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The Steady-State Two-Phase Stefan Problem with an Internal Energy and Some Related Problems (**).

1. – Introduction.

We consider the problem of the steady temperature distribution of a body or a container with a fluid, which is submited to an internal energy g.

We assume the body to be a bounded domain $\Omega \subset \mathbf{R}^n$, with a sufficently regular boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, Γ_1 and Γ_2 being disjoint portions of $\partial\Omega$ of positive $(n-1)$ dimensional measure. Assuming a phase-change temperature of 0 °C for the material occupying Ω, keep Γ_1 at the temperature $\theta = b > 0$ and maintain a heat flux q on Γ_2 and a null heat flux on Γ_3. Assuming a stady-state problem, we can expect a phase change to take place in Ω if the internal energy g in Ω and the outflow of heat q through Γ_2 are small and large enough respectively. This paper is devoted to obtain necessary and/or sufficient conditions for q and g such that θ takes negative and positive values in Ω (two phases are present).

The temperature $\theta = \theta(x)$ can be represented in the following way

$$(1.1)\qquad \theta(x) = \begin{cases} \theta_1(x) < 0, & x \in \Omega_1 \text{ (solid phase)}, \\ 0, & x \in \mathcal{L} \text{ (free boudary)}, \\ \theta_2(x) > 0, & x \in \Omega_2 \text{ (liquid phase)}, \end{cases}$$

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where $\Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{L}$, and satisfies the conditions below

$$(1.2)\qquad \begin{cases} -k_i \Delta\theta_i = g & \text{in } \Omega_i \ (i=1,2)\,, \\ \theta_1 = \theta_2 = 0, \quad k_1 \dfrac{\partial \theta_1}{\partial n} = k_2 \dfrac{\partial \theta_2}{\partial n} & \text{on } \mathcal{L}\,, \\ \theta_2 \big|_{\Gamma_1} = b\,, \quad \dfrac{\partial \theta}{\partial n}\Big|_{\Gamma_3} = 0\,, & \\ -k_2 \dfrac{\partial \theta_2}{\partial n} = q & \text{if } \theta > 0 \text{ on } \Gamma_2\,, \\ -k_1 \dfrac{\partial \theta_1}{\partial n} = q & \text{if } \theta < 0 \text{ on } \Gamma_2\,, \end{cases}$$

where $k_i > 0$ is the thermal conductivity of phase i ($i = 1$ solid phase, $i = 2$ liquid phase).

If we define the new unknown function u as follows [3, 7, 14]

$$(1.3)\qquad u = k_2 \theta^+ - k_1 \theta^- \quad \text{in } \Omega\,,$$

we obtain the problem

$$(1.4)\qquad -\Delta u = g \quad \text{in } \Omega\,, \qquad u|_{\Gamma_1} = B\,, \qquad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q\,, \qquad \frac{\partial u}{\partial n}\Big|_{\Gamma_3} = 0\,,$$

with

$$(1.5)\qquad B = k_2 b > 0\,.$$

The notation above and in the sequel is the following: n is the outer normal to Γ_2 (or Γ_3); $|\Omega|$ denotes n-dimensional Lebesgue measure of Ω; $|\Gamma|$ denotes $(n-1)$ dimensional Lebesgue measure of Γ; Γ_1, Γ_2 and Γ_3 are assumed to be smooth such that the solutions of some elliptic problems, which appeared in this paper, are functions of $H^2(\Omega) \cap C^0(\bar{\Omega})$.

In section 2 we present the variational formulation of (1.4), whose solution is well known [9, 14]. We give several monotonic properties of the solution u to (1.4) and some comparison theorems that will allow us to estimate, from above and below (section 3), the critical flux function $q = q_c(B, g)$ such that (following [2]):

— for (q, g) with $q \leq q_c(B, g)$, $\theta > 0$ in Ω (only one phase is present),

— for (q, g) with $q > q_c(B, g)$, θ takes negative and positive values in Ω (two phases are present).

We also obtain (following [15]) some useful functional derivatives.

In section 4 we solve some heat flux optimization problems with temperature constrains (following [8]) and we give three steady-state examples [15] with explicit solution (section 5) which illustrate all previous theoretical results. Moreover, we can exhibit and example with a mushy region.

This work has grown out as a generalization of the results obtained in [2, 8, 15] in the case $g = 0$.

2. – Variational formulation and some general properties.

We recall the variational formulation of (1.4). Let:

$$\begin{cases} a(u, v) = \int_\Omega \nabla u \nabla v \, dx \,, \\ L(v) = \int_\Omega gv \, dx - \int_{\Gamma_2} qv \, d\gamma \ \ (= L_{qg}(v)) \,, \\ V = H^1(\Omega) \,, \quad V_0 = \{v \in V : v|_{\Gamma_1} = 0\} \,, \\ K = \{v \in V : v|_{\Gamma_1} = B\} \ \ (= K_B) \,, \end{cases} \tag{2.1}$$

with $L \in V_0'$ (*e.g.* $q \in L^2(\Gamma_2)$, $g \in L^2(\Omega)$) and $B \in H^{1/2}(\Gamma_1)$, then the unique solution $u = u_{Bqg}$ of (1.4) is characterized by [9, 14]:

$$u \in K_B \,, \quad a(u, v) = L(v) \,, \quad \forall v \in V_0 \,, \tag{2.2}$$

and also by the minimum problem

$$u \in K_B \,, \quad J(u) \leqslant J(v) \,, \quad \forall v \in K_B \,, \tag{2.3}$$

with

$$J(v) = \frac{1}{2} a(v, v) - L(v) \,. \tag{2.4}$$

Linearity implies that the unique solution u_{Bqg} of (1.4) is also characterized by

$$u_{Bqg} = u_B + u_q + u_g \quad \text{in } \Omega \,, \tag{2.5}$$

where u_B, u_q and u_g are respectively defined by

$$u_B \in K_B \,, \quad a(u_B, v) = 0 \,, \quad \forall v \in V_0 \,, \tag{2.6}$$

$$u_q \in V_0 \,, \quad a(u_q, v) = -\int_{\Gamma_2} qv \, d\gamma \,, \quad \forall v \in V_0 \,, \tag{2.7}$$

$$u_g \in V_0 \,, \quad a(u_g, v) = \int_\Omega gv \, dx \,, \quad \forall v \in V_0 \,, \tag{2.8}$$

