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Remarks on a One-Dimensional Stefan Problem Related to the Diffusion-Consumption Model

Wir untersuchen das Verhalten des freien Randes im Crank-Gupta-Modell für die Diffusion und den Verbrauch von Sauerstoff im lebenden Körpergewebe, wobei wir annehmen, daß die Anfangsbedingung der stationären Lösung entspricht und der Zustrom von Sauerstoff sich auf dem fixierten Rand verringert.

We study the behaviour of the free boundary of the Crank-Gupta model for the diffusion-consumption of oxygen in a living tissue, assuming an initial condition corresponding to the stationary solution and with a decreasing input of oxygen at the fixed boundary.

Мы исследуем поведение свободной границы в модели Крэнка-Гупты для диффузии и расхода кислорода в живой ткани. При этом предполагаем, что начальное условие соответствует стационарному решению и, дальше, что приток кислорода на фиксированной границе уменьшается.

1 Introduction

The Crank-Gupta model for the one-dimensional diffusion of oxygen in a living tissue [4] is given by the following equations

$$u_{xx} - u_t = 1\,, \quad \text{in} \quad D_T = \{(x, t): 0 < x < s(t), 0 < t < T\}\,,$$
$$s(0) = 1\,,$$
$$u(x, 0) = h(x)\,, \qquad 0 < x < 1; \qquad u_x(0, t) = f(t)\,, \qquad 0 < t < T;$$
$$u\big(s(t), t\big) = 0\,, \qquad 0 < t < T; \qquad u_x\big(s(t), t\big) = 0\,, \quad 0 < t < T\,,$$

where $u(x, t)$ denotes the oxygen concentration and the source term in the heat equation accounts for the oxygen consumption in the tissue.

It is well known that this problem admits a stationary solution $u(x, t) = \frac{1}{2}(1 - x)^2$, $s(t) = 1$, in correspondence with the obvious initial condition and the boundary condition $f(t) = -1$, which means that a constant unitary input of oxygen is maintained at $x = 0$.

In this paper we study the behaviour of the free boundary $x = s(t)$ when, given an initial condition corresponding to the stationary solution, the oxygen input at $x = 0$ decreases in time (with possible change of sign), i.e. when $u_x(0, t)$ is a monotonically increasing function of the time, with $u_x(0, 0) = -1$.

To this aim we will consider the equations for the time derivative $z(x, t) = u_t(x, t)$. For $z(x, t)$ we have then a one-phase Stefan problem, similar to that of a supercooled liquid, i.e. with the free boundary conditions $z\big(s(t), t\big) = 0$, $z_x\big(s(t), t\big) = -\dot{s}(t)$, and with $z(x, t) < 0$ in $0 < x < s(t)$, for $t > 0$.

According to the general scheme of [5], we know that only one of the following three cases can occur:

(A) the solution exists for arbitrarily large T;
(B) a time T_B exists such that $s(T_B) = 0$;
(C) a time T_C exists such that $\lim_{t \to T_C} s(t) > 0$ and $\lim_{t \to T_C} \dot{s}(t) = -\infty$.

In section 2 the general case of a monotonically increasing $u_x(0, t)$, implying $z_x(0, t) > 0$, is considered. A necessary and sufficient condition for the case (A) is also determined: this condition is quite intuitive in terms of the oxygen problem as it states that a global solution exists if and only if the oxgyen input at $x = 0$ remains positive (i.e. $u_x(0, t) < 0$) for any time $t > 0$.

In section 3 the case in which $z_x(0, t)$ is constant (i.e. $u_x(0, t)$ increases linearly in time) is considered in detail.

2. The general case of increasing flux

Let us consider the following problem: find a triple (T, s, z) such that

(i) $T > 0$;
(ii) $s(t)$ is a positive continuous function in $[0, T)$, $s \in C^1(0, T)$;
(iii) $z(x, t)$ is a bounded function, continuous in $0 \leq x \leq s(t)$, $0 \leq t < T$, such that $z_x(x, t)$ is bounded in the same domain and continuous, with the possible exception of a finite numer of points on the parabolic boundary, $z_{xx}(x, t)$ and $z_t(x, t)$ are continuous in $0 < x < s(t)$, $0 < t < T$;
(iv) the following conditions are satisfied:

$$z_{xx} - z_t = 0\,, \qquad D_T = \{(x, t): 0 < x < s(t), 0 < t < T\} \tag{2.1}$$
$$s(0) = 1; \qquad z(x, 0) = 0\,, \qquad 0 < x < 1; \qquad (2.2); (2.3)$$
$$z_x(0, t) = g(t)\,, \qquad 0 < t < T; \tag{2.4}$$
$$z\big(s(t), t\big) = 0\,, \qquad 0 < t < T; \qquad z_x\big(s(t), t\big) = -\dot{s}(t)\,, \qquad 0 < t < T\,. \qquad (2.5); (2.6)$$

We are concerned with the case in which $g(t)$ satisfies the following hypothesis, which will be assumed throughout sec. 2:

(H) $g(t)$ is a non negative piecewise continuous function in $(0, +\infty)$, bounded in every interval $(0, t)$, $t > 0$.

