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SIMILARITY SOLUTIONS IN A CLASS OF THAWING PROCESSES

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A study on similarity solutions for a mathematical model for thawing in a saturated porous medium is considered when change of phase induces a density jump and the influence of pressure on the melting temperature is considered. The mathematical analysis is made for different cases, depending on the sign of the three physical parameters.

1. Introduction

In this paper we consider the problem of thawing of a partially frozen porous medium, saturated with an incompressible liquid, with the aim of constructing similarity solutions.

More specifically we deal with the following situations (for a detailed exposition of the physical background we refer to Ref. 6):

(i) a sharp interface between the frozen part and the unfrozen part of the domain exists (sharp, in the macroscopic sense);
(ii) the frozen phase is at rest with respect to the porous skeleton, which will be considered to be undeformable;
(iii) due to the density jump between the liquid and solid phases, thawing can induce either desaturation or water movement in the melting region. We will consider the latter situation, assuming that liquid is continuously supplied to keep the medium saturated.

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Although thawing has received less attention than freezing, our investigation is in the same spirit as Nakano[4,7] (see also Refs. 10 and 11 for further references) with the simplification due to the absence of ice lenses and frozen fringes.

The unknowns of the problem are a function $x = s(t)$, representing the free boundary separating $Q_1 \equiv \{(x,t) : 0 < x < s(t),\ t > 0\}$ and $Q_2 \equiv \{(x,t) : s(t) < x,\ t > 0\}$, and the two functions $u(x,t)$ and $v(x,t)$ defined in Q_1 and Q_2 respectively, representing the temperature of the unfrozen and of the frozen zone. Besides standard regularity requirements, $s(t)$, $u(x,t)$ and $v(x,t)$ fulfil the following conditions (we refer to Ref. 5 for a detailed explanation of the model):

$$u_t = a_1 u_{xx} - b\rho\dot{s}(t)u_x\,, \qquad \text{in } Q_1 \tag{1.1}$$

$$v_t = a_2 v_{xx}\,, \qquad \text{in } Q_2 \tag{1.2}$$

$$u(s(t),t) = v(s(t),t) = d\rho s(t)\dot{s}(t)\,, \qquad t > 0\,, \tag{1.3}$$

$$k_F v_x(s(t),t) - k_U u_x(s(t),t) = \alpha\dot{s}(t) + \beta\rho s(t)\dot{s}^2(t)\,, \quad t > 0\,, \tag{1.4}$$

$$u(0,t) = B > 0\,, \qquad t > 0 \tag{1.5}$$

$$v(x,0) = v(+\infty,t) = -A < 0\,, \qquad x > 0,\ t > 0\,, \tag{1.6}$$

$$s(0) = 0\,, \tag{1.7}$$

with

$$\begin{cases} a_1 = \alpha_1^2 = \dfrac{k_U}{\rho_U c_U}\,, \quad a_2 = \alpha_2^2 = \dfrac{k_K}{\rho_F c_F}\,, \quad b = \dfrac{\varepsilon\rho_W c_W}{\rho_U c_U}\,, \\ d = \dfrac{\varepsilon\gamma\mu}{K}\,, \quad \rho = \dfrac{\rho_W - \rho_I}{\rho_W}\,, \quad \alpha = \varepsilon\rho_I\lambda\,, \\ \beta = \dfrac{\varepsilon^2\rho_I(c_W - c_I)\gamma\mu}{K} = \varepsilon d\rho_I(c_W - c_I)\,, \end{cases} \tag{1.8}$$