REMARK 2.1. In the case B, q, g are constant on Γ_1, Γ_2 and in Ω respectively, it results

$$u_{Bqg} = B - qu_1 + gu_2 \quad \text{in } \Omega , \tag{2.9}$$

where

$$u_1 \in V_0 , \quad a(u_1 , v) = \int_{\Gamma_2} v \, d\gamma , \quad \forall v \in V_0 , \tag{2.10}$$

and

$$u_2 \in V_0 , \quad a(u_2 , v) = \int_{\Omega} v \, dx , \quad \forall v \in V_0 . \tag{2.11}$$

We recall the maximum principle [9, 13] to prove:

$$u_1 \quad \text{and} \quad u_2 \text{ are positive functions in } \Omega . \tag{2.12}$$

Next, we give a monotonicity property of the solution to problem (2.2) as a function of the data B (or b), q and g.

LEMMA 2.1. If $u = u_{Bqg}$ is the unique solution to problem (2.2) for data functions B $(= k_2 b)$, q and g then, we have:

i) If $B_1 \leqslant B_2$ (or $b_1 \leqslant b_2$) on Γ_1, $q_2 \leqslant q_1$ on Γ_2, and $g_1 \leqslant g_2$ in Ω, then

$$u_1 = u_{B_1 q_1 g_1} \leqslant u_{B_2 q_2 g_2} = u_2 \quad \text{in } \bar{\Omega} . \tag{2.13}$$

ii) A strict inequality is obtained for u_i if either of the inequalities for B_i, q_i or g_i is strict.

We consider for fixed $B > 0$, the unique solution u to (2.2). It is easy to prove that

$$\int_{\Omega} g u^- \, dx + a(u^- , u^-) = \int_{\Gamma_2} q u^- \, d\gamma \tag{2.14}$$

and this result leads immediately to the following:

LEMMA 2.2. The unique solution u to (2.2) for fixed positive $B \in H^{1/2}(\Gamma_1)$, positive $q \in L^2(\Gamma_2)$ and non negative $g \in L^2(\Omega)$ verifies:

$$u^- \neq 0 \quad \text{in } \Omega \Leftrightarrow u^- \neq 0 \quad \text{on } \Gamma_2 . \tag{2.15}$$

REMARK 2.2. In other words, if $g \geq 0$ in Ω, there will be a change of phase in Ω if and only if u takes negative values on Γ_2. We can also achieve this fact by using the maximum principle [9, 13].

Now, we define the real functions $F_{Bg}\colon \mathbf{R}\to\mathbf{R}$ and $F_{Bq}\colon \mathbf{R}\to\mathbf{R}$ such that

$$F_{Bg}(q) = J(u_{Bqg}) = \frac{1}{2}a(u_{Bqg}, u_{Bqg}) - \int_{\Omega} g u_{Bqg}\,dx + q\int_{\Gamma_2} u_{Bqg}\,d\gamma, \tag{2.16}$$

for fixed positive $B \in H^{1/2}(\Gamma_1)$ and $g \in L^2(\Omega)$, and

$$F_{Bq}(g) = J(u_{Bqg}) = \frac{1}{2}a(u_{Bqg}, u_{Bqg}) - g\int_{\Omega} u_{Bqg}\,dx + \int_{\Gamma_2} q u_{Bqg}\,d\gamma, \tag{2.17}$$

for fixed positive $B \in H^{1/2}(\Gamma)$ and $q \in L^2(\Gamma_2)$. Then, we obtain the following properties:

THEOREM 2.3. The functions F_{Bg} and F_{Bq} verify:

i) $F_{Bg} \in \mathcal{C}^1(\mathbf{R})$ with

$$F'_{Bg}(q) = \int_{\Gamma_2} u_{Bqg}\,d\gamma, \tag{2.18}$$

and F'_{Bg} is a strictly decreasing function.

ii) $F_{Bq} \in \mathcal{C}^1(\mathbf{R})$ with

$$F'_{Bq}(g) = -\int_{\Omega} u_{Bqg}\,dx, \tag{2.19}$$

and F'_{Bq} is a strictly decreasing function.

PROOF. We shall only include a proof for ii). The item i) is proved using similar techniques and the same result is obtained in [15] for the gase $g = 0$. First we prove that the function $g \to \int_{\Omega} u_{Bqg}\,dx$ is continuous.

Let α be the coercivity constant on V_0 of the bilinear form a, and $h \in \mathbf{R}$. Taking into account the Cauchy-Schwarz and the Poincarè-Friedrichs inequalities (with constant $\mathcal{C}(\Omega)$) we obtain:

$$\|u_{Bq(g+h)} - u_{Bqg}\|_V \leqslant \frac{|\Omega|^{1/2}\mathcal{C}(\Omega)}{\alpha}|h|, \tag{2.20}$$

and

$$\left|\int_{\Omega}(u_{Bq(g+h)} - u_{Bqg})\,dx\right| \leqslant \frac{|\Omega|\mathcal{C}(\Omega)}{\alpha}|\eta|. \tag{2.21}$$

It is easy to verify that:

$$\frac{F_{Bq(g+h)} - F_{Bqg}}{h} = -\frac{1}{2}\int_{\Omega} (u_{Bq(g+h)} + u_{Bqg})\, dx , \tag{2.22}$$

and hence, $F_{Bq} \in \mathcal{C}^1(\mathcal{R})$ with F'_{Bq} given by (2.19).

Moreover F'_{Bq} is a strictly decreasing function as a consequence of Lemma 2.1.

