{\partial F}{\partial n} + \alpha F\right)\right|_{\Gamma_2} \ge \left.(\nabla F \times n)\right|_{\Gamma_2} + \alpha q \quad \forall \alpha > 0$$

and

$$|\nabla F \times n| \le \|\nabla F\| = \left[(-x_1 + p_1)^2 + \sum_{i \ne 1} p_i^2\right]^{\frac{1}{2}} \le A \quad \text{on } \partial\Omega.$$

If we choose $q > A$, it results $\left.\left(\dfrac{\partial F}{\partial n} + \alpha F\right)\right|_{\Gamma_2} \ge -A + \alpha q > 0$, $\forall \alpha > 1$.

Then, we obtain from Proposition 2

$$F(x) = \Lambda\left(\alpha, 1, \frac{1}{\alpha}\left.\left(\frac{\partial F}{\partial n} + \alpha F\right)\right|_{\Gamma_2}, F|_{\Gamma_1}\right) \ge \Lambda(\alpha, 1, 0, 0) = U_{2\alpha} \quad \text{a.e. in } \overline{\Omega}.$$

Therefore $U_{2\alpha} \in L^\infty\left(\overline{\Omega}\right)$.

To prove viii) we recall that, from (v) we have $\forall \alpha > 1$

$$\int_\Omega U_{2\alpha}\, dx \ge \int_\Omega u_{100}\, dx = a(u_{100}, u_{100}) = a_1(u_{100}, u_{100}) \ge \lambda_1 \|u_{100}\|_V^2.$$

Therefore $\exists \Omega_1 \subset \Omega$ with $|\Omega_1| = \int\limits_{\Omega_1} dx > 0$ such that

$$U_{2\alpha} \ge C_3 = \frac{\lambda_1 \|u_{100}\|_V^2}{|\Omega|} \quad \text{a.e. in } \overline{\Omega}_1.$$

□

Theorem 2. *Let* g, T, B *be constant and* $B > 0$. *If* $g < -\dfrac{\max(T, B)}{c_3}$, *then* $u_{\alpha g T B}$ *is of non constant sign in* Ω $\forall \alpha > 1$.

Proof. By lemma 7 we have

$$u_{\alpha g T B} = T + (B - T) U_{1\alpha} + g U_{2\alpha}$$

and $\exists \Omega_1 \subset \Omega$ with $|\Omega_1| > 0$ such that $U_{2\alpha} \ge c_3$ a.e. in Ω_1.

For $g < 0$ we obtain

i) If $T \le B$, $u_{\alpha g T B} \le B + g U_{2\alpha} \le B + g C_3$ a.e. in $\overline{\Omega}_1$.

ii) If $T > B$, $u_{\alpha gTB} \leq T + gU_{2\alpha} \leq T + gC_3$ a.e. in $\overline{\Omega}_1$.

Then, if $\max(T, B) < -gC_3$ it results $u_{\alpha gTB}|_{\overline{\Omega}_1} < 0$ and as we have $u_{\alpha gTB}|_{\Gamma_1} = B > 0$ it follows that u is of non constant sign in $\overline{\Omega}$. □

An immediate consequence of this theorem and the monotone property 2 is the following:

Corollary 1. *Let $\alpha > 0$, $g \in L^2(\Omega)$, $B \in H^{\frac{1}{2}}(\Gamma_1)$ and $T \in H^{\frac{1}{2}}(\Gamma_2)$ be such that $g \leq g_0$ a.e. in Ω, $0 < B \leq B_0$ a.e. on Γ_1 and $T \leq B_0$ a.e. on Γ_2, where the constants g_0 and B_0 satisfy $g_0 < -\frac{B_0}{C_3}$ then $u_{\alpha gTB}$ is of non constant sign in $\overline{\Omega}$.*

Theorem 3. *Let $g, T, B > 0$ be constant. Then, for all fixed $0 < \varepsilon < 1$ and $\alpha \geq \frac{C_1}{\varepsilon^2}$, $u_{\alpha gTB}$ changes sign in $\overline{\Omega}$ if one of the following conditions are satisfied:*

i) $g \leq 0$ and $T < -\frac{\varepsilon B}{1-\varepsilon}$

or

ii) $g > 0$ and $T < -\frac{\varepsilon B + gH}{1-\varepsilon}$, with $H = ess \sup_{\overline{\Omega}} |F(x)|$ where $F(x)$ is the polynomial defined in lemma 7 vii).