where

$$\begin{cases} \varepsilon > 0 : \text{porosity}\,, \quad \rho_w \text{ and } \rho_I > 0\text{: density of water and ice (g/cm}^3\text{)}, \\ c > 0 : \text{specific heat at constant density (cal/g}\cdot{}^\circ\text{C)}, \\ k_U \text{ and } k_F > 0\text{: conductivity of the unfrozen and frozen zones} \\ \quad \text{(cal/s cm}\cdot{}^\circ\text{C)}, \\ u : \text{temperature of the unfrozen zone } ({}^\circ\text{C}), \\ v : \text{temperature of the frozen zone } ({}^\circ\text{C}),\ u = v = 0 \text{ being the} \\ \quad \text{melting point at atmospheric pressure}, \\ \lambda > 0 : \text{latent heat at } u = 0 \text{ (cal/g)}, \\ \gamma : \text{coefficient in the Clausius–Clapeyron law (s}^2\text{cm }{}^\circ\text{C/g)}, \\ \mu > 0 : \text{viscosity of liquid (g/s cm)}, \\ K > 0 : \text{hydraulic permeability (cm}^2\text{)}, \\ B > 0 : \text{boundary temperature at the fixed face } x = 0\ ({}^\circ\text{C}), \\ -A < 0 : \text{initial temperature } ({}^\circ\text{C}). \end{cases} \tag{1.9}$$

Remark 1.1. We recall the Clausius–Clapeyron law on the dependence of melting temperature upon pressure, which, in the presence of capillarity has the form[8]

$$\frac{\pi}{\rho_I} + p\frac{\rho}{\rho_I} = -\lambda\frac{T - T_0}{T_0}\,. \tag{1.10}$$

In (1.10) π is the capillarity at the interface, T is the absolute temperature, and T_0 is the melting point under normal conditions (°C), and p is the pressure of the liquid phase ($p = 0$ being the atmospheric pressure).

The usual Clapeyron law corresponds to setting $\pi = 0$, thus implying that

$$u(s(t),t) = v(s(t),t) = -\gamma p(s(t),t)\,, \tag{1.11}$$

where $\gamma = \rho T_0/\rho_I\lambda$ has the same sign as ρ.

On the other hand, if the frozen region is in equilibrium with the environment at atmospheric pressure, $\pi = -p$ and (1.10) still lead to (1.11), but with

$$\gamma = \frac{T_0}{\rho_W\lambda}\,, \tag{1.12}$$

which is always positive.

In the following we will use (1.11) with no sign restriction on γ.

Remark 1.2. In order to fulfil (1.3), a pressure gradient is needed. Darcy's law implies that p_x is independent of x and, assuming that $p(0,t) = 0$, we obtain

$$p(s(t),t) = -\frac{\varepsilon\rho\mu}{K}s(t)\dot{s}(t)\,, \quad t > 0\,, \tag{1.13}$$

and hence, substituting (1.13) into (1.11), we obtain (1.3).

Remark 1.3. The free boundary problem (1.1)–(1.7) reduces to the usual Stefan problem when $d\rho = 0$, since in that case we have the classical Stefan condition on $x = s(t)$:

$$u(s(t),t) = v(s(t),t) = 0\,, \qquad t > 0\,, \tag{1.3'}$$

$$k_F v_x(s(t),t) - k_U u_x(s(t),t) = \alpha\dot{s}(t)\,, \quad t > 0\,. \tag{1.4'}$$

For this reduced problem a similarity solution (often referred to as Neumann's solution) is known.[2,9,12,13] From now on we assume that $d\rho \neq 0$.

2. Similarity Solutions

Now, we will look for similarity solutions to problem (1.1)–(1.7) by considering eight different cases depending on the sign of the parameters ρ, β and d (or equivalently of γ). Our results include the cases considered in Refs. 3 and 5.

First of all we note that the function

$$u(x,t) = \phi(\eta) \quad \text{with } \eta = \frac{x}{2\alpha_1\sqrt{t}} \tag{2.1}$$

is a solution of Eq. (1.1) if and only if ϕ satisfies the following equation:

$$\frac{1}{2}\phi''(\eta) + \left(\eta - \frac{b\rho}{\alpha_1}\dot{s}(t)\sqrt{t}\right)\phi'(\eta) = 0\,. \tag{2.2}$$

If we assume

$$s(t) = 2\xi\alpha_1\sqrt{t} \tag{2.3}$$

we obtain that

$$\phi(\eta) = C_1 + C_2\int_0^\eta \exp(-r^2 + 2b\rho\xi r)\,dr\,, \tag{2.4}$$

where ξ, C_1 and C_2 are constant values.