It is well known that the problem (i)–(iv), with the hypothesis (H), has a unique solution (see [5]). Moreover, the behaviour of the solution is ruled by Theorem 8 of the above mentioned reference. The statement of this Theorem implies that one of the following cases must occur (see also [6]):

(A) the problem (i)–(iv) has a solution with arbitrarily large T;

(B) there exists a time $T_B > 0$ such that $\lim_{t\to T_B} s(t) = 0$;

(C) there exists a time $T_C > 0$ such that

$$\inf_{t\in(0,T_C)} s(t) > 0 \quad \text{and} \quad \liminf_{t\to T_C} \dot{s}(t) = -\infty\,.$$

As we shall see, any of these cases can actually occur with an appropriate choice of the function $g(t)$ in (2.4). The first simple proprieties of the solution of (i)–(iv) are summarized in the following

Lemma 2.1: *If the hypothesis* (H) *holds, we have*:

$$z(x,t) \leqq 0 \quad \text{in } D_T \tag{2.7}$$

$$\dot{s}(t) = -z_x(s(t),t) \leqq 0\,, \qquad 0 < t < T\,, \tag{2.8}$$

$$z_x(x,t) \geqq 0 \quad in\ D_T\,. \tag{2.9}$$

Moreover, if $G(t) = \sup_{\tau\in(0,t)} g(\tau)$

$$z(x,t) \geqq G(t)\,(x-1) \quad in\ D_T\,. \tag{2.10}$$

Proof: (2.7) and (2.10) are direct consequences of the maximum principle, and (2.8) follows immediately from (2.7) and (2.5). (2.9) follows from a generalized version of the maximum principle (see for instance [2]).

Lemma 2.2: *If* (T, s, z) *solves* (i)–(iv) *then*

$$s(t) = 1 - \int_0^t g(\tau)\,d\tau - \int_0^{s(t)} z(x,t)\,dx\,, \qquad t\in(0,T)\,, \tag{2.11}$$

$$\frac{s^2(t)-1}{2} = \int_0^t z(0,\tau)\,d\tau - \int_0^{s(t)} xz(x,t)\,dx\,, \qquad t\in(0,T); \tag{2.12}$$

$$\frac{s^3(t)-1}{3} = 2\int_0^t \tau\dot{s}(\tau)\,d\tau + 2\int_0^t \tau g(\tau)\,d\tau - \int_0^{s(t)} (x^2-2t)\,z(x,t)\,dx\,, \qquad t\in(0,T)\,, \tag{2.13}$$

and

$$\frac{s^3(t)-1}{3} = 2\int_0^t d\tau \int_0^{s(\tau)} z(x,\tau)\,dx - \int_0^{s(t)} x^2 z(x,t)\,dx\,, \qquad t\in(0,T)\,. \tag{2.14}$$

Proof: (2.11), (2.12) and (2.13) follow from Green's identity

$$\iint_{D_t} (vLu - uL^*v)\,dx\,d\tau = \int_{\partial D_t} [(u_xv - uv_x)\,d\tau + uv\,dx]$$

where L denotes the heat operator and L^* its adjoint, with $u = z(x,t)$ and $v = 1$, $v = x$, $v = x^2 - 2t$, respectively (2.14) follows immediately from (2.13) and (2.11).

We can now state a first important consequence of (2.11).

Lemma 2.3: *Suppose* $\bar{t} \leqq T$ *and let* $\lim_{t\to\bar{t}} s(t) > 0$ *and* $\int_0^{\bar{t}} g(t)\,dt < 1$. *Let*

$$\eta(t) = \begin{cases} \max\,\{x\in[0,s(t)]: z(x,t) \leqq -1\} \\ 0 \quad \text{if} \quad z(x,t) > -1\,, \qquad x\in[0,s(t)] \end{cases}$$

Then

$$\limsup_{t\to\bar{t}} \eta(t) < \lim_{t\to\bar{t}} s(t)\,.$$

Proof: Notice first that $\lim_{t\to\bar{t}} s(t)$ exists because of (2.8). From (2.9) we have $z(x,t) \leqq -1$ in $[0,\eta(t)]$ and $-1 < z(x,t) \leqq 0$ in $(\eta(t), s(t)]$ for $t < \bar{t}$.

Let $\bar{s} = \limsup_{t\to\bar{t}} \eta(t)$ and $\{t_n\}$ a sequence such that $t_n \to \bar{t}$ and $\eta_n = \eta(t_n) \to \bar{s}$, then, from (2.11)

$$s(t_n) = 1 - \int_0^{t_n} g(t)\,dt - \int_0^{\eta_n} z(x,t_n)\,dx - \int_{\eta_n}^{s(t_n)} z(x,t_n)\,dx > 1 - \int_0^{t_n} g(t)\,dt + \eta_n\,.$$

Performing the limit with $n \to +\infty$

$$\lim s(t_n) \geq 1 - \int_0^{\bar{t}} g(t)\,dt + \bar{s} > \bar{s}\,.$$

The following Lemma is an adapted, simplified version of Lemma 2.4 of [6]. We repeat the proof for sake of completeness.