Proof.

$$u_{\alpha gTB} = T + (B - T) U_{1\alpha} + gU_{2\alpha}$$

We consider $T < 0$ and from lemma 7 iii), if $0 < \varepsilon < 1$ and $\alpha \geq \frac{C_1}{\varepsilon^2}$ there exists $\Gamma^1_{\alpha,\varepsilon}$ such that

$$u_{\alpha gTB} \leq T + (B - T)\varepsilon + gU_{2\alpha} \quad \text{a.e. on } \Gamma^1_{\alpha,\varepsilon}.$$

Then, we have

a) If $g \leq 0$

$$u_{\alpha gTB} \leq (1 - \varepsilon) T + \varepsilon B \quad \text{a.e. on } \Gamma^1_{\alpha,\varepsilon}$$

and therefore, for $T < -\frac{\varepsilon B}{1-\varepsilon}$ it results $u_{\alpha gTB}|_{\Gamma^1_{2,\alpha}} < 0$.

b) If $g > 0$, we have proved in lemma 7 vii) that

$$|U_{2\alpha}|_{L^\infty(\overline{\Omega})} \leq |F(x)|_{L^\infty(\Omega)} = H$$

where H depends only on $\overline{\Omega}$, then

$$u_{\alpha gTB} \leq (1 - \varepsilon) T + \varepsilon B + gH < 0 \quad \text{a.e. on } \Gamma^1_{\varepsilon\alpha} \Longleftrightarrow T < -\frac{\varepsilon B + gH}{1-\varepsilon}.$$

□

Again, an immediate consequence of this theorem and the monotone property 2 is the following

Corollary 2. *For any fixed $0 < \varepsilon < 1$ and $\alpha \geq \frac{C_1}{\varepsilon^2}$, let $g \in L^2(\Omega)$, $B \in H^{\frac{1}{2}}(\Gamma_1)$ and $T \in H^{\frac{1}{2}}(\Gamma_2)$ be such that*

i) $g \leq g_0 \leq 0$ a.e. in Ω, $0 < B \leq B_0$ a.e. on Γ_1 and $T \leq T_0 < -\frac{\varepsilon B}{1-\varepsilon}$.

or

ii) $0 \leq g \leq g_0$ *a.e. in* Ω, $0 < B \leq B_0$ *a.e. on* Γ_1 *and* $T \leq T_0 < -\dfrac{\varepsilon B + H}{1-\varepsilon}$

then $u_{\alpha gTB}$ *is of non constant sign in* $\overline{\Omega}$.

5. Example

We consider $\Omega = (0, x_0) \times (0, y_0)$, $\Gamma_1 = \{0\} \times [0, y_0]$, $\Gamma_2 = \{x_0\} \times [0, y_0]$ and $\Gamma_3 = (0, x_0) \times \{0\} \cup (0, x_0) \times \{y_0\}$.

Let g, T and $B > 0$ be constant, then

$$U_{1\alpha} = 1 - \frac{\alpha}{1+\alpha x_0} x, \qquad U_{2\alpha} = -\frac{x^2}{2} + \frac{x_0 (2 + \alpha x_0)}{2 (1 + \alpha x_0)} x \tag{15}$$

and

$$u_{\alpha gTB} = -\frac{g}{2} x^2 + \frac{(T - B)\alpha + \frac{g x_0}{2} (2 + \alpha x_0)}{(1 + \alpha x_0)} x + B. \tag{16}$$

In this case we can obtain a necessary and sufficient condition for $u_{\alpha gTB}$ to be of non constant sign in Ω. From (16) we have

$$u_{\alpha gTB} = -\frac{g}{2}(x - x_{v_\alpha})^2 + y_{v_\alpha}$$

$$x_{v_\alpha} = \frac{(T - B)\alpha + \frac{g x_0}{2} (2 + \alpha x_0)}{g (1 + \alpha x_0)} x + B \tag{17}$$

$$y_{v_\alpha} = B + \frac{g x_{v_\alpha}^2}{2}.$$

Then $u_{\alpha gTB}$ is of non constant sign in Ω if and only if

$$g \geq 0 \text{ and } u_{\alpha gTB}(x_0) < 0 \tag{18}$$

or

$$g < 0,\ y_{v_\alpha} < 0, \text{ and } 0 < x_{v_\alpha} \leq x_0 \text{ or } g \geq 0 \text{ and } u_{\alpha gTB}(x_0) < 0. \tag{19}$$