COROLLARY 2.4. i) If $q_0 \in \mathrm{R}$, such that $F'_{B_0 g_0}(q_0) < 0$ for a fixed positive $B_0 \in H^{1/2}(\Gamma_1)$ and non negative $g_0 \in L^2(\Omega)$, then u_{Bqg} is of non-constant sign in Ω (two phases are present) for all $B \in H^{1/2}(\Gamma_1)$, $0 < B \leqslant B_0$; $g \in L^2(\Omega)$, $0 \leqslant g \leqslant g_0$ and $q \in L^2(\Gamma_2)$ such that $\operatorname{Inf}_{x \in \Gamma_2} q(x) \geqslant q_0$.

ii) If $g_1 \in \mathrm{R}$ such that $F'_{B_1 q_1}(g_1) > 0$ for a fixed positive $B_1 \in H^{1/2}(\Gamma_1)$ and $q_1 \in L^2(\Gamma_2)$, then u_{Bqg} is of non-constant sign in Ω for all $B \in H^{1/2}(\Gamma_1)$, $0 < B \leqslant B_1$; $q \in L^2(\Gamma_2)$, $q \geqslant q_1$ and $g \in L^2(\Omega)$ such that $\sup_{x \in \Omega} g(x) \leqslant g_1$.

PROOF. The results follow from the above Theorem and Lemmas 2.1 and 2.2 (the last one only in case i)). We can also use the maximum principle.

3. – Some estimates for the critical heat flux which characterizes a two-phase steady Stefan problem.

We shall now consider the case $B > 0$ and constant on Γ_1, q constant on Γ_2 and $g \in L^2(\Omega)$. We define a critical heat flux function

$$q_c : \mathrm{R}^+ L^2(\Omega) \to \mathrm{R}, \qquad (B, g) \to q_c(B, g) \tag{3.1}$$

such that

— for each $B > 0$ and $q \leqslant q_c(B, g)$, $u_{Bqg} \geqslant 0$ in Ω (no phase change),

— for each $B > 0$ and $q > q_c(B, g)$, u_{Bqg} is a function of non-constant sign in Ω (two phases are present).

THEOREM 3.1. q_c is a non decreasing function, that is for all $0 < B_1 \leqslant B_2$ and for all $g_1, g_2 \in L^2(\Omega)$, $g_1 \leqslant g_2$ it results

$$q_c(B_1, g_1) \leqslant q_c(B_2, g_2). \tag{3.2}$$

PROOF. From Lemma 2.1, $0 \leqslant u_{B_1 q_c(B_1, g_1) g_1} \leqslant u_{B_2 q_c(B_1, g_1) g_2}$ in $\bar{\Omega}$ and hence we have the thesis.

In next theorem we follow the idea of [2] and we give some estimates for $q_c(B, g)$.

THEOREM 3.2. i) Set $g \geq 0$ and w_{Bg} denote the solution of

$$-\Delta w_{Bg} = g \text{ in } \Omega, \qquad w_{Bg}|_{\Gamma_1} = B, \qquad w_{Bg}|_{\Gamma_2} = 0, \qquad \frac{\partial w_{Bg}}{\partial n}\bigg|_{\Gamma_3} = 0. \tag{3.3}$$

Define q_i: $\mathbf{R}^+ (L^2(\Omega))^+ = \mathbf{R}^+ \{g \in L^2(\Omega): g > 0 \text{ in } \Omega\} \to \mathbf{R}$ such that

$$q_i(B, g) = \operatorname*{Inf}_{\Gamma_2}\left(-\frac{\partial w_{Bg}}{\partial n}\bigg|_{\Gamma_2}\right). \tag{3.4}$$

Then, for all $B > 0$ and $g \geqslant 0$, $q \leqslant q_i(B, g)$ implies $u_{Bqg} \geqslant w_{Bg}$ in Ω.

ii) Let $P_2 \in \Gamma_2$ and we define, if it is possible, the affine function π_B to be such that

$$\pi_B|_{\Gamma_1} \geqslant B, \qquad \pi_B(P_2) = 0, \qquad \pi_B|_{\Gamma_2} \geqslant 0, \qquad \frac{\partial \pi_B}{\partial n}\bigg|_{\Gamma_3} \geqslant 0 \tag{3.5}$$

and

$$z_{Bg} = \pi_B + \tilde{w}_g \qquad \text{in } \Omega, \tag{3.6}$$

where $\tilde{w}_g$ denotes the solution of

$$-\Delta \tilde{w}_g = g, \qquad \tilde{w}_g|_{\Gamma_1} = 0, \qquad \tilde{w}_g|_{\Gamma_2} = 0, \qquad \frac{\partial \tilde{w}_g}{\partial n}\bigg|_{\Gamma_3} = 0. \tag{3.7}$$

Let q_s: $\mathbf{R}^+ (L^2(\Omega))^+ \to \mathbf{R}$ be such that

$$q_s(B, g) = \operatorname*{Sup}_{\Gamma_2}\left(-\frac{\partial z_{Bg}}{\partial n}\bigg|_{\Gamma_2}\right). \tag{3.8}$$

Then, for all $B > 0$ and $g \geqslant 0$, $q > q_s(B, g)$ implies $u_{Bqg} < z_{Bg}$ in Ω.

iii) We have

$$q_i(B, g) \leqslant q_c(B, g) \leqslant q_s(B, g), \qquad \forall g \in L^2(\Omega),\ g \geqslant 0,\ B \in \mathbf{R}^+. \tag{3.9}$$

PROOF. We use the maximum principle.

REMARK 3.1. We note that $w_g \leqslant \pi_B + \tilde{w}_g$ in Ω, and, if $w_g \neq \pi_B + \tilde{w}_g$, we have $q_i(B,g) < q_s(B,g)$. Sufficient conditions for π_B to exist are the same as in the case $g=0$ and can be seen in [2].