Lemma 2.4: *Let (T, s, z) be a solution of* (i)–(iv) *and $T' \leq T$. If there exist two constant $z_0 \in (0, 1)$ and $d \in (0, s')$, $s' = \inf_{t \in (0, T')} s(t)$, such that*

$$z\big(s(t) - d, t\big) > -z_0\,, \qquad 0 < t < T'\,,$$

then

$$\dot{s}(t) \geq d^{-1} \ln (1 - z_0)\,, \qquad 0 < t < T'\,. \tag{2.15}$$

Proof: Fix ε and let $\Omega_\varepsilon = \{s(t) - d < x < s(t), 0 < t < T' - \varepsilon\}$. On Ω_ε consider the function

$$w(x, t) = -z_0(1 - e^{-ad})^{-1}\,(1 - e^{a(x - s(t))})$$

where a is a positive arbitrary constant.

We have immediately $w(x, t) \leq z(x, t)$ on the parabolic boundary of Ω_ε, and $w\big(s(t), t\big) = z\big(s(t), t\big) = 0$, $0 < t < T' - \varepsilon$. Moreover, if we chose $a = a(\varepsilon) = -\inf_{t \in (0, T' - \varepsilon)} \dot{s}(t)$, we have $Lw \geq 0$ and, from the maximum principle $w(x, t) \leq z(x, t)$ on Ω_ε.

It follows that $z_x\big(s(t), t\big) \leq w_x\big(s(t), t\big) = az_0(1 - e^{-ad})^{-1}$ and, with our choice $a = a(\varepsilon)$

$$a(\varepsilon) \leq a(\varepsilon)\, z_0(1 - e^{-a(\varepsilon) d})^{-1}$$

from which

$$\dot{s}(t) \geq -a(\varepsilon) \geq d^{-1} \ln (1 - z_0)\,, \qquad 0 < t < T' - \varepsilon\,,$$

and (2.15) follows with $\varepsilon \to 0$.

A first immediate consequence of these two Lemmas is

Corollary 2.5: *If (T_C, s, z) solves* (i)–(iv) *and*

$$\liminf_{t \to T_C} \dot{s}(t) = -\infty\,, \qquad \lim_{t \to T_C} s(t) > 0$$

then there exists a $\bar{t} \leq T_C$ such that

$$\int_0^{\bar{t}} g(t)\,dt \geq 1\,. \tag{2.16}$$

Actually the strict inequality holds in (2.16) as stated in the following Proposition.

Proposition 2.6: *If the case* (C) *occurs, then*

$$\int_0^{T_C} g(t)\,dt > 1\,. \tag{2.17}$$

Proof: Lemmas (2.3) and (2.4) imply that, if (C) occurs, there exists a unique level curve $z(x, t) = -1$, say $\eta(t)$, which starts from the fixed boundary $x = 0$ at some $t_C < T_C$, and hits the free boundary at $(s(T_C), T_C)$, where $s(T_C) = \lim_{t \to T_C} s(t)$.

Applying the strong maximum principle in $0 < x < \eta(t)$, $t_C < t < T_C$, we have $z(x, t) < -1$ and also $z(x, T_C) = \lim_{t \to T_C} z(x, t) < -1$ (this limit exists for classical arguments). Substituting in (2.11) we obtain (2.17).

Concerning the case (B) we have the following results.

Proposition 2.7: *If (T_B, s, z) solves* (i) – (iv) *and $\lim_{t \to T_B} s(t) = 0$, then*

$$\int_0^{T_B} g(t)\,dt = 1\,.$$

Proof: Let $G = \sup_{t \in (0, T_B)} g(t)$ then, from (2.10), $|z(x, t)| < G(1 - x)$ and we can perform the limit for $t \to T_B$ in (2.11).

As a partial converse of Preposition 2.7 we have

Proposition 2.8: *If there exists $T_0 > 0$ such that $\int_0^{T_0} g(t)\,dt = 1$ and $g(t) \leq 1$, $0 < t < T_0$, then* (B) *occurs with $T_B = T_0$.*

Proof: Proposition 2.5 ensures that (C) did not occur up to T_0. Moreover, as $g(t) \leq 1$, $0 < t < T_0$, we have $z(x, t) > x - 1$ and performing the limit in (2.11)

$$s(T_0) \leq \int_0^{s(T_0)} (1 - x)\,dx = s(T_0) - \frac{s^2(T_0)}{2}\,,$$

so that $s(T_0) = 0$.

The necessary condition for both the cases (C) and (B), given by Corollary 2.5 and Proposition 2.7, respectively, gives an immediate sufficient condition for the case (A).

Moreover, this condition is also necessary.