We have, for $g \geq 0$

$$u_{\alpha gTB}(x_0) < 0 \Leftrightarrow T < -\frac{x_0}{2\alpha} g - \frac{B}{\alpha x_0}. \tag{20}$$

If $g < 0$, we obtain

$$y_{v_\alpha} < 0 \Leftrightarrow T > B - \frac{g x_0}{2\alpha}(2 + \alpha x_0) + (1 + \alpha x_0)\sqrt{(-2Bg)} \tag{21}$$

$$0 < x_{v_\alpha} \leq x_0 \Leftrightarrow B + \frac{g x_0^2}{2} \leq T < B - \frac{g x_0}{2\alpha}(2 + \alpha x_0) \tag{22}$$

$$u_{\alpha gTB}(x_0) < 0 \Leftrightarrow T < -\frac{x_0}{2\alpha} g - \frac{B}{\alpha x_0}. \tag{23}$$

Taking into account $(18)-(23)$, the necessary and sufficient conditions for $u_{\alpha gTB}$ to be of non constant sign in Ω become, $\forall\alpha>0$:

(24)
$$\begin{array}{ll} T<T_{1\alpha}(g) & \text{if } g\geq 0 \\ T<T_{1\alpha}(g) \text{ or } \begin{cases} T_2(g)\leq T \\ T_{3\alpha}(g)>T \end{cases} & \text{if } -\frac{2B}{x_0^2}<g<0 \\ T<T_{3\alpha}(g) & \text{if } g\leq -\frac{2B}{x_0^2}. \end{array}$$

where $T_{1\alpha}(g)=-\dfrac{x_0}{2\alpha}g-\dfrac{B}{\alpha x_0}$, $T_2(g)=B+\dfrac{gx_0^2}{2}$, and $T_{3\alpha}(g)=B-\dfrac{gx_0}{2\alpha}(2+\alpha x_0)$.

This conditions can also be expressed as:

(25)
$$\begin{array}{llll} g & < & g_{1\alpha}(T)=-\dfrac{2\alpha}{x_0}T-\dfrac{2B}{x_0^2} & \text{if } T<0 \\ g & \leq & g_2(T)=\frac{2(T-B)}{x_0^2} & \text{if } 0\leq T<B \\ g & < & g_{3\alpha}(T)=-\frac{2(T-B)}{x_0(2+\alpha x_0)} & \text{if } T\geq B. \end{array}$$

Then we obtain the following property:

Theorem 4. *If g,T and $B>0$ are constant and $\Omega=(0,x_0)\times(0,y_0)$, $\Gamma_1=\{0\}\times[0,y_0]$, $\Gamma_2=\{x_0\}\times[0,y_0]$ and $\Gamma_3=(0,x_0)\times\{0\}\cup(0,x_0)\times\{y_0\}$, then the necessary and sufficient conditions in order the temperature $u_{\alpha gTB}$ is of non-constant sign in Ω (we have a steady-state two-phase Stefan problem in Ω) is given by (24) or equivalently by (25).*

We compare (24) or (25) with the sufficient conditions that we have obtained in Theorems 8 and 10.

From Lemma 7, (15) and (16), we can take $C_3=\dfrac{x_0^2}{a}$, with $a>8$, $C_1=\dfrac{1}{4x_0}$, $H=\dfrac{x_0^2}{2}$ and we have:

* From Theorem 8, $\forall\alpha>0$

(26)
$$g<g_4(T)=\begin{cases} -\dfrac{aB}{x_0^2} \text{ if } T\leq B \\ -\dfrac{aT}{x_0^2} \text{ if } T>B. \end{cases}$$

* From Theorem 8, $\forall 0<\varepsilon<1$,

(27)
$$\text{i) } T < T_4(\varepsilon)=-\frac{B}{\frac{1}{\varepsilon}-1} \quad \text{if } \alpha\geq\frac{1}{4x_0\varepsilon^2},\ g\leq 0$$

$$\text{ii) } T < T_5(\varepsilon)=-\frac{B}{\frac{1}{\varepsilon}-1}-\frac{x_0^2}{2(1-\varepsilon)}g \quad \text{if } \alpha\geq\frac{1}{4x_0\varepsilon^2},\ g>0$$

It is easy to verify that $(26)\Longrightarrow(25)$ and $(27)\Longrightarrow(24)$.

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