THEOREM 3.3. Let F_{Bg} be defined as in (2.16), then we have

(3.10) i) $$F'_{Bg}(q) < 0 \Leftrightarrow q > q_0(B) + \frac{c_g}{c_1},$$

with

(3.11) $$c_1 = a(u_1, u_1) = \int_{\Gamma_2} u_1 \, d\gamma > 0,$$

(3.12) $$c_g = a(u_g, u_1) = \int_{\Omega} g u_1 \, dx = \int_{\Gamma_2} u_g \, d\gamma,$$

(3.13) $$q_0(B) = \frac{B|\Gamma_2|}{c_1} > 0,$$

(u_1 and u_g are defined by (2.10) and (2.8) respectively).

(3.14) ii) $$q_0(B) + \frac{c_g}{c_1} > q_c(B,g), \quad \forall g \in L^2(\Omega), \ g \geqslant 0, \ B \in \mathrm{R}^+,$$

that is, if $q > q_0(B) + c_g/c_1$ then u_{Bqg} is of non-constant sign in Ω.

PROOF. i) We use (2.5)-(2.10) in (2.16) and we obtain

(3.15) $$F_{Bg}(q) = -\frac{1}{2} a(u_g, u_g) + q a(u_g, u_1) - \frac{q^2}{2} a(u_1, u_1) + \\ + B\left[q|\Gamma_2| - \int_{\Omega} g \, dx\right].$$

Therefore

(3.16) $$F'_{Bg}(q) = a(u_g, u_1) + B|\Gamma_2| - q a(u_1, u_1) = c_g + B|\Gamma_2| - q c_1.$$

It results from (2.12) that $c_1 > 0$ and then

$$F'_{Bg}(q) < 0 \Leftrightarrow q > \frac{B|\Gamma_2|}{c_1} + \frac{c_g}{c_1} = q_0(B) + \frac{c_g}{c_1}.$$

ii) It results from i) and Lemma 2.2, taking into account that, for $g \geq 0$, $c_g = \int_{\Omega} g u_1 \, dx > 0$ and then $q_0(B) + c_g/c_1 > 0$.

REMARK 3.2. In the case g is constant, we have $c_g = gc_{12}$ with

$$c_{12} = a(_1, u_2) = \int_\Omega u_1\, dx = \int_{\Gamma_2} u_2\, d\gamma > 0, \quad (u_2 \text{ defined by (2.11)}) \tag{3.17}$$

and

$$q_0(B) + \frac{c_{12}}{c_1} g > q_c(B, g), \qquad \forall g \geqslant 0. \tag{3.18}$$

In a similar way, we shall consider now the case $B > 0$ and constant on Γ_1, $q \in L^2(\Gamma_2)$ and g constant in Ω. It results:

$$F_{Bq}(g) = -\frac{g^2}{2} a(u_2, u_2) - g[a(u_2, u_q) + B|\Omega|] + \\ + B\int_{\Gamma_2} q\, d\gamma + \frac{1}{2}\int_{\Gamma_2} q u_q\, d\gamma, \tag{3.19}$$

$$F'_{Bq}(g) = -a(u_2, u_2) g - a(u_2, u_q) - B|\Omega|, \tag{3.20}$$

where u_2 and u_q are defined by (2.11) and (2.7) respectively. We obtain the following

THEOREM 3.4. Let F_{Bq} be defined by (2.17), then we have:

$$\text{i)} \qquad F'_{Bq}(g) > 0 \Leftrightarrow g < \frac{c_q}{c_2} - g_0(B), \tag{3.21}$$

with

$$c_2 = a(u_2, u_2) = \int_\Omega u_2\, dx > 0, \tag{3.22}$$

$$c_q = -a(u_2, u_q) = -\int_\Omega u_q\, dx = \int_{\Gamma_2} q u_2\, d\gamma, \tag{3.23}$$

$$g_0(B) = \frac{B|\Omega|}{c_2} > 0. \tag{3.24}$$

ii) If $g < -g_0(B) + c_q/c_2$ then u_{Bqg} is of non-constant sign in Ω.

REMARK 3.3. Note that in the above theorem neither q nor g are required to be positive or non negative, as they were in the previous theorems.

REMARK 3.4. In the case q is constant on Γ_2, we have $c_q = qc_{12}$ and

$$q_1(B) + \frac{c_2}{c_{12}} g > q_c(B,g), \qquad \forall g \in \mathbf{R}, \tag{3.25}$$

with

$$q_1(B) = g_0(B)\frac{c_2}{c_{12}}. \tag{3.26}$$

From now on, we shall consider the case $B > 0$, q and g are constant on Γ_1, Γ_2 and in Ω, respectively.

The function F: $\mathbf{R}^+ \mathbf{R}^2 \to \mathbf{R}$, defined by

$$F(B,q,g) = J(u_{Bqg}) = F_{Bg}(q) = F_{Bq}(g), \tag{3.27}$$

verifies

$$F(B,q,g) = -\frac{c_1}{2} q^2 + c_{12} qg - \frac{c_2}{2} g^2 + B|\Gamma_2| q - B|\Omega| g, \tag{3.28}$$

and

$$\begin{cases} \dfrac{\partial F}{\partial q}(B,q,g) = -c_1 q + c_{12} g + B|\Gamma_2|, \\ \dfrac{\partial F}{\partial g}(B,q,g) = c_{12} q - c_2 g - B|\Omega|, \\ \dfrac{\partial F}{\partial B}(B,q,g) = |\Gamma_2| q - |\Omega| g. \end{cases} \tag{3.29}$$

THEOREM 3.5. i) We have that $D = c_1 c_2 - c_{12}^2 > 0$, that is, for fixed $B > 0$

$$\text{graph } F = \{(q,g,z) \in \mathbf{R}^3 / z = F(B,q,g)\}$$

is an elliptic paraboloid.

ii) The straight-lines in the plane g, q:

(3.30) l_1) $\quad -c_1 q + c_{12} g + B|\Gamma_2| = 0, \qquad l_2) \quad c_{12} q - c_2 g - B|\Omega| = 0,$

meet together at the point (q^*, g^*) where

$$q^* = B\frac{|\Gamma_2|\, c_2 - |\Omega|\, c_{12}}{D}, \qquad g^* = B\frac{|\Gamma_2|\, c_{12} - |\Omega|\, c_1}{D}. \tag{3.31}$$

PROOF. We have

$$0 < c_{12} = a(u_1, u_2) = \int_{\Omega} \nabla u_1 \nabla u_2 \, dx \leqslant \|\nabla u_1\|_{L^2(\Omega)} \|\nabla u_2\|_{L^2(\Omega)} = \sqrt{c_1}\sqrt{c_2},$$

and $\operatorname{div} \nabla u_1 = \Delta u_1 = 0$ in Ω, $\operatorname{div} \nabla u_2 = \Delta u_2 = -1$ in Ω, and then $\nabla u_1 \neq c \nabla u_2$ for any constant c. Therefore, the equality doesn't hold and it is $D = c_1 c_2 - c_{12} > 0$. After that, we simply solve the linear system (3.30) to obtain (3.31).

