MATHEMATICAE NOTAE - AÑO XXX (1983) (67 - 79)

ON THE OBSTACLE PROBLEM

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ABSTRACT.

We study the classical formulation of the obstacle problem for an homogeneous elastic string and for an homogeneous elastic membrane, streched across a rigid circular frame, without external forces.

We consider the simplest problems where the obstacle Ψ is an even function (for the string) or a function depending only on r (for the membrane).

We define an inverse problem and this leads us to a correspondence between a family of functions G_2 and the family of obstacles which give connected coincidence sets. This correspondence allows us to construct examples where the coincidence set Ω^* is a connected prescribed domain in Ω.

We give a list of examples in which the free boundary $\partial\Omega^*$ is computed exactly or it is given as the unique solution of an algebraic equation.

I. INTRODUCTION.

In this paper we study the classical formulation of the obstacle problem for an homogeneous elastic string and for an homogeneous elastic membrane, streched across a rigid circular frame, without external forces.

Existence, uniqueness and continuity of its first derivatives are well stablished for the solution of the obstacle problem. Lewy Stampacchia |1| and Kinderlehrer |2| have also shown that, for n=2 under the hypothesis of convexity of Ω and strong concavity of Ψ, Ω^* is simply connected.

We give a necessary and sufficient conditions for the coincidence set Ω^* to be simply connected. These conditions lead us to define a correspondence between a family of functions G_2 and the family of obstacles which gives simply connected coincidence set.

(*) Este trabajo fue presentado en la XXX Reunión Anual de la Unión Matemática Argentina, Salta, 11-13 agosto 1980.-

This correspondence allows us to construct examples where Ω^* is a simply connected prescribed domain in Ω .

II. THE OBSTACLE PROBLEM FOR AN HOMOGENEOUS ELASTIC STRING

Consider $\Omega = (-R,R)$, $R > 0$ and $\Psi : \overline{\Omega} \to R$ an even function. This can be done without loss of generality.

Let $A = \{ \Phi \in C^2[0,R) \cap C[0,R] \ / \ \Phi(R) < 0 \ , \ \Phi'(0) = 0, \Phi(0) > 0 \}$ and let, in the sequel, $\Psi \in A$. We will also note Ψ the even extension to $\overline{\Omega}$.

In this case, the obstacle problem is:

$$(P_1) \begin{cases} \text{Find } u \in C^1(\overline{\Omega}) \text{ , even function such that} \\ u'' \leq 0 \quad \text{a.e. in } [0,R) \\ u''(u - \Psi) = 0 \quad \text{a.e. in } [0,R) \\ u(x) \geq \Psi(x) \ , \ x \in [0,R] \\ u(R) = 0 \\ u' = \Psi' \quad \text{on } \partial\Omega^* \end{cases}$$

where $\Omega^* = \{ x \in \Omega \ / \ u(x) = \Psi(x) \}$ is the coincidence set.

II.1. NECESSARY AND SUFFICIENT CONDITIONS FOR THE COINCIDENCE SET Ω^* TO BE CONNECTED.

It is easily verified that:

Every solution u of (P_1) for which the coincidence set Ω^* is connected must satisfy

$$(1) \qquad u(x) = \begin{cases} \Psi(x) & , \ x \in [0,\xi] \\ \Psi'(\xi)(x - R) & , \ x \in [\xi,R] \end{cases}$$

with $\xi \in (0,R)$ such that $g(\xi) = 0$, where

$$(2) \qquad g(x) = \Psi(x) + \Psi'(x)(R - x) \quad .$$

REMARK II.1.

The above conditions are not sufficient for Ω^* to be connected. In fact, if one considers $\Psi(x) = 4\pi - x + \text{sen } x$, $x \in [0, 4\pi + 1]$, it results

$$u(x) = \begin{cases} 4\pi - x + \text{sen } x & , \quad x \in \left[0, \frac{\pi}{2}\right] \\ -x + 4\pi + 1 & , \quad x \in \left[\frac{\pi}{2}, 4\pi+1\right] \end{cases}$$

and $\Omega^* \cap \left[0, 4\pi + 1\right] = \left[0, \frac{\pi}{2}\right] \cup \left\{\frac{5}{2}\pi\right\}$.

THEOREM II.1.

Let $\psi \in A$. *The following propositions are equivalent:*

(i) *The solution* u *of* (P_1) *satisfies* (1).