Theorem 2.9. *The problem* (i)—(iv) *has a (unique) solution for arbitrarily large T if and only if*

$$\int_0^t g(\tau)\,d\tau < 1 \quad \textit{for any} \quad t > 0 .$$

To complete the proof of Theorem 2.9 we need the following comparison Lemma.

Lemma 2.10: *Let (T_j, s_j, z_j), $j = 1, 2$, be two solutions, in the sense of* (i)—(iii), *of*

$$z_{jxx} - z_{jt} = 0 , \qquad D_{T_j} = \{(x, t): 0 < x < s_j(t), 0 < t < T_j\}; \tag{2.18a}$$

$$s_j(0) = b_j; \qquad z_j(x, 0) = h_j(x) , \qquad 0 < x < b_j; \tag{2.18b); (2.18c}$$

$$z_{jx}(0, t) = g_j(t) , \qquad 0 < t < T_j; \tag{2.18d}$$

$$z_j(s_j(t), t) = 0 , \qquad 0 < t < T_j; \qquad z_{jx}(s_j(t), t) = -\dot{s}_j(t) , \qquad 0 < t < T_j , \tag{2.18e); (2.18f}$$

where both g_1 and g_2 satisfy the hypotesis (H) *and h_1, h_2 are piecewise continuous functions satisfying appropriate conditions for the local existence* (see, for instance [7]), *and let be*

$$b_1 < b_2; \tag{2.19}$$

$$g_2(t) \leqq g_1(t) , \qquad 0 < t; \qquad h_1(x) \leqq h_2(x) , \qquad 0 < x < b_1; \tag{2.20); (2.21}$$

$$\int_x^{b_2} (h_2(y) + 1)\,dy \geqq 0 , \qquad 0 < x < b_2 . \tag{2.22}$$

Define $q_j = b_j + \int_0^{b_j} h_j(x)\,dx$, $T_{0,2} = \min\{T_2, \sup\{\bar{t} > 0: \int_0^t g_2(\tau)\,d\tau < q_2, t < \bar{t}\}\}$ *then* $s_1(t) < s_2(t)$, $0 < t <$ $< \min\{T_1, T_{0,2}\}$.

Proof: Define

$$u_j(x, t) = \int_x^{s_j(t)} d\xi \int_\xi^{s_j(t)} dy\big(z_j(y, t) + 1\big) . \tag{2.23}$$

By straightforward computation one verifies that, if (T_j, s_j, z_j) solves (2.18a)—(2.18f), (T_j, s_j, u_j) solves:

$$u_{jxx} - u_{jt} = 1 , \quad \text{in } D_{T_j}; \qquad s_j(0) = b_j; \tag{2.24a); (2.24b}$$

$$u_j(x, 0) = \int_x^{b_j} d\xi \int_\xi^{b_j} dy(h_j(y) + 1), 0 < x < b_j; \tag{2.24c}$$

$$u_{jx}(0, t) = -q_j + \int_0^t g_j(\tau)\,d\tau , \qquad 0 < t < T_j; \tag{2.24d}$$

$$u_j(s_j(t), t) = 0 , \qquad 0 < t < T_j; \qquad u_{jx}(s_j(t), t) = 0 , \qquad 0 < t < T_j . \tag{2.24e); (2.24f}$$

Consider now $u_{2x}(x, t)$: from (2.22) we have $u_{2x}(x, 0) \leqq 0$, $0 < x < b_2$. Moreover, $u_{2x}(s_2(t), t) = 0$, $0 < t < T_2$, from (2.24f) and, from the definition of $T_{0,2}$ and (2.24d), $u_{2x}(0, t) \leqq 0$, $0 < t < T_{0,2}$. We can apply the maximum principle in $D_{T_{0,2}} = \{(x, t): 0 < x < s_2(t), 0 < t < T_{0,2}\}$ so that $u_{2x}(x, t) < 0$ in $D_{T_{0,2}}$. This inequality and (2.24e) imply

$$u_2(x, t) > 0 \quad \text{in } D_{T_{0,2}} . \tag{2.25}$$

Moreover, if $0 < x < b_1$, using (2.21) and (2.22)

$$\begin{aligned} u_2(x, 0) - u_1(x, 0) &= \int_{b_1}^{b_2} d\xi \int_\xi^{b_2} dy\,(h_2(y) + 1) + \int_x^{b_1} d\xi \Big(\int_\xi^{b_2} dy(h_2(y) + 1) - \int_\xi^{b_1} dy(h_1(y) + 1)\Big) \geqq \\ &\geqq \int_x^{b_1} d\xi \Big(\int_\xi^{b_1} dy(h_2(y) - h_1(y)) + \int_{b_1}^{b_2} dy(h_2(y) + 1)\Big) \geqq 0 . \end{aligned} \tag{2.26}$$

Suppose now that a $t_0 < \min\{T_1, T_{0,2}\}$ exists such that $s_1(t) < s_2(t)$ in $0 < t < t_0$ and $s_1(t_0) = s_2(t_0)$. Let $D_{t_0} = \{(x, t): 0 < x < s_1(t), 0 < t < t_0\}$ and consider $w(x, t) = u_2(x, t) - u_1(x, t)$ in D_{t_0}.