THEOREM 3.6. Let be

$$Q_0(B) = \frac{B}{\operatorname{Inf}_{\Gamma_2} u_1} \geqslant \frac{B}{\operatorname{Sup}_{\Gamma_2} u_1} = \tilde{q}_0(B) > 0, \tag{3.32}$$

$$M = \frac{\operatorname{Sup}_{\Gamma_2} u_2}{\operatorname{Inf}_{\Gamma_2} u_1} \geqslant \frac{\operatorname{Inf}_{\Gamma_2} u_2}{\operatorname{Sup}_{\Gamma_2} u_1} = m > 0. \tag{3.33}$$

Then, we have

$$\tilde{q}_0(B) + mg \leqslant q_c(B, g) \leqslant Q_0(B) + Mg, \qquad \forall g \geqslant 0. \tag{3.34}$$

PROOF. If u_{Bqg} is of non-constant sign in Ω, then by Lemma 2.2, for $B > 0$, $g \geq 0$ and $q > 0$, there exists $x \in \Gamma_2$ such that $u_{Bqg}(x) = B - - qu_1(x) + gu_2(x) < 0$. Therefore

$$q > \frac{B + gu_2(x)}{u_1(x)} > \tilde{q}_0(B) + mg > 0,$$

that is, $\tilde{q}_0 + mg \leqslant q_c(B, g)$, $\forall g \geqslant 0$. If $g \geqslant 0$ and

$$q > Q_0(B) + Mg = \frac{B + \operatorname{Sup}_{\Gamma_2} u_2}{\operatorname{Inf}_{\Gamma_2} u_1} > 0,$$

then, $u_{Bqg}(x) < 0$, $\forall x \in \Gamma_2$ and $q_c(B, g) \leqslant Q_0(B) + Mg$, $\forall g \geqslant 0$.

COROLLARY 3.7. i) If $u_1|_{\Gamma_2} = \alpha_1$ is constant, then $q_c(B, 0) = B/\alpha_1 = = \tilde{q}_0(B) = Q_0(B)$.

ii) If $u_1|_{\Gamma_2} = \alpha_1$ and $u_2|_{\Gamma_2} = \alpha_2$ are constant, then

$$aq_c(B, g) = \frac{B}{\alpha_1} + \frac{\alpha_2}{\alpha_1} g, \qquad \forall g \geqslant 0. \tag{3.35}$$

4. – Some heat flux optimization problems with temperature constrains.

Following the idea of [8], we consider, for $B \in H^{3/2}(\Gamma_1)$, $B > 0$; $q \in H^{1/2}(\Gamma_2)$, and $g \in L^2(\Omega)$, $g \geq 0$, the following optimization problem: For a fixed $B > 0$ and $g \in L^2(\Omega)$, $g \geq 0$; find q that produces the total maximum heat flux on Γ_2, within the outflow of heat for which $u_{Bqg} \geq 0$ in $\bar{\Omega}$ (only one phase is present). That is

$$(4.1) \qquad (P_1)\text{: Find } q_g \in Q_g^+ \text{ such that } N(q_g) = \operatorname*{Sup}_{q \in Q_q^+} N(q),$$

where

$$(4.2) \qquad N: Q \to \mathbf{R}, \qquad q \to N(q) = \int_{\Gamma_2} q \, d\gamma,$$

and

$$(4.3) \qquad \begin{cases} S_g = \left\{ v \in K_B : -\Delta v = g \text{ in } \Omega, \left. \dfrac{\partial v}{\partial n} \right|_{\Gamma_3} = 0 \right\}, \\ S_g^+ = \{ v \in S_g : v \geq 0 \text{ in } \Omega \}, \\ Q_g = H^{1/2}(\Gamma_2), \qquad Q_g^+ = T^{-1}(S_g^+) = \{ q \in Q_g : u_{Bqg} \geq 0 \text{ in } \Omega \}, \end{cases}$$

with

$$(4.4) \qquad T: Q_g \to S_g, \qquad q \to T(q) = u_{Bqg},$$

where u_{Bqg} is the unique solution of (2.2). Then, there will not exist a phase change in Ω for any heat flux $q \in Q_q^+$.

LEMMA 4.1. i) T is an affine and monotone decreasing operator, that is, there exist $u_1 \in S_g$ and T_1, T_2 so that $T = T_1 + T_2$, where

$$(4.5) \qquad \begin{cases} T_1 : Q_g \to S_g / T_1(q) = \tilde{u}_1 \in S_g, \qquad \forall q \in Q_g, \\ T_2 : Q_g \to V_0 / T_2 \text{ is linear and continuous.} \end{cases}$$

ii) Q_g^+ is a convex set and N is a linear (then, convex) functional.

PROOF. The proof is similar to that given in [8], for the case $g = 0$, if we define $\tilde{u}_1 \in K_B$ and $\tilde{u}_q \in V_0$ such that:

$$(4.6) \qquad \tilde{u}_1 \in K_B, \quad a(\tilde{u}_1, v - \tilde{u}_1) = \int_\Omega g(v - \tilde{u}_1)\, dx, \quad \forall v \in K_B,$$

$$(4.7) \qquad \tilde{u}_q \in V_0, \quad a(\tilde{u}_q, v) = -\int_{P_2} qv\, d\gamma, \quad \forall v \in V_0,$$

and we put

$$(4.8) \qquad T_1(q) = \tilde{u}_1 \quad \text{and} \quad T_2(q) = \tilde{u}_q \quad \forall q \in Q.$$

If we consider, by hypothesis, that the solution u of (1.4) or (2.2) verifies the condition $u \in H^2(\Omega) \cap C^0(\bar{\Omega})$ (For $n \leqslant 3$, it is sufficient that $u \in H^2(\Omega)$), we can obtain the following existence and uniqueness property.