(ii) Ψ *satisfies:*

$$\psi''(x) \leq 0 \quad , \forall x \in [0,\eta]$$
$$\psi'(x) \geq \psi'(\eta) \quad , \forall x \in N$$

where $N = \{x \in (0,R) \ / \ g(x) = 0\}$ *and* $\eta = \min N$.

PROOF.

It is easily verified that (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (i). Let η be as in (ii). Consider

$$u(x) = \begin{cases} \Psi(x) & , x \in [0,\eta] \\ \Psi'(\eta)(x - R) & , x \in [\eta,R] \end{cases} .$$

It will be proved that u is the solution of (P_1). It is sufficient to prove that $u(x) \geq \Psi(x)$, $\forall x \in (\eta,R)$, provided the other conditions are clearly verified.

Suppose there exists $x \in (\eta,R)$ such that $u(x) < \Psi(x)$. Then, there exist $\eta \leq w_1 < w_2 < R$ such that

$$u(w_1) = \Psi(w_1), u(w_2) = \Psi(w_2) \quad \text{and}$$
$$u(x) < \Psi(x) \quad \forall x \in (w_1,w_2) .$$

It follows from the mean value theorem that there exists $w \in (w_1,w_2)$ such that $\Psi'(w) = \Psi'(\eta)$.

Besides $g(w) = \Psi(w) - u(w) > 0$ and $g(w_2) = (R-w_2)(\Psi'(w_2) - \Psi'(\eta)) \leq 0$.

If $g(w_2) < 0$, there exists $\dot{x} \in N \cap (w_1,w_2)$ and this is a contradiction.

If $g(w_2) = 0$, that is, $\Psi'(w_2) = \Psi'(\eta)$, it can be given $\varepsilon > 0$ such that $\Psi''(x) > 0$, $\forall x \in (w_2-\varepsilon,w_2)$. Hence $g'(x) > 0$ and $g(x) < 0$, $\forall x \in (w_2-\varepsilon,w_2)$, which also gives place to a contradiction. ▮

Similary, it is proved the following:

THEOREM II.2.

Let $\Psi \in A$. *The following propositions are equivalent:*

(i) *The solution* u *of* (P_1) *has a connected coincidence set.*

(ii) *There exists* $\varepsilon \geq 0$ *such that:*

$$\Psi''(x) \leq 0 \quad , x \in [0,\eta]$$
$$\Psi'(x) = \Psi'(\eta) \ , x \in [\eta,\eta+\varepsilon]$$
$$\Psi'(x) > \Psi'(\eta) \ , x \in N \cap (\eta+\varepsilon,R).$$

II.2. AN INVERSE PROBLEM.

In II.1., it was characterized a family of obstacles which give solutions u of (P_1) satisfaying (1), or whose coincidence sets are connected. This was done taking into account the positive zeros of the function g defined in (2).

This fact leads to consider an inverse problem in the following sense:

Find obstacles Ψ of the type above characterized from a given family of functions g .

In fact, given $g \in C[0,R)$, the initial value problem:

$$(3) \quad \begin{cases} \text{i)} \ \Psi(x) + \Psi'(x)(R-x) = g(x) \\ \text{ii)} \ \Psi(0) = g(0) \end{cases}$$

has one and only one solution $\Psi \in C^1[0,R)$ and

$$(4) \qquad \Psi(x) = \frac{g(0)}{R}(R-x) + (R-x)\int_0^x \frac{g(t)}{(R-t)^2}\,dt. \ .$$

If $\Psi \in A$, the function g verifies

$$(5) \quad \begin{cases} g \in C^1([0,R)) \\ g(0) = \Psi(0) > 0 \\ \exists \lim\limits_{x \to R^-} (R-x)\int_0^x \frac{g(t)}{(R-t)^2}\,dt = \Psi(R) < 0 \end{cases} \quad .$$

Conversely, let

$$(6) \quad G = \left\{ g \in C^1[0,R)/g(0) > 0 \text{ and } \exists \lim_{x \to R^-} (R-x)\int_0^x \frac{g(t)}{(R-t)^2}\,dt < 0 \right\}.$$

The correspondence

$$(7) \quad \begin{aligned} &T : G \to A \\ &g \to T(g) = \Psi \end{aligned}$$

, where Ψ is defined as in (4), is bi-yective.