As $Lw = 0$ in D_{t_0} and $w_x(0, t) = \int_0^t (g_2(\tau) - g_1(\tau))\,d\tau - (q_2 - q_1) \leqq 0$, $w(x, t)$ assumes its minimum on

$$\{(x, t): 0 < x < b_1, t = 0\} \cup \{(x, t): x = s_1(t), 0 < t < t_0\}$$

and from (2.25) and (2.26) $w(x, t) > 0$ in D_{t_0}.

As $w(s_1(t_0), t_0) = 0$, $(s_1(t_0), t_0)$ is an isolated minimum for w in D_{t_0} and the parabolic Hopf's Lemma [1] gives $w_x(s(t_0), t_0) < 0$ contrary to the hypothesis that $s_1(t_0) = s_2(t_0)$.

Remark 2.11: Note that the lemma ensures also the monotonic behaviour of the solutions $u_j(x, t)$. On the contrary, no monotonic dependence holds for the functions z_j. Such a difference in the behaviour relates to the fact that the functions u_j, and not z_j, appear in real physical models [4, 6].

Proof of Theorem 2.9: It remains to prove that (A) implies that $\int_0^t g(\tau)\,d\tau < 1$, $t > 0$.

Suppose that T_0 exists such that $\int_0^{T_0} g(t)\,dt = 1$ and $s(T_0) > 0$, $\dot{s}(T_0) > -\infty$. Then either:

(a) $g(t) = 0$ for any $t > T_0$, or

(b) $g(t) \geqq 0$, $t > T_0$ and not identically zero. Recall that, from (2.11)

$$s(T_0) + \int_0^{s(T_0)} z(x, T_0)\,dx = 0\,. \tag{2.27}$$

In the case (a) we can apply Theorem 2.9 of [6] as the condition $z_x(x, t) \geqq 0$, see Lemma 2.1, ensures that the equation $z(x, T_0) = -1$ has a unique solution. It follows that (B) must occur at a time $T_B > T_0$.

Consider now the case (b): Let be $u(x, t) = \int_x^{s(t)} d\xi \int_\xi^{s(t)} dy\big(z(y, t) + 1\big)$ as in (2.23) and apply the maximum principle to $u_x(x, t)$ in $D_{T_0} = \{(x, t): 0 < x < s(t), 0 < t < T_0\}$. As $u_x(x, 0) = x - 1 \leqq 0$, we have

$$u_x(x, T_0) = -\int_x^{s(T_0)} dy\big(z(y, T_0) + 1\big) \leqq 0\,. \tag{2.28}$$

To simplify the notation, change the origin time letting $T_0 = 0$, so that $z(x, t)$ and $s(t)$ satisfy equations (2.18a) to (2.18f) with $b_i = s(T_0)$, $h_i(x) = z(x, T_0)$ and $g_i(t) = g(t + T_0)$.

Choose now a monotonically decreasing sequence b_n with $b_n > s(T_0)$, $\lim_n b_n = s(T_0)$ and let (T_n, s_n, z_n) be the solution of (2.18a)–(2.18f) corresponding to the data $b_i = b_n$, $g_i(t) = 0$, $h_i(x) = h_n(x)$ where

$$h_n(x) = \begin{cases} z(x, T_0)\,, & 0 < x < s(T_0) \\ 0\,, & s(T_0) < x < b_n\,. \end{cases}$$

Note that $b_n + \int_0^{b_n} h_n(x)\,dx > 0$ and the above mentioned Theorem 2.9 of [6] implies that $T_n = +\infty$. Moreover,

$$\int_x^{b_n} \big(h_n(y) + 1\big)\,dy = b_n - s(T_0) + \int_x^{s(T_0)} \big(z(y, T_0) + 1\big)\,dy > 0$$

because of (2.28).

We can now apply Lemma 2.10 with $z_1 = z$, $s_1 = s$ and $z_2 = z_n$, $s_2 = s_n$ so that

$$s(t) < s_n(t) \tag{2.29}$$

for any time t for which $s(t)$ exists.

As we have assumed that $\dot{s}(T_0) > -\infty$, $z(x, T_0)$ satisfies $|z(x, T_0)| \leqq K\big(s(T_0) - x\big)$. Then Theorem 5 of [5] applies and $s_n(t)$ tends (uniformely in t) to the free boundary $\bar{s}(t)$ of the solution of (2.18a)–(2.18f) with $b_i = s(T_0)$, $h_i(x) = z(x, T_0)$, $g_i(t) = 0$, for any time for which $\bar{s}(t)$ exists.

Like in the case (a), (2.27) implies that a T_B exists such that $\bar{s}(T_B) = 0$.

Performing the limit in (2.29), $s(t) \leqq \bar{s}(t)$, and $s(t)$ cannot exist for $t > T_B$.