THEOREM 4.2. There exists a unique $q_g \in Q_g^+$ such that

$$(4.9) \qquad N(q_g) = \operatorname*{Max}_{q \in Q_g^+} N(q).$$

Moreover, the element q_g is defined by

$$(4.10) \qquad q_g = -\frac{\partial w_{Bg}}{\partial n}\bigg|_{\Gamma_2},$$

where w_g is given by (3.3).

PROOF. We consider $v_q = u - w_{Bg}$, and we have, from the maximum principle:

$$N(q_g) - N(q) = \int_{\Gamma_2} (q_g - q)\, d\gamma = -\int_{\Gamma_1} \frac{\partial v_q}{\partial n}\, d\gamma \geqslant 0.$$

Uniqueness will be proved later on. Let $I: S_g \to \mathbf{R}$ be the linear functional defined by

$$(4.11) \qquad I(v) = -\int_{\Gamma_2} \frac{\partial v}{\partial n}\, d\gamma, \quad \forall v \in S_g.$$

We now consider a new formulation P_1' of P_1: Find $v_g \in S_g^+$ such that

$$(4.12) \qquad I(v_g) = \operatorname*{Max}_{v \in S_g^+} I(v).$$

It is clear that

$$v_g \text{ is a solution to } (P_1') \Rightarrow q_g = -\frac{\partial v_g}{\partial n}\Big|_{\Gamma_2} \text{ is a solution to } (P_1),$$

$$q_g \text{ is a solution to } (P_1) \Rightarrow v_g = T(q_g) \text{ is a solution to } (P_1'),$$

We can also define (P_1') in the following way: Find $v_g \in S_g^+$ such that

(4.13) $$I(v_g) = \text{Max}\,\{I(v): v \in S_g, G(v) \leqslant 0\},$$

where

(4.14) $$G: S_g \to \mathcal{C}^0(\Gamma_2), \qquad v \to G(v) = -v|_{\Gamma_2}.$$

We obtain, following [1, 4, 8], that there exists a Lagrange multiplier $\mu \in (\mathcal{C}^0(\Gamma_2))'$ (dual of $\mathcal{C}^0(\Gamma_2)$) with $\mu \geqslant 0$ (i.e. $\langle \mu, p\rangle = \int_{\Gamma_2} \mu p\, d\gamma \geqslant 0$, $\forall p \in \mathcal{C}^0(\Gamma_2)$, $p \geqslant 0$ on Γ_2) that satisfies

(4.15) $$-I(v) + \langle \mu, G(v)\rangle \geqslant -I(u), \qquad \forall v \in S_g,$$

(4.16) $$\langle \mu, G(u)\rangle = 0,$$

and then, we can prove that

(4.17) $$u = w_{Bg} \quad \text{and} \quad \mu = \frac{\partial v_0}{\partial n}\Big|_{\Gamma_2},$$

where v_0 is the unique solution of

(4.18) $$\Delta v_0 = 0 \quad \text{in } \Omega, \quad v_0|_{\Gamma_1} = 0, \quad v_0|_{\Gamma_2} = 1, \quad \frac{\partial v_0}{\partial n}\Big|_{\Gamma_3} = 0.$$

REMARK 4.1. The Lagrange multiplier μ, defined by (4.17), is independent from g, so we have the same μ for each solution w_{Bg}, $\forall g \geqslant 0$.

We now consider a second optimization problem:

(P_2) For fixed $B \in H^{3/2}(\Gamma_1)$, $B > 0$; $g \in L^2(\Omega)$, $g \geqslant 0$, find the maximum constant flux q_M such that $u_{Bqg} \geqslant 0$ in Ω, that is: Find $q_M = q_M(B,g) > 0$ such that

(4.19) $$u_{Bqg} \geqslant 0 \quad \text{in } \Omega \text{ for all constant } q \leqslant q_M.$$

From (3.1), it is obvious that, for fixed constant B, we have that $q_M(B,g) = q_c(B,g)$, $\forall g \geqslant 0$, $g \in L^2(\Omega)$.

THEOREM 4.3. The solution to (P_2) is given by

$$q_M = q_M(B, g) = \operatorname*{Inf}_{x \in \Gamma_2} \frac{u_B(x) + u_g(x)}{u_1(x)} \tag{4.20}$$

where u_B, u_g and u_1 are given respectively by (2.6), (2.8) and (2.10).

PROOF. It follows immediately from (2.5) and Lemma 2.2.

The monotonicity of u_{Bqg} (Lemma 2.1) implies

COROLLARY 4.4. We consider $B_0 \in H^{3/2}(\Gamma_1)$, $B_0 > 0$, $g_0 \in L^2(\Omega)$, $g_0 \geqslant 0$ and $q \in L^2(\Gamma_2)$ such that $\operatorname*{Sup}_{x \in \Gamma_2} q(x) \leqslant q_M(B_0, g_0)$, then $u_{Bqg} \geqslant 0$ in Ω for all $B \geqslant B_0$ and $g \geqslant g_0$.