Moreover

$$\Psi'(x) = \int_0^x \frac{g'(t)}{R-t}\,dt \quad \text{and} \quad \Psi''(x) = \frac{g'(x)}{R-x} \quad , \quad \forall x \in [0,R) \quad .$$

REMARK II.2.

If $g \in \tilde{G} = \{ g \in C^1[0,R) \cap C[0,R] \ / \ g(0) > 0 \text{ and } g(R) < 0 \}$ then

$$\begin{cases} g \in G \\ g(R) = \lim_{x \to R^-} (R-x) \int_0^x \frac{g(t)}{(R-t)^2}\,dt = \Psi(R) < 0 \end{cases}$$

and $\tilde{G} \subsetneq G$. In fact, if $R > \frac{2}{\pi}$, $g(x) = \cos\left(\frac{1}{R-x}\right) - c$, with $0 < c < \cos\frac{1}{R}$ is such that $g \in G$ and $g \notin \tilde{G}$.

Further $T(\tilde{G}) = \{ \Psi \in A \ / \ \lim_{x \to R^-} \Psi'(x)(R-x) = 0 \}$.

Let

(8) $$N_g = \{ x \in (0,R) \ / \ g(x) = 0 \}$$

(9) $$G_1 = \left\{ g \in G/g'(x) \le 0 \quad \forall x \in [0,\xi] \quad \text{and} \quad \int_\xi^\eta \frac{g(t)}{(R-t)^2}\,dt \le 0 \quad \forall \eta \in N_g \quad , \quad \xi = \min N_g \right\} .$$

G_1 is well defined provided that, $g \in G$ implies $N_g \neq \emptyset$ and $\xi = \min N_g$ exists since g is continuos.

THEOREM II.3.

Let $\tilde{G}_1 = \left\{ g \in G/g'(x) \le 0 \quad \forall x \in [0,\xi] \quad \textit{and} \quad \int_\xi^x \frac{g(t)}{(R-t)^2}\,dt \le 0 \quad \forall x \in [\xi,R) \quad , \quad \xi = \min N_g \right\}$.

Then

(i) $G_1 = \tilde{G}_1$

(ii) $T(G_1) = \{ \Psi \in A \ / \ \textit{the solution } u \textit{ of } (P_1) \textit{ satisfies } (1) \}$.

PROOF.

(i) It is easily verified that $\tilde{G}_1 \subset G_1$.

Conversely, let $g \in G_1$ and $x > \xi$.

If $g(x) < 0$, there exists $\eta_1 \in N_g$ such that $g(t) < 0$ $\forall t \in (\eta_1, x]$, hence $\int_\xi^x \frac{g(t)}{(R-t)^2}\,dt = \int_\xi^{\eta_1} \frac{g(t)}{(R-t)^2}\,dt + \int_{\eta_1}^x \frac{g(t)}{(R-t)^2}\,dt < 0$.

If $g(x) > 0$, there exists $\eta_2 \in N_g$ such that $g(t) > 0$

$\forall t \in [x,\eta_2)$, therefore $\int_\xi^x \frac{g(t)}{(R-t)^2}\,dt = \int_\xi^{\eta_2} \frac{g(t)}{(R-t)^2}\,dt - \int_x^{\eta_2} \frac{g(t)}{(R-t)^2}\,dt < 0$.

(ii) If $g \in G_1$, there exists $\xi = \min N_g$, and $T(g) = \Psi$ satisfies:

$$\begin{cases} \Psi \in A \\ \Psi''(x) = \frac{g'(x)}{R-x} \le 0 \ , \quad x \in [0,\xi] \\ \Psi'(\xi) - \Psi'(\eta) = \int_\xi^\eta \frac{g(t)}{(R-t)^2}\,dt \le 0 \ , \quad \eta \in N_g \ . \end{cases}$$

Thus, from theorem II.1, one obtain (ii). ∎

Similarly, the proof of the following theorem is a consequence of theorem II.2.

THEOREM II.4.

Let $G_2 = \left\{ g \in G / g' \le 0 \text{ in } [0,\xi] \ , \ g' = 0 \text{ in } [\xi,\xi+\varepsilon] \ , \int_\xi^\eta \frac{g(t)}{(R-t)^2}\,dt < 0 \ \forall \eta \in N_g \cap (\xi+\varepsilon,R) \textit{ with } \xi = \min N_g \textit{ and } \varepsilon \ge 0 \right\}$.