A solution of (1.1), (1.2) and (1.6) is given by (2.3) and

$$u(x,t) = C_1 + C_2\int_0^{x/(2\alpha_1\sqrt{t})} \exp(-r^2 + 2b\rho\xi r)\,dr\,, \tag{2.5}$$

$$v(x,t) = C_3 + C_4\,\mathrm{erf}\left(\frac{x}{2\alpha_2\sqrt{t}}\right)\,, \tag{2.6}$$

where erf is the error function defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-r^2)\,dr\,, \quad \mathrm{erfc}(x) = 1 - \mathrm{erf}(x)\,, \tag{2.7}$$

and C_1, C_2, C_3, C_4 and ξ are constant values to be determined.

From conditions (1.3), (1.5) and (1.6) we deduce that

$$C_1 = B\,, \quad C_2 = \frac{2d\rho\alpha_1^2\xi^2 - B}{g(2b\rho,\xi)}\,, \tag{2.8}$$

$$C_3 = \frac{2d\rho\alpha_1^2\xi^2 + A\,\mathrm{erf}(\alpha_1\xi/\alpha_2)}{\mathrm{erfc}(\alpha_1\xi/a_2)}\,, \quad C_4 = -\frac{2d\rho\alpha_1^2\xi^2 + A}{\mathrm{erfc}(\alpha_1\xi/\alpha_2)}\,, \tag{2.9}$$

where $g = g(p,y)$ is defined by

$$g(p,y) = \int_0^y \exp(pyr - r^2)\,dr\,. \tag{2.10}$$

Therefore, the similarity solution is completely determined once the constant ξ is chosen. This is done by imposing condition (1.4) which yields that ξ is a root of the following equation:

$$K_1(B - my^2)H(p,y) - K_2F(m,y) = \delta y + \nu y^3\,, \quad y > 0\,, \tag{2.11}$$

where

$$H(p,y) = \frac{\exp((p-1)y^2)}{(2\sqrt{\pi})g(p,y)} \tag{2.12}$$

and

$$F(m,y) = (A + my^2)\frac{\exp(-\gamma_0^2 y^2)}{\operatorname{erfc}(\gamma_0 y)} \tag{2.13}$$

and the constants K_1, K_2, γ_0, δ, p, m and ν are defined as follows:

$$K_1 = \frac{k_U}{\alpha_1\sqrt{\pi}} > 0\,, \quad K_2 = \frac{k_F}{\alpha_2\sqrt{\pi}} > 0\,, \quad \gamma_0 = \frac{\alpha_1}{\alpha_2} > 0\,, \quad \delta = \alpha\alpha_1 > 0\,, \tag{2.14}$$

$$p = 2b\rho\,, \quad m = 2d\rho\alpha_1^2\,, \quad \nu = 2\beta\rho\alpha_1^3\,. \tag{2.15}$$

Then, we have obtained

Theorem 2.1. *The free boundary problem* (1.1)–(1.7) *has the similarity solution*

$$s(t) = 2\xi\alpha_1\sqrt{t}\,, \tag{2.16}$$

$$u(x,t) = B + \frac{m\xi^2 - B}{g(p,\xi)}\int_0^{x/(2\alpha_1\sqrt{t})} \exp(-r^2 + p\xi r)\,dr\,, \tag{2.17}$$

$$v(x,t) = \frac{m\xi^2 \operatorname{erfc}\left(\frac{x}{2\alpha_2\sqrt{t}}\right) + A\left(\operatorname{erf}(\gamma_0\xi) - \operatorname{erf}\left(\frac{x}{2\alpha_2\sqrt{t}}\right)\right)}{\operatorname{erfc}(\gamma_0\xi)} \tag{2.18}$$

if and only if the coefficient $\xi > 0$ *satisfies Eq.* (2.11).

3. Preliminary Results

To analyze (2.11) in the different cases, we need to study the properties of the functions involved. First, we consider F, defined by (2.13) where A and γ_0 are given positive constants, while m can be positive or negative. We have

Proposition 3.1. *If* $m > 0$, *then* F *grows from* A *to* $+\infty$ *when* y *grows from* 0 *to* $+\infty$ *and it tends to* $+\infty$ *like* $\sqrt{\pi}\gamma_0 m y^3$. *If* $m < 0$, F *grows from* A *to a positive maximum, then it decreases to* $-\infty$ *like* $\sqrt{\pi}\gamma_0 m y^3$.