We give a third optimization problem as a generalization of problem (P_2):

(P_3) For fixed $B \in H^{3/2}(\Gamma_1)$, $B > 0$; $q_* \in L^2(\Gamma_2)$, $q_* > 0$ and $g \in L^2(\Gamma_2)$, $g \geqslant 0$; find $Q_M > 0$ such that

$$u_{Bqg} \geqslant 0 \text{ in } \Omega, \qquad \text{for all } Q \leqslant Q_M \quad \text{and } q = Qq_*. \tag{4.21}$$

THEOREM 4.5. The solution to (P_3) is given by

$$Q_M = \operatorname*{Inf}_{x \in \Gamma_2} \frac{u_B(x) + u_g(x)}{q_{q_*}(x)} \tag{4.22}$$

where u_B, u_g and u_{q_*} are given respectively by (2.6), (2.8) and

$$\Delta u_{q_*} = 0 \text{ in } \Omega, \quad u_{q_*}|_{\Gamma_1} = 0, \quad \frac{\partial u_{q_*}}{\partial n}\bigg|_{\Gamma_2} = q_* > 0, \quad \frac{\partial u_{q_*}}{\partial n}\bigg|_{\Gamma_3} = 0. \tag{4.23}$$

5. - Examples.

We shall give three examples in which the solution is explicitly known, for $B > 0$, q and g constant [6, 15]:

EXAMPLE 1. We consider

$$\begin{cases} n = 2, \quad \Omega = (0, x_0) \times (0, y_0), \quad x_0 > 0, \; y_0 > 0, \\ \Gamma_1 = \{0\} \times [0, y_0], \quad \Gamma_2 = \{x_0\} \times [0, y_0], \\ \Gamma_3 = (0, x_0) \times \{0\} \cup (0, x_0) \times \{y_0\}. \end{cases} \tag{5.1}$$

We obtain that (u_1, u_2, c_1, c_2, c_{12}, l_1 and l_2, (g^*, q^*), q_i, q_s and q_c are defined by (2.10), (2.11), (3.11), (3.22), (3.17), (3.30), (3.31), (3.4), (3.8) and (3.1) respectively):

$$(5.2)\quad \begin{cases} u_1(x,y) = x\,, \qquad u_2(x,y) = x_0 x - \dfrac{x^2}{2}\,, \\[2ex] c_1 = x_0 y_0\,, \qquad c_2 = \dfrac{x_0^3 y_0}{3}\,, \qquad c_{12} = \dfrac{x_0^2 y_0}{2}\,, \\[2ex] D = c_1 c_2 - c_{12}^2 = \dfrac{x_0^4 y_0^2}{12}\,, \qquad |\Omega| = x_0 y_0\,, \qquad |\Gamma_2| = y_0\,, \\[2ex] l_1)\quad q - \dfrac{x_0}{2} g - \dfrac{B}{x_0} = 0\,, \qquad l_2)\ q - \dfrac{x_0}{3} g - \dfrac{B}{x_0} = 0\,, \\[2ex] (g^*, q^*) = \left(-\dfrac{2B}{x_0}, -\dfrac{6B}{x_0^2}\right), \qquad q_i(B,g) = q_s(B,g) = \dfrac{B}{x_0} + \dfrac{x_0}{2} g\,, \\[2ex] q_c(B,g) = \dfrac{B}{x_0} + \dfrac{x_0}{2} g\,, \qquad \text{if } g \geqslant -\dfrac{2B}{x_0^2}\,, \\[2ex] q_c(B,g) = g x_0 + \sqrt{-2Bg}\,, \qquad \text{if } g \leqslant -\dfrac{2B}{x_0^2}\,. \end{cases}$$

EXAMPLE 2. We consider

$$(5.3)\quad \begin{cases} n = 2\,, \qquad 0 < r_1 < r_2\,, \qquad r, \omega\text{: polar coordinates in } \mathbb{R}^2\,, \\ \Omega = \{(r,\omega)\colon r_1 < r < r_2\}\,, \qquad \Gamma_1 = \{(r,\omega)\colon r = r_1, 0 \leqslant \omega < 2\pi\}\,, \\ \Gamma_2 = \{(r,\omega)\colon r = r_2, 0 \leqslant \omega < 2\pi\}\,, \qquad \Gamma_3 = \emptyset\,. \end{cases}$$

We obtain that

$$u_1(r) = r_2 \log \frac{r}{r_1}\,, \qquad u_2(r) = \frac{r_2^2}{2} \log \frac{r}{r_1} - \frac{r^2 - r_1^2}{4}\,.$$

We define

$$c = \frac{r_2}{r_1}\,, \qquad \alpha(c) = (c^2 - 1) - 2 \log c\,,$$

$$\beta(c) = (1 + c^2) \log c - (c^2 - 1)\,, \qquad \gamma(c) = 2c^2 \log c - (c^2 - 1)\,,$$

and then

$$
(5.4)\quad
\begin{cases}
c_1 = 2\pi r_2^2 \log c , \qquad c_{12} = \dfrac{\pi r_1^3 c}{2}[2c^2 \log c - (c^2-1)] , \\
c_2 = \dfrac{\pi r_1^4}{8}[4c^4 \log c - 2c^2(c^2-1) - (c^2-1)^2] , \\
D = c_1 c_2 - c_{12}^2 = \dfrac{\pi^2}{4} r_1^6 c^2 (c^2-1)\beta(c) , \\
|\Omega| = \pi(r_2^2 - r_1^2) , \qquad |\Gamma_2| = 2\pi r_2 , \\
l_1) \quad q - \dfrac{B}{r_2 \log c} - \left[\dfrac{r_2}{2} - \dfrac{r_1(c^2-1)}{4c\log c}\right] g = 0 , \\
l_2) \quad q - \dfrac{2(c^2-1)B}{[2c^2\log c - (c^2-1)]r_1 c} - \\
\qquad\qquad - \dfrac{r_1[4c^4 \log c - (c^2-1)(3c^2-1)]}{4c[2c^2\log c - (c^2-1)]} g = 0 , \\
(g^*, q^*) = \left(-\dfrac{4B\alpha(c)}{r_1^2(c^2-1)\beta(c)} , \dfrac{B}{r_2}\left[\dfrac{1}{\log c} - \dfrac{\gamma(c)\alpha(c)}{(c^2-1)\beta(c)\log c}\right]\right) , \\
q_i(B,g) = \dfrac{B}{r_2 \log c} + \left[\dfrac{r_2}{2} - \dfrac{r_1(c^2-1)}{4c \log c}\right] g , \\
q_s(B,g) = \dfrac{B}{r_2 - r_1} + \left[\dfrac{r_2}{2} - \dfrac{r_1(c^2-1)}{4c \log c}\right] g , \\
q_c(B,g) = \dfrac{B}{r_2 \log c} + \left[\dfrac{r_2}{2} - \dfrac{r_1(c^2-1)}{4c \log c}\right] g , \qquad \text{if } g \geqslant -\dfrac{4B}{r_1^2\gamma(c)} , \\
q_c(B,g) = -\dfrac{g r_1}{2c}[Q^2(g) - c^2] , \qquad \text{if } g < -\dfrac{4B}{r_1^2 \gamma(c)} ,
\end{cases}
$$