It results:

(i) $G_2 = \left\{ g \in G/g' \le 0 \textit{ in } [0,\xi] \ , \ g' = 0 \textit{ in } [\xi,\xi+\varepsilon] \ , \int_\xi^x \frac{g(t)}{(R-t)^2}\,dt < 0 \ \forall x \in (\xi+\varepsilon,R) \textit{ with } \xi = \min N_g \textit{ and } \varepsilon \ge 0 \right\}$.

(ii) $T(G_2) = \{ \Psi \in A$ / *the solution* u *of* (P_1) *has a connected coincidence set* $\}$.

III. THE OBSTACLE PROBLEM FOR AN HOMOGENEOUS ELASTIC MEMBRANE.

Consider: $\Omega = \{(r,\theta)/0 \le r < R \ , \ 0 \le \theta < 2\pi\}$ and $\psi : \bar{\Omega} \to \mathbb{R}$ a function only of r , that is $\Psi(r,\theta) = \Psi(r)$.

Under this assumption, the solution of the obstacle problem is also a function only of r , that is $u(r,\theta) = u(r)$ and everything follows as in the case of an homogeneous elastic string.

Let A be as in paragraph II, and let in the sequel $\Psi \in A$.

In this case, the obstacle problem is:

$$(P_2)\quad\left\{\begin{array}{l}\text{Find}\quad u\in C^1[0,R]\quad\text{and such that}\\ u'(0)=0\\ u''+\frac{1}{r}u'\le 0\qquad\text{a.e. in}\quad(0,R)\\ (u''+\frac{1}{r}u')(u-\Psi)=0\quad\text{a.e. in}\quad(0,R)\\ u(r)\ge\Psi(r)\quad,\quad r\in[0,R]\\ u(R)=0\\ u'=\Psi'\quad\text{on}\quad\partial\Omega_1\quad.\\ \text{Where}\quad\Omega_1=\{r\in[0,R)\,/\,u(r)=\Psi(r)\}\ .\end{array}\right.$$

In this case the coincidence set is $\Omega^*=\Omega_1\times[0,2\pi)$.

III.1. NECESSARY AND SUFFICIENT CONDITIONS FOR THE COINCIDENCE SET Ω^* TO BE SIMPLY CONNECTED.

It is verified, as in paragraph II, that:

Every solution u of (P_2) whose coincidence set is simply connected must satisfy

$$(10)\qquad u(r)=\begin{cases}\Psi(r) & ,\quad r\in[0,\xi]\\ \Psi'(\xi)\xi\log\frac{r}{R} & ,\quad r\in[\xi,R]\end{cases}$$

with $\xi\in(0,R)$ such that $g(\xi)=0$, where

$$(11)\qquad g(r)=\Psi(r)+r\Psi'(r)\log\frac{R}{r}\quad.$$

As in II.1, one can easily prove:

THEOREM III.1.

Let $\Psi\in A$. *The following propositions are equivalent:*

(i) *The solution* u *of* (P_2) *satisfies* (10).

(ii) Ψ *satisfies:*

$$\psi'(r)+r\psi''(r)\le 0\quad,\quad r\in[0,\eta]$$
$$r\,\psi'(r)\ge\eta\psi'(\eta)\quad,\quad r\in N$$

where $N=\{r\in(0,R)\ /\ g(r)=0\}$ *and* $\eta=\min N$.

THEOREM III.2.

Let $\Psi\in A$. *The following propositions are equivalent:*

(i) *The solution* u *of* (P_2) *has a simply connected coincidence set.*

(ii) *There exists* $\varepsilon\ge 0$ *such that:*

$$\Psi'(r) + r\Psi''(r) \le 0 \quad , \qquad r \in [0,\eta]$$
$$r\Psi'(r) = \eta\Psi'(\eta) \quad , \qquad r \in [\eta,\eta+\varepsilon]$$
$$r\Psi'(r) > \eta\Psi'(\eta) \quad , \qquad r \in N \cap (\eta+\varepsilon,R) \quad .$$

III.2. AN INVERSE PROBLEM.

As in II.2, we consider the following problem:

Find obstacles Ψ of the types above characterized from a given family of functions g .

LEMMA III.3.

Let $g \in C[0,R]$. *The initial value problem*

$$(12) \quad \begin{cases} \text{(i)} & \Psi(r) + r\Psi'(r)\log \frac{R}{r} = g(r) \\ \text{(ii)} & \Psi(0) = g(0) \end{cases}$$

has only one solution $\Psi \in C[0,R) \cap C^1(0,R)$ *given by*

$$(13) \qquad \Psi(r) = \log \frac{R}{r} \int_0^r \frac{g(t)}{t \log^2 \frac{R}{t}}\, dt \quad .$$

PROOF.