Proof. The first part is trivial. To prove the second part of the proposition, we only have to show that the derivative $F'(y)$ has only one zero. Setting $z = \gamma_0 y$, it is easy to see that the equation

$$[\sqrt{\pi}z\exp(z^2)\operatorname{erfc}(z)]^{-1} = 1 + \frac{|m|}{\gamma_0^2 A - |m|z^2}\,, \quad z \neq \gamma_0\sqrt{\frac{A}{|m|}} \tag{3.1}$$

has one unique root, since the L.H.S. increases from $1 - m/(A\gamma_0)^2$ to $+\infty$ as z goes from 0 to $\gamma_0\sqrt{A/|m|}$. □

Now we study $g(p,y)$ defined by (2.10) or by its equivalent form[1]

$$\frac{2}{\sqrt{\pi}}g(p,y) = \exp\left(\frac{p^2y^2}{4}\right)\left(\operatorname{erf}\left(\frac{py}{2}\right) + \operatorname{erf}\left(\frac{(2-p)y}{2}\right)\right), \quad p \in \mathbf{R},\ y > 0\,. \tag{3.2}$$

When $p > 0$, we have from elementary calculation:

Proposition 3.2. *For all $p > 0$, we have*

$$g(p,y) \geq \frac{1}{py}\left(\exp((p-1)y^2) - \exp(-y^2)\right), \quad y > 0, \tag{3.3}$$

$$g_y(p,y) \geq \exp((p-1)y^2) + \frac{p}{2}\left(1 - \exp(-y^2)\right), \quad y > 0, \tag{3.4}$$

$$g(p,0) = 0, \quad g_y(p,0) = 1, \quad g(p,+\infty) = +\infty. \tag{3.5}$$

From Proposition 3.2 and (2.12), we immediately have

Proposition 3.3. *For all $p > 0$, the function $H(p,y)$ has the following properties:*

$$\lim_{y\to 0} H(p,y) = +\infty, \quad \forall\, p > 0; \tag{3.6}$$

$$\lim_{y\to+\infty} H(p,y) = \begin{cases} 0 & if\ 0 < p < 2, \\ 1 & if\ p = 2, \\ +\infty & if\ p > 2; \end{cases} \tag{3.7}$$

$$\lim_{y\to+\infty} \frac{y}{H(p,y)} = \frac{2}{(p-2)\sqrt{\pi}}, \ \forall\, p > 2; \tag{3.8}$$

$$\lim_{y\to+\infty} y^2 H(p,y) = \begin{cases} 0 & if\ 0 < p < 2, \\ +\infty & if\ p \geq 2; \end{cases} \tag{3.9}$$

$$\frac{\partial H}{\partial y}(p,y) < 0, \quad \forall\, y > 0,\ \forall\, 0 < p \leq 2. \tag{3.10}$$

In order to study $H(p,y)$ for $p < 0$, it is useful to introduce the variable

$$q = -p/2, \tag{3.11}$$

and to write

$$\begin{gathered} H(p,y) = \exp(-(1+q)^2y^2)[\operatorname{erf}((1+q)y) - \operatorname{erf}(qy)]^{-1}, \\ y > 0,\ q = -p/2 > 0. \end{gathered} \tag{3.12}$$

The proofs of the propositions that follow are based on elementary though lengthy calculations which will not be reproduced here. If we denote by $N(q,y)$ the function in square brackets in (3.12), we have

Proposition 3.4. *Function N has the following properties for any $q > 0$:*

$$N(q,0) = 0, \quad \lim_{y\to+\infty} N(q,y) = 0, \tag{3.13}$$

$$\frac{2y}{\sqrt{\pi}}\exp\big(-(1+q)^2y^2\big) \le N(q,y) \le \frac{2y}{\sqrt{\pi}}\exp(-q^2y^2)\,, \quad \forall\, y>0,\ q>0\,, \tag{3.14}$$

$$\frac{\partial N}{\partial y}(q,y)\begin{cases} >0 & if\ 0<y<\overline{y}(q)\,,\\ =0 & if\ y=\overline{y}(q)\,,\\ <0 & if\ y>\overline{y}(q)\,,\end{cases} \tag{3.15}$$