Corollary 2.12: *Suppose that the solution of* (i)–(iv) *exists for arbitrarily large T and let*

$$D = \{(x, t): 0 < x < s(t), t > 0\}$$

then $z \in L^1(D)$ and $\|z\|_{L^1(D)} \leqq 2/3$.

Proof: From (2.14) and (2.8)

$$\int_0^t d\tau \int_0^{s(t)} |z(x, \tau)|\,dx \leq 1/6 - 1/2 \int_0^{s(t)} x^2 z(x, t)\,dx\,.$$

Moreover, from (2.11) and (2.8)

$$\int_0^{s(t)} -x^2 z(x, t)\,dx < \int_0^{s(t)} -z(x, t)\,dx \leqq \int_0^t g(\tau)\,d\tau$$

and, from Theorem 2.9, $\|z\|_{L^1(D)} \leqq 2/3$.

We conclude this section giving a criterium to find an upper bound for the maximal time of existence in the case that $g(t)$ is a monotone non decreasing function (not identically zero). Notice that in this case (A) cannot occur.

Proposition 2.13: *Let $g(t)$ be a monotone non decreasing function and (T, s, z) the corresponding solution of* (i)–(iv), *then every $t < T$ satisfies*

$$2g(t) \int_0^t g(\tau)\,d\tau \leqq 1 + g^2(t)\,. \tag{2.30}$$

Proof: As $g(t)$ is non decreasing, (2.10) gives

$$z(x, t) \geqq g(t)\,(x - 1)\,.$$

Then, using (2.11) we have

$$\frac{g(t)}{2}\, s^2(t) + \big(1 - g(t)\big)\, s(t) + \int_0^t g(\tau)\, d\tau - 1 \leqq 0 \tag{2.31}$$

for any $t < T$. (2.30) is the condition to have a real solution $s(t)$ of (2.31).

3. The case $g(t) = \text{const.}$

In this section we consider the case in which the flux $z_x(0, t)$ is constant in time, say $g(t) = K$.

As a trivial consequence of Theorem 2.9 no global solutions exist in this case, so that either (B) or (C) must occur. Moreover, one can easily prove that the solution, for a given K, exists for any $t' < 1/K$: see [8] (Theorem 2.9 could be used in this case choosing a function $g(t)$ such that $g(t) = K$, $t \leqq t'$ and $g(t) = 0$, $t > t'$).

In [8] it is also proved that $K < 1$ implies (B) (this follows immediately from (2.10) and (2.17)), and we assume $K > 1$ throughout this section.

All the estimates on the behaviour of $s(t)$ are based upon the following inequalities.

Lemma 3.1: *Let $z(x, t)$ be a solution of* (i)–(iv) *with $g(t) = K$, then*

$$z(x, t) \geqq -K/\sqrt{\pi} \int_0^t \frac{\exp\big(-x^2/(4(t-\tau))\big)}{(t-\tau)^{1/2}}\, d\tau , \tag{3.1}$$

$$z(x, t) \geqq -K\left((1 - x) - 2/\pi \cos\frac{\pi x}{2} \exp(-\pi^2 t/4)\right), \tag{3.2}$$

$$z(x, t) \geqq -K\left((1 - x) - 8/\pi^2 \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} \cos\frac{(2n+1)\pi x}{2} \exp\big(-(2n+1)^2 \pi^2 t/4\big)\right), \tag{3.3}$$

$$z(0, t) \geqq -K \min\{2\sqrt{t}/\sqrt{\pi},\ 1 - 8/\pi^2 \exp(-\pi^2 t/4)\} = \begin{cases} -K\, 2\sqrt{t}/\sqrt{\pi}, & t < t_0, \\ -K(1 - 8/\pi^2 \exp(-\pi^2 t/4)), & t > t_0 \end{cases} \quad t_0 \sim 0.213033 . \tag{3.4}$$

Proof: (3.1)–(3.3) follow from the maximum principle applied to $z(x, t) - w(x, t)$, where $w(x, t)$ is the solution of the heat equation in the following domains and with the following boundary conditions (b.c.):
for (3.1), $w(x, t)$ is the solution in the first quadrant with b.c. $w(x, 0) = 0$, $w_x(0, t) = K$;
for (3.2), $w(x, t)$ is the solution in $0 < x < 1$, $t > 0$ with b.c. $w(x, 0) = -K\left(1 - x - 2/\pi \cos\frac{\pi x}{2}\right) \leqq 0$, $w_x(0, t) = K$, $w(1, t) = 0$;
for (3.3), $w(x, t)$ is the solution in $0 < x < 1$, $t > 0$ with b.c. $w(x, 0) = 0$, $w_x(0, t) = K$, $w(1, t) = 0$ ([3], pag 102).

From (3.3) we have

$$z(0, t) \geqq -K\left(1 - 8/\pi^2 \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} \exp\big(-(2n+1)^2\pi^2 t/4\big)\right), \tag{3.5}$$

so that $z(0, t) \geqq -K\big(1 - 8/\pi^2 \exp(-\pi^2 t/4)\big)$, from which (3.4) follows by comparing with (3.1) in $x = 0$.