As g is bounded in $[0,R]$, it results $\Psi \in C^1(0,R)$.

One as

$$\lim_{r \to 0} \Psi(r) = \lim_{r \to 0}\left[-\log \frac{r}{R} \int_0^r \frac{g(t)}{t \log^2 \frac{R}{t}}\, dt\right] = \lim_{r \to 0} g(r) = g(0) \quad .$$

and defining $\Psi(0) = g(0)$, the lemma is proved.

REMARK III.1.

From the diferential equation (12i), it results that, if $\Psi \in A$, the function g verifies:

$$(14) \quad \begin{cases} g \in C^1[0,R) \\ g(0) = \Psi(0) > 0 \\ \displaystyle\lim_{r \to R^-} \log \frac{R}{r} \int_0^r \frac{g(t)}{t \log^2 \frac{R}{t}}\, dt = \Psi(R) < 0 \quad . \end{cases}$$

Conversely, it results the following:

THEOREM III.4.

Let

$$(15) \quad G = \left\{ g \in C^1[0,R)/g(0) > 0 \ , \ \exists \lim_{r \to R^-} \log \frac{R}{r} \int_0^r \frac{g(t)}{t \log^2 \frac{R}{t}}\, dt < 0 \right.$$

and $\lim_{r \to 0} \frac{g'(r)}{r \log \frac{R}{r}} \in \mathbb{R} \Big\}$.

The correspondence

(16) $$T : G \to A$$
$$g \to T(g) = \Psi$$, *where* Ψ *is defined as in* (13), *is biyective.*

PROOF.

It is sufficient to prove that Ψ', Ψ'' are continuous in $r=0$.

$$\lim_{r \to 0} \Psi'(r) = \lim_{r \to 0} \frac{1}{r}\int_0^r \frac{g'(t)}{\log \frac{R}{t}}\, dt = \lim_{r \to 0} \frac{g'(r)}{\log \frac{R}{t}} = 0 .$$

Hence $\Psi'(0) = 0$ and $\Psi \in C^1([0,R))$.

$$\lim_{r \to 0} \Psi''(r) = \lim_{r \to 0} \left[-\frac{1}{r^2}\int_0^r \frac{g'(t)}{\log \frac{R}{t}}\, dt + \frac{g'(r)}{r \log \frac{R}{r}} \right] = \lim_{r \to 0} \frac{g'(r)}{2r \log \frac{R}{r}}$$

and $\Psi''(0) = \lim_{r \to 0} \frac{\Psi'(r)}{r} = \lim_{r \to 0} \frac{g'(r)}{2r \log \frac{R}{r}}$, thus $\Psi \in C^2[0,R)$. ▮

REMARK I.ΓI.2.

If $g \in \widetilde{G} = \Big\{ g \in C^1[0,R) \cap C[0,R] \;/\; g(0) > 0 ,\; g(R) < 0$ and $\exists \lim_{r \to 0} \frac{g'(r)}{r \log \frac{R}{r}} \in \mathbb{R} \Big\}$

then

$$g \in G$$
$$g(R) = \lim_{r \to R^-} \log \frac{R}{r} \int_0^r \frac{g(t)}{t \log^2 \frac{R}{t}}\, dt = \Psi(R)$$

and $\widetilde{G} \subsetneq G$. In fact, if $R > e^{\frac{2}{\pi}}$, $g(r) = \cos\left(\frac{1}{\log \frac{R}{r}}\right) - C$ with $0 < C \lneq 1$, is such that $g \in G$ and $g \notin \widetilde{G}$.

Moreover

$$T(\widetilde{G}) = \Big\{ \Psi \in A \;/\; \lim_{r \to R^-} r\Psi'(r) \log \frac{R}{r} = 0 \Big\} .$$ ▮

Let

(17) $$N_g = \{ r \in (0,R) \;/\; g(r) = 0 \}$$

(18) $$G_1 = \Big\{ g \in G \;/\; g' \le 0 \text{ in } [0,\xi] \text{ and } \int_\xi^\eta \frac{g(t)}{t \log^2 \frac{R}{t}}\, dt \le 0 \;\; \forall\, \eta \in N_g ,\; \xi = \min N_g \Big\} .$$

As in II.2, there are easily proved the following:

THEOREM III.5.