$$\frac{\partial^2 N}{\partial y^2}(q,y)=0 \;\Leftrightarrow\; y=\sqrt{3}\overline{y}(q)\,, \tag{3.16}$$

where $\overline{y}$ *corresponds of the maximum of* $\partial N/\partial y$ *for fixed* q, *i.e.*

$$\overline{y}(q)=\sqrt{\frac{\log(1+1/q)}{1+2q}}\,. \tag{3.17}$$

Consequently, we have

Proposition 3.5. *For any* $p<0$, *function* H *has the following properties*:

$$\lim_{y\to 0} H(p,y)=+\infty\,, \quad \forall\, p>0 \qquad \lim_{y\to+\infty} H(p,y)=0\,, \tag{3.18}$$

$$\frac{\partial H}{\partial y}(p,y)<0\,, \quad \forall\, y>0\,, \tag{3.19}$$

$$\frac{\sqrt{\pi}}{2}\frac{\exp(-(1+2q)y^2)}{y} \le H(p,y) \le \frac{\sqrt{\pi}}{2y}\,, \quad \forall\, y>0\,, \tag{3.20}$$

$$\lim_{y\to+\infty} y^2 H(p,y)=0\,. \tag{3.21}$$

At this point, we use Propositions 3.3 and 3.5 to study the behavior of the first term on the L.H.S. of Eq. (2.11). Setting for convenience

$$I(y)=(B-my^2)H(p,y)\,, \tag{3.22}$$

we have

Proposition 3.6. *For every value of constants* m *and* p *we have*

$$\lim_{y\to+0} I(y)=+\infty\,. \tag{3.23}$$

Moreover

(i) *if* $m>0$, $y_0=\sqrt{B/m}$ *is such that*

$$I(y_0)=0\,, \quad I(y)\le 0\,, \quad \forall\, y\ge y_0 \tag{3.24}$$

and

(a) *if* $p<2$, *then* $I'(y)<0$ *in* $(0,y_0)$ *and* $\lim_{t\to+\infty} I(y)=0$,
(b) *if* $p>2$, *then* $\lim_{t\to+\infty} I(y)=-\infty$, *and* $I(y)\simeq -m(p/2-1)\sqrt{\pi}y^3$,
(c) *if* $p=2$, *then* $I'(y)<0$ *in* $(0,y_0)$, $\lim_{t\to+\infty} I(y)=+\infty$, *and* $I(y)\simeq -my^2$.

(ii) *if* $m < 0$,

(a) *if* $p < 2$, *then* $\lim_{t\to+\infty} I(y) = 0$,
(b) *if* $p > 2$, *then* $\lim_{t\to+\infty} I(y) = +\infty$, *and* $I(y) \simeq -m(p/2-1)\sqrt{\pi}y^3$,
(c) *if* $p = 2$, *then* $\lim_{t\to+\infty} I(y) = +\infty$, *and* $I(y) \simeq -my^2$.

4. Similarity Solutions

Now we are in a position to discuss the solvability of (2.11), which we rewrite as follows:

$$K_1 I(y) - K_2 F(y) = \delta y + \nu y^3\,, \quad y > 0\,. \tag{4.1}$$

It will be convenient to denote by q_i, $i = 1, 2, \ldots, N$ the zeroes of the L.H.S. of (4.1), if they exist, i.e.

$$K_1 I(q_i) - K_2 F(q_i) = 0\,, \quad i = 1, 2, \ldots, N\,. \tag{4.2}$$

First we consider the case $m > 0$ and we prove

Theorem 4.1. *If* $m > 0$ *and* $\nu \geq 0$, (4.1) *always admits solutions. Moreover,*

(a) *if* $p \leq 2$, *there exists a unique* ξ *satisfying* (4.1),
(b) *if* $p > 2$, *there is one and only one solution* ξ *in* $(0, q_1)$.