Remark 3.2: Obviously (3.3) is a sharper estimate than (3.1) or (3.2), but it is not suited for computation as, for $x > 0$, its terms are not defined in sign. Nevertheless, for $x = 0$, all the terms are positive, so one can obtain other estimates, like (3.4), for $z(0, t)$, keeping a finite number of terms in (3.5). For instance, if we keep the first two terms, we have

$$z(0, t) \geqq -K \min\{2\sqrt{t}/\sqrt{\pi};\ 1 - 8/\pi^2 \exp(-\pi^2 t/4) - 8/(9\pi^2) \exp(-9\pi^2 t/4)\} \tag{3.6}$$

the first function being lesser when $t < t_1 \sim 0.125122$.

As a consequence of Lemma 3.1, we have the following estimate on K for the case (C).

Proposition 3.3: *A K_1 exists such that $K > K_1$ implies* (C) *for the solution of* (i)–(iv). *The following estimate holds for K_1:*

$$K_1 < 2.221297 . \tag{3.7}$$

Proof: From Proposition 2.7 it follows that (B) implies that $T_B = 1/K$. Performing the limit $t \to 1/K$ in (2.12) we have

$$1/2 = -\int_0^{1/K} z(0, t)\, dt .$$

Using (3.4) we have

$$1/2 \leqq \int_0^{t_0} 2K\sqrt{t}/\sqrt{\pi}\, dt + \int_{t_0}^{1/K} K(1 - 8/\sqrt{\pi} \exp(-\pi^2 t/4)\, dt$$

which can be written as

$$\alpha K - 32/\pi^4 K \exp\left(-\pi^2/(4K)\right) \leqq 1/2 \tag{3.9}$$

where $\alpha = t_0 - 4t_0^{3/2}/(3\sqrt{\pi}) + 32/\pi^4 \exp(-\pi^2 t_0/4)$.

Letting $\xi = \pi^2/(4K)$, (3.9) becomes

$$\xi \geqq \pi^2\alpha/2 - 16/\pi^2 \exp(-\xi)\,. \tag{3.10}$$

(3.10) is satisfied for any $\xi > \xi_1$, where ξ_1 is the unique solution of the equality in (3.10), so that (3.9) holds for any $K \leqq K_1 = \pi^2/(4\xi_1)$. Using the numerical value for t_0 in (3.4) we obtain the estimate (3.7) for K_1.

Remark 3.4: A scarsely better estimate of the lower bound for K to have the case (C) can be obtained in the same way using (3.6) instead of (3.4). In this case the numerical value is 2.220008.

Remark 3.5: In [8] a first estimate was given using only (3.1) as lower bound for $z(x, t)$. In this way one has that (C) must occur if $K > 64/(9\pi) \sim 2.263537$.

Lemma 3.6: *Let $T_{\max}(K)$ be the greatest time for which the solution exists for a given K. Then the following estimates hold*:

$$T_{\max}(K) \leqq 1/2 + 1/(2K^2)\,, \qquad T_{\max}(K) \leqq T_K\,, \tag{3.11), (3.12}$$

where T_K is the unique solution of

$$\left(1 - K + 4K/\pi^2 \exp(-\pi^2 t/4)\right)^2 + 2(1 - Kt)\,K = 0\,. \tag{3.13}$$

Proof: (3.11) follows immediately from (2.30) letting $g(t) = K$.

Using the estimate (3.2), from (2.11) we obtain

$$s(t) \leqq 1 - Kt + K\left(s(t) - s^2(t)\right) - 4K/\pi^2 \exp(-\pi^2 t/4) \sin\frac{\pi s(t)}{2}\,.$$

As $0 < s(t) < 1$, then $\sin\frac{\pi s(t)}{2} \geqq s(t)$ and $s(t)$ must satisfy the inequality

$$K/2\, s^2(t) + \left(1 - K + 4K/\pi^2 \exp(-\pi^2 t/4)\right) s(t) + (Kt - 1) \leqq 0$$

which implies

$$\Delta(K, t) = \left(1 - K + 4K/\pi^2 \exp(-\pi^2 t/4)\right)^2 - 2K(Kt - 1) \geqq 0\,.$$

As $\Delta(K, 0) > 0$ for any K, $\frac{d}{dt}\Delta(K, t) < 0$ for any K and t, and $\lim_{t\to+\infty} \Delta(K, t) = -\infty$ for any K, the equation (3.13) has a unique solution T_K and $\Delta(K, t) \geqq 0$ for $t \leqq T_K$ for any K.

In the appendix, T_K and the value of $1/2 + 1/(2K^2)$ are confronted and an estimate for T_K is given.

Proposition 3.7: *Let K_2 be the solution of $K\left(1 - 8/\pi^2 \exp(-\pi^2/8(1 + 1/K^2))\right) = 1$. Then $K \leqq K_2$ implies* (B). *An estimate for K_2 is $K_2 > 1.091465$.*

Proof: From Lemma 2.3, if (C) holds, there exists a level curve $z(x, t) = -1$. As $z(x, t) > z(0, t)$, if $z(0, t) > -1$ for any t, $0 < t < T_{\max}(K)$, then the case (B) occurs.