It results:

(i) $G_1 = \left\{ g \in G \,/\, g' \leq 0 \text{ in } [0,\xi] \text{ and } \int_\xi^r \frac{g(t)}{t \log^2 \frac{R}{t}} dt \leq 0 \ \forall\, r \in [\xi,R) \,,\ \xi = \min N_g \right\}$.

(ii) $T(G_1) = \left\{ \Psi \in A / \right.$ *the solution* u *of* (P_2) *satisfies* $\left. (10) \right\}$.

THEOREM III.6.

Given

(19) $G_2 = \left\{ g \in G \,/\, g' \leq 0 \text{ in } [0,\xi] \,,\ g' = 0 \text{ in } [\xi,\xi+\varepsilon] \text{ and } \int_\xi^\eta \frac{g(t)}{t \log^2 \frac{R}{t}} dt < 0 \ \forall\, \eta \in N_g \cap (\xi+\varepsilon,R) \,,\ \text{with } \xi = \min N_g \text{ and } \varepsilon \geq 0 \right\}$.

It results:

(i) $G_2 = \left\{ g \in G \,/\, g' \leq 0 \text{ in } [0,\xi] \,,\ g' = 0 \text{ in } [\xi,\xi+\varepsilon] \text{ and } \int_\xi^r \frac{g(t)}{t \log^2 \frac{R}{t}} dt < 0 \ \forall\, r \in (\xi+\varepsilon,R) \,,\ \text{with } \xi = \min N_g \text{ and } \varepsilon \geq 0 \right\}$.

(ii) $T(G_2) = \left\{ \Psi \in A / \right.$ *the solution* u *of* (P_2) *has a simply connected coincidence set* $\left. \right\}$.

IV. EXAMPLES.

We develope several examples of the obstacle problem for $n=1$ and $n=2$, where we consider:

$\Psi \in A$: the obstacle.

g : the function defined by (2) if $n=1$ and by (11) if $n=2$.

ξ : a constant such that the free boundary is $\partial\Omega^* = \{-\xi,\xi\}$ if $n=1$ and $\partial\Omega^* = \{\xi\} \times [0,2\pi)$ if $n = 2$. It is given, in general , as a solution of an algebraic equation $f_1(\xi) = f_2(\xi)$.

u : the solution of the obstacle problem.

UNIDIMENSIONAL CASE.

EXAMPLE 1.

$\psi(x) = \alpha(\beta^n - x^n)$, $0 < \beta < R$, $\alpha > 0$, $n \geq 2$

$g(x) = \alpha\left[(n-1)x^n - nRx^{n-1} + \beta^n\right]$

$0 < \xi = R - \left(R^2 - \beta^2\right)^{\frac{1}{2}} < \beta$ if $n = 2$

$$-(n-1)\xi + nR = \frac{\beta}{\xi^{n-1}} \qquad \text{if} \quad n \geq 3$$

$$u(x) = -\alpha n \xi^{n-1}(x - R) \quad , \quad x \in [\xi,R] \quad .$$

EXAMPLE 2.

$$\psi(x) = \alpha e^{-\beta^2 x^2} - \gamma \ , \ 0 < \gamma < \alpha \quad , \quad R > \frac{1}{\beta}\left(\log\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}$$

$$g(x) = \alpha e^{-\beta x^2}[1 - 2\beta^2 x(R - x)] - \gamma$$

$$\frac{\gamma}{\alpha} e^{-\beta^2\xi^2} = 2\beta^2\xi - 2\beta^2 R\xi + 1 \quad , \quad 0 < \xi \leq \min\left(\frac{R}{2}, \frac{1}{\beta}\left(\log\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}, \frac{1}{\sqrt{2}\,\beta}\right)$$

$$u(x) = -2\alpha\beta^2\xi e^{-\beta^2\xi^2}(x - R) \quad , \quad x \in [\xi,R] \quad .$$

EXAMPLE 3.