Proof. The theorem is a straightforward consequence of the results of Sec. 3. We note that the monotonicity of both sides of (4.1) guarantees uniqueness in case (a) and that Proposition 3.6(i) yields the following inequality:

$$\xi < q_1 < y_0 = \sqrt{\frac{B}{m}}\,. \tag{4.3}$$

Result (b) is weaker since the monotonicity of $I(y)$ is no longer guaranteed. □

Next, we have

Theorem 4.2. *If* $m > 0$, $\nu < 0$, *a solution to* (4.1) *exists whenever*

$$\frac{B}{m} < \frac{\delta}{|\nu|}\,. \tag{4.4}$$

Proof. It is sufficient to recall that q_1 is always less than y_0 and that the R.H.S. of (4.1) is positive in $(0, \sqrt{\delta/|\nu|})$. □

Remark 4.3. It is obvious that (4.4) can be replaced by the weaker requirement

$$q_1 < \sqrt{\frac{\delta}{|m|}}\,, \tag{4.4'}$$

but we preferred to state Theorem 4.2 as above, since (4.4) involves explicitly the parameters entering the model.

Remark 4.4. Under some additional requirements, it can be seen that no more than one solution exists in the interval $(0, q_1)$. This is true, e.g. if $p > 2$ and

$$q_1 < \sqrt{\frac{\delta}{3|m|}}, \tag{4.5}$$

where the R.H.S. of (4.5) is the abscissa of the maximum of the function on the R.H.S. of (4.1).

Remark 4.5. If (4.4) is not satisfied, examples of non-existence of a solution can actually be found.

Now we consider cases in which $m < 0$.

Theorem 4.6. *If $m < 0$, $\nu \geq 0$, sufficient conditions for the existence of solutions are the existence of a root of (4.2) or*

$$\nu > \left(K_1\left(\frac{p}{2} - 1\right)^+ + K_2\gamma_0\right)|m|\sqrt{\pi}, \tag{4.6}$$

where $(\ )^+$ denotes the positive part of the quantity in bracket.

Proof. It is sufficient to note that (4.6) guarantees that, as $y \to +\infty$, the L.H.S. of (4.1) tends to $+\infty$ slower than the R.H.S. (see Proposition 3.6(ii) and Proposition 3.1). The first case is obvious. □

Now, we pass to consider the last case. The proof of the following theorem is trivial.

Theorem 4.7. *If $m < 0$, $\nu < 0$ (4.1) has at least two solutions if (4.2) has roots and*

$$q_1 < \sqrt{\frac{\delta}{|\nu|}}. \tag{4.7}$$

Remark 4.8. When $c_w \neq c_I$, the temperature

$$u^* = \frac{\lambda}{c_I - c_w} \tag{4.8}$$

is the intersection of the two lines representing the energy of the solid and the liquid as a function of temperature.

We note that the similarity solution is such that

$$u(s(t), t) = v(s(t), t) = m\xi^2. \tag{4.9}$$

Therefore it seems appropriate to say that a similarity solution is physically acceptable if

$$m\xi^2 < \frac{\lambda}{c_I - c_w} \quad \text{when } c_I > c_w,$$

$$m\xi^2 > \frac{\lambda}{c_I - c_w} \quad \text{when } c_I < c_w,$$

i.e.

$$2d\rho\alpha_1^2(c_I - c_w)\xi^2 < \lambda\,. \tag{4.10}$$

(I) If $d\rho > 0$ and $c_I < c_w$ (i.e. $m > 0$, $\nu > 0$) (4.10) is always satisfied and hence all solutions whose existence is guaranteed by Theorem 4.1 (with no restrictions on the data and the coefficients) are physically acceptable.

(II) If $d\rho > 0$ and $c_I > c_w$ (i.e. $m > 0$, $\nu < 0$) condition (4.4) of Theorem 4.2 guarantees that at least a solution exists such that (4.10) is satisfied.

(III) If $d\rho < 0$ and $c_I < c_w$ (i.e. $m < 0$, $\nu > 0$) (4.10) is always satisfied. Although Theorem 4.6 does not guarantee existence for any value of the parameters, nevertheless it does not impose any restriction on the size of ξ.

(IV) If $d\rho < 0$ and $c_w < c_I$ (i.e. $m < 0$, $\nu < 0$) condition (4.7) of Theorem 4.7 guarantees that at least a solution exists such that (4.10) is satisfied.

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