From the estimates (3.4) and (3.11) for $z(0, t)$ and $T_{\max}$, respectively, we have

$$z(0, t) \geqq -F(K) = -K\left(1 - 8/\pi^2 \exp(-\pi^2/8(1 + 1/K^2))\right)$$

because the minorant of z in (3.4) is monotonically decreasing. As $F(0) = 0$, $F(+\infty) = +\infty$, and $F'(K) > 0$ we have $z(0, t) > -1$ for all the values of K lesser than K_2, where K_2 is the unique solution of $F(K) = 1$.

Remark 3.8: If we replace the estimate (3.4) with (3.6), we have no numerical advantage, the new term in (3.6) being approximatively $8/(9\pi^2) \exp(-9\pi^2/4) \sim 2.0\ 10^{-11}$ as $T_{\max} \sim 1$. On the contrary a little progress can be made, using the same method, using a sharper estimate for $T_{\max}$ (see appendix). The numerical value of the new estimate is $K_2 \sim 1.092283$.

Note, however, that one cannot expect a very good estimate for the case (B) only by looking at the values of $z(0, t)$. In fact, as a consequence of the motion of the free boundary towards $x = 0$, the function z is, in general, non monotone w.r.t. the variable t, so that a level curve $z(x, t) = -1$ can actually exist for the same value of K, which starts for the same $\bar{t}$ at $x = 0$, and comes back to $x = 0$, remaining, for any t, at a distance from the moving boundary greater than some constant, which implies the case (B).

We conclude giving some estimates for $s(t)$.

Proposition 3.9: *The following inequalities hold*:

$$s(t) \geqq \max\left\{(1 - 8(3\sqrt{\pi})\, Kt^{3/2})^{1/2},\ 1 - Kt,\ 0\right\} \tag{3.14}$$

where the greatest function is, respectively, the first one in $0 < t < \alpha_K$, the second one in $\alpha_K < t < 1/K$, and 0 elsewhere,

$$\alpha_K = \left(-4/(3K\sqrt{\pi}) + 1/K\left(16/(9\pi) + 2K\right)\right)^2\,,$$

$$s(t) \leqq 1 - Kt + K/2 - 16/\pi^3 K \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(2n+1)^3} \exp\left(-(2n+1))^2 \pi^2 t/4\right)\,. \tag{3.15}$$

Proof: From (2.11) we have immediately $s(t) \geqq 1 - Kt$; using (3.1) with $x = 0$, from (2.12) we get $s^2(t) \geqq \geqq 1 - 8/(3\sqrt{\pi})\,Kt^{3/2}$; α_K is the unique solution of the equation $1 - 8/(3\sqrt{\pi})\,Kt^{3/2} = (1 - Kt)^2$ in $0 < t < 1/K$. (3.15) follows immediately inserting (3.3) in (2.11) and integrating the series term by term in (0, 1).

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Appendix

Here we give some estimates for the time T_K, which appears in (3.12). First of all, we rewrite (3.13) as

$$g_1^2(K, t) = g_2(K, t)\,, \qquad t > 1/K\,, \tag{A.1}$$

where

$$g_1(K, t) = K(1 - 4/\pi^2 \exp(-\pi^2 t/4)) - 1\,, \qquad g_2(K, t) = 2K^2(t - 1/K)\,. \tag{A.2}$$

The functions g_1 and g_2 have the following **properties**:

- *g_1 and g_2 are increasing function of both the arguments for $K > 0$ and $t > 1/K$;*
- $g_1(K, 0) \geqq 0$ *iff* $K \geqq \pi^2/(\pi^2 - 4)$;
- $g_1(K, t) \leqq 0$ *iff* $t \leqq 4/\pi^2(\log(4K) - \log(\pi^2(K - 1))) = \tilde{t}(K)$;
- $0 < \tilde{t}(K) < 1/K$ *iff* $\widetilde{K} < K < \pi^2/(\pi^2 - 4)$
  where $\widetilde{K}$ is the unique solution of $\tilde{t}(K) = 1/K$ with $K > 1$, $\widetilde{K} \sim 1.039$;
- $g_1(K, +\infty) = K - 1, \quad g_2(K, 1/K) = 0\,, \quad g_2(K, 1/2 + 1/(2K^2)) = K - 1\,.$

From the above properties it follows that, for $K > \widetilde{K}$,

$$T_K < 1/2 + 1/(2K^2)\,,$$

and that T_K can be approximated by the following iterative sequence

$$1/2 + 1/(2K^2) > t_1(K) > \dots > t_n(K) > t_{n+1}(K) \to T_K$$

for any $K > \widetilde{K}$, where $t_{n+1}(K) = 1/K + 1/(2K^2)\, g_1^2(K, t_n)$.
The estimate for K_2 in Remark 3.8 uses $t_1(K)$ as upper bound for $T_{\max}$.