$$\psi(x) = \frac{\beta}{1-\alpha^2x^2} - \gamma \quad , \quad 0 < \gamma < \beta \quad , \quad \alpha > 0 \quad , \quad R > \frac{1}{\alpha}\left(\frac{\beta}{\gamma} - 1\right)^{\frac{1}{2}}$$

$$g(x) = \beta\,\frac{1 - \alpha^2x^2 + 2\alpha^2 Rx}{(1 + \alpha^2x^2)^2} - \gamma$$

$$\xi^4 + \left(\frac{\beta + 2\gamma}{\gamma\alpha^2}\right)\xi^2 - \left(\frac{2\beta R}{\gamma\alpha^2}\right)\xi - \left(\frac{\beta-\gamma}{\gamma\alpha^2}\right) = 0 \quad , \quad 0 < \xi < R \quad .$$

$$u(x) = -\frac{2\beta\alpha^2\xi}{(1 + \alpha^2\xi^2)^2}(x-R) \quad , \quad x \in [\xi,R] \quad .$$

EXAMPLE 4.

$$\psi(x) = \alpha\cos\beta x \quad , \quad \alpha > 0 \quad , \quad \beta > 0 \quad , \quad \frac{\pi}{2\beta} < R < \frac{3\pi}{2\beta}$$

$$g(x) = \alpha[\cos\beta x + \beta(x-R)\,\text{sen}\,\beta x] \quad .$$

$$\text{cotg}\,\beta\xi = \beta(R-\xi) \quad , \quad \text{arc cotg}\,\beta R < \xi < \text{arc cotg}\,\frac{\pi}{2} \quad .$$

$$u(x) = \alpha\beta(\text{sen}\,\beta\xi)(R-x) \quad , \quad x \in [\xi,R] \quad .$$

EXAMPLE 5.

$$\psi(x) = -\alpha\cosh\beta x + \gamma \ , \ \gamma > \alpha > 0 \quad , \quad \beta > 0 \quad , \quad R > \frac{1}{\beta}\,\text{arg cosh}\,\frac{\gamma}{\alpha}$$

$$g(x) = -\alpha[\cosh\beta x + \beta(R-x)\,\text{senh}\,\beta x] + \gamma \ .$$

$$1 + \beta(R-\xi)\,\text{tgh}\,\beta\xi = \frac{\gamma}{\alpha}\,\frac{1}{\cosh\beta\xi} \quad , \quad 0 < \xi < R \quad .$$

$$u(x) = -\alpha\beta(\text{senh}\,\beta\xi)(x-R) \quad , \quad x \in [\xi,R] \quad .$$

EXAMPLE 6.

$$\psi(x) = -(x-\alpha)^4 - 4\alpha^3 x + \beta, \alpha > 0, R > 0, \alpha^4 < \beta < \alpha^4 + R^2[(R-2\alpha)^2 + 2\alpha^2]$$

$$g(x) = -(x-\alpha)^3[x-\alpha+4(R-x)] + \beta - 4\alpha^3 R$$

$$\xi^4 - \frac{4}{3}(R+2\alpha)\xi^3 + \alpha(3+4R-\alpha)\xi^2 - 4\alpha^2 R\xi + \frac{\beta-\alpha^4}{3} = 0 \quad , \quad 0 < \xi < R$$

$$u(x) = -4[(\xi-\alpha)^3 + \alpha^3\xi](x-R) \quad , \quad x \in [\xi,R] \quad .$$

EXAMPLE 7.

$$\psi(x) = \begin{cases} \alpha-\beta x-\beta(R-x)\log\frac{R-x}{R} & , \quad 0 \le x < R \\ \alpha-\beta R & , \quad x=R \end{cases} \qquad \alpha > 0 \ , \ \beta > 0 \ , \ R > \frac{\alpha}{\beta} \ .$$

$$g(x) = \alpha - \beta x$$
$$\xi = \frac{\alpha}{\beta}$$
$$u(x) = \beta(x-R)\log\left(1 - \frac{\alpha}{\beta R}\right) \quad , \quad x \in \left[\frac{\alpha}{\beta} , R\right] \ .$$

EXAMPLE 8.

$$\psi(x) = \begin{cases} \alpha-2\beta Rx+\beta x^2-2\beta R(R-x)\log\frac{R-x}{R} & , \quad 0\le x<R \\ \alpha-\beta R^2 & , \quad x = R \end{cases} \qquad \alpha>0 \ , \ \beta>0 \ , \ R>\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}$$

$$g(x) = \alpha - \beta x^2$$
$$\xi = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}$$
$$u(x) = 2\left[(\alpha\beta)^{\frac{1}{2}} + R\log\left(1 - \left(\frac{\alpha}{\beta R^2}\right)^{\frac{1}{2}}\right)\right](x-R) \quad , \quad x \in \left[\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} R\right] \ .$$