where $Q(g)$ is defined implicitely by

$$
(5.5)\qquad H(g, Q) = B + g\frac{r_1^2}{4}[Q^2[2\log Q - 1] + 1] = 0 .
$$

EXAMPLE 3. We consider the same data as in example 2 but now for $n=3$. We obtain

$$(5.7)\quad \begin{cases} u_1(r) = r_2^2\left(\dfrac{1}{r_1} - \dfrac{1}{r}\right), u_2(r) = -\dfrac{r_2^2}{3}\left(\dfrac{1}{r_1} - \dfrac{1}{r}\right) - \dfrac{r^2 - r_1^2}{6}, \\[2ex] c_1 = 4\pi r_1^3 c^3 (c-1), \qquad c_{12} = \dfrac{2}{3}\pi r_1^4 c^2 (c-1)^2 (2c+1), \\[2ex] c_2 = \dfrac{4}{45}\pi r_1^5 (c-1)^2 (5c^3 + 6c^2 + 3c + 1), \\[2ex] D = c_1 c_2 - c_{12}^2 = \dfrac{4}{45}\pi^2 r_1^8 (c-1)^4 c^3 (4c^2 + 7c + 4), \\[2ex] |\Omega| = \dfrac{4}{3}\pi (r_2^3 - r_1^3), \qquad |\Gamma| = 4\pi r_2^2, \\[2ex] l_1) \quad q - \dfrac{B}{r_1 c(c-1)} - \dfrac{r_1 (c-1)(2c+1)}{6c} g = 0, \\[2ex] l_2) \quad q - \dfrac{2B(c^2 + c + 1)}{r_1 c^2 (c-1)(2c+1)} - \dfrac{2r_1 (5c^3 + 6c^2 + 3c + 1)}{15c^2 (2c+1)} g = 0, \\[2ex] g^* = -\dfrac{180B(c+1)}{r_1^2 (c-1)^2 (4c^2 + 7c + 4)}, \\[2ex] q^* = -\dfrac{12B}{r_1} \dfrac{(c^2 + 3c + 1)}{c(c-1)(4c^2 + 7c + 4)}, \\[2ex] q_i(B,g) = \dfrac{B}{r_1 c(c-1)} + \dfrac{r_1 (c-1)(2c+1)}{6c} g, \\[2ex] q_s(B,g) = \dfrac{B}{r_1 (c-1)} + \dfrac{r_1 (c-1)(2c+1)}{6c} g, \\[2ex] q_c(B,g) = \dfrac{B}{r_1 c(c-1)} + \dfrac{r_1 (c-1)(2c+1)}{6c} g, \qquad \text{if } g \geq g_1, \\[2ex] q_c(B,g) = g \dfrac{r_1}{3} (c - \tilde{Q}(g)), \qquad \text{if } g \leq g_1, \end{cases}$$

where

$$g_1 = -\frac{6B}{r_1^2(c-1)^2(2c+1)}, \tag{5.8}$$

and $\tilde{Q}(g)$ is implicitely defined by

$$\tilde{H}(g,\tilde{Q}) = B + g\frac{r_1^2}{6}\left[2c\tilde{Q}\left(c-\left(\frac{c}{\tilde{Q}}\right)^{1/3}\right) - c^{4/3}\tilde{Q}^{2/3} + 1\right] = 0. \tag{5.8}$$

We recall that, in the three examples, for $g \geq 0$, Lemma 2.2 is valid and, for $g<0$, $u_{Bqg}(x_0) = \operatorname*{Min}_{x\in\bar{\Omega}} u_{Bqg}(x)$ with $x_0 \in \Omega$ [6].

REMARK 5.1. The formulation (1.4) can exhibit the existence of a mushy region [5, 10, 11, 12] (*i.e.* the zone where u is equals to zero has positive measure) in Ω for a suitable source g. We consider data (5.1) in Example 1. Then, if data $x_0 > 0$, $B > 0$, $q > 0$, $g_2 > 0$ and $g_1 < 0$ satisfy the following condition

$$\left(-\frac{2B}{g_1}\right)^{1/2} < x_0 - \frac{q}{g_2},$$

then the function

$$u(x,y) = u(x) = \begin{cases} B + g_1\xi x - \dfrac{g_1}{2}x^2, & 0 \leqslant x \leqslant \xi, \\ 0, & \xi \leqslant x \leqslant \eta, \\ -\dfrac{g_2\eta^2}{2} + g_2\eta x - \dfrac{g_2}{2}x^2, & \eta \leqslant x \leqslant x_0, \end{cases}$$

exhibits a mushy region, given by the interval (ξ, η), where

$$0 < \xi = \left(-\frac{2B}{g_1}\right)^{1/2} < \eta = x_0 - \frac{q}{g_2} < x_0,$$

and the source term $g \in L^2(\Omega)$ is given by

$$g(x,y) = \begin{cases} g_1 < 0, & 0 < x < \xi, & 0 < y < y_0, \\ 0, & \xi \leqslant x \leqslant \eta, & 0 < y < y_0, \\ g_2 > 0, & \eta < x < x_0, & 0 < y < y_0. \end{cases}$$

Moreover, we can easily verify that the right size inequality in (3.10) is hold.

An open problem remains to verify the asymptotic behaviour of the evolution mushy region model.

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