EXAMPLE 9

$$\psi(x) = \begin{cases} \frac{1}{\alpha} - \beta - \frac{x}{\alpha(R+\alpha)} + \frac{R-x}{(R+\alpha)^2}\log\frac{1+\frac{x}{\alpha}}{1-\frac{x}{R}} & , \quad 0 \le x < R \\ \frac{1}{R+\alpha} - \beta & , \quad x = R \end{cases}$$

$$\alpha>\theta \quad , \quad \beta>0 \quad , \quad 1-R\beta<\alpha\beta<1$$

$$g(x) = \frac{1}{x+\alpha} - \beta$$
$$\xi = \frac{1}{\beta} - \alpha$$
$$u(x) = -\xi\left[\frac{\xi}{\alpha(R+\alpha)(\xi+\alpha)} + \frac{1}{(R+\alpha)^2}\log\frac{1+\frac{\xi}{\alpha}}{1-\frac{\xi}{R}}\right](R-x) \quad , \quad x \in \left[\frac{1}{\beta} - \alpha, R\right] \ .$$

BIDIMENSIONAL CASE.

EXAMPLE 10.

$$\psi(r) = \alpha\left(\beta^n - r^n\right) \quad n \ge 2 \quad , \quad \alpha>0 \quad , \quad \beta>0 \quad , \quad R>\beta$$
$$g(r) = \alpha\left[\beta^n - r^n - nr^n\log\frac{R}{r}\right]$$
$$\log\left(\frac{R}{\xi}\right)^n = \left(\frac{\beta}{R}\right)^n\left(\frac{R}{\xi}\right)^n - 1 \quad , \quad 0 < \xi < R$$
$$u(r) = \alpha n\xi^n\log\frac{R}{r} \quad , \quad r \in [\xi,R] \ .$$

EXAMPLE 11.

$$\psi(r) = \beta e^{-\alpha^2 r^2} - \gamma \quad , \quad \beta > \gamma > \beta e^{-\alpha R^2} > 0$$
$$g(r) = \beta e^{-\alpha r^2}\left[1 - 2\alpha^2 r^2\log\frac{R}{r}\right] - \gamma$$
$$e^{\alpha^2\xi^2} = \frac{\beta}{\gamma}\left(1-2\alpha^2\xi^2\log\frac{R}{\xi}\right) \quad , \quad 0 < \xi < R$$
$$u(r) = 2\beta\alpha^2\xi^2\log\frac{R}{r} \quad , \quad r \in [\xi,R] \ .$$

EXAMPLE 12.

$$\Psi(r) = \alpha \cos \beta r \quad , \quad \alpha>0 \ , \ \beta>0 \ , \ \frac{\pi}{2\beta} < R < \frac{3\pi}{2\beta}$$

$$g(r) = \alpha\left[\cos \beta r - \beta r \,\mathrm{sen}\, \beta r \log \frac{R}{r}\right]$$

$$\mathrm{cotg}\, \beta\xi = \beta\xi \log \frac{R}{\xi} \quad , \quad \frac{1}{\beta} \mathrm{arc\ cotg} \frac{\beta R}{e} < \xi < \frac{\pi}{2\beta}$$

$$u(r) = \alpha\beta\xi(\mathrm{sen}\, \beta\xi)\log \frac{R}{r} \quad , \quad r \in [\xi,R] \ .$$

AGRADECIMIENTO.

Parte de este trabajo ha sido realizado mediante un subsidio que la SUBCYT (Argentina) otorgó al proyecto "*Problemas de frontera libre de la Física-Matemática*".

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RESUMEN.

En el presente trabajo se estudia el problema del obstáculo a través de su formulación clásica, para un hilo elástico homogéneo y para una membrana circular elástica homogénea, no sometidos a cargas externas.

Se consideran los casos particulares donde el obstáculo Ψ es una función par (en el caso del hilo) o una función radial (en el caso de la membrana).

Se define un problema inverso y se establece una correspondencia entre una familia de funciones G_2 y la familia de obstáculos cuyos conjuntos de contacto son conexos. Esto permite construir, a partir de un conexo Ω^* prefijado en Ω, problemas que admitan al mismo como conjunto de contacto.

Se proporciona una lista de ejemplos en los que la frontera libre $\partial\Omega^*$ está calculada exactamente o se expresa como la única solución de una ecuación algebraica.

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