



SIMULTANEOUS DETERMINATION OF TWO UNKNOWN THERMAL COEFFICIENTS THROUGH A MUSHY ZONE MODEL WITH AN OVERSPECIFIED CONVECTIVE BOUNDARY CONDITION

Andrea N. Ceretani^{1,2} and Domingo A. Tarzia¹

¹CONICET - Departamento de Matemática

Facultad de Ciencias Empresariales

Universidad Austral

Paraguay 1950

S2000FZF Rosario, Argentina

e-mail: aceretani@austral.edu.ar

dtarzia@austral.edu.ar

²Departamento de Matemática

Facultad de Ciencias Exactas, Ingeniería y Agrimensura

Universidad Nacional de Rosario

Pellegrini 250

S2000BTP Rosario, Argentina

Abstract

We have considered the simultaneous determination of two unknown thermal coefficients for a semi-infinite material which was under a phase-change process with a mushy zone according to the model of Solomon, Wilson and Alexiades. It was assumed that the material was initially liquid at its melting temperature and that the solidification process began when a heat flux was imposed at the fixed face. The

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associated free boundary value problem was overspecified with a convective boundary condition aiming at the simultaneous determination of the temperature in the solid region, one of the two free boundaries of the mushy zone and two thermal coefficients. These were chosen among the latent heat by unit mass, the thermal conductivity, the mass density, the specific heat and the two coefficients that characterize the mushy zone. It was assumed that the other free boundary of the mushy zone, the bulk temperature, the heat flux and heat transfer coefficients at the fixed face were known. Depending on the choice of the unknown thermal coefficients, fifteen phase-change problems arose.

In this paper, we present those fifteen problems and obtain necessary and sufficient conditions on data for each of them in order to obtain their solutions. We show that there are twelve cases in which it is possible to find a unique solution and that there are infinite solutions for the remaining three cases. Moreover, for each problem, we give explicit formulae for the temperature of the material, the unknown free boundary and the two unknown thermal coefficients.

1. Introduction

Heat transfer problems with a phase-change such as melting and freezing have been studied in the last century in view of their wide scientific and technological applications [1, 2, 4, 12]. Especially, inverse problems related to the determination of thermal coefficients have attracted many scientists because they are often ill-posed problems [3, 6-11, 13-15, 17-19, 22].

In our recent work [5], we studied the determination of one unknown thermal coefficient for a semi-infinite material which is under a solidification process ensued from a heat flux imposed at the fixed boundary. In that work, we overspecified the associated free boundary value problem by a convective boundary condition [21] aiming at the simultaneous determination of the temperature in the solid region, the two free boundaries of the mushy zone and one unknown thermal coefficient. In this paper, we study the same physical phenomenon with two unknown thermal coefficients and some additional information given on it.

In the following, we restate the main characteristics of the phase-change process considered in [5, 16]. The material is assumed to be initially liquid at a melting temperature of 0°C and it is considered the existence of the following three different regions in the solidification process, according to the model of Solomon et al. [16, 19]:

1. liquid region at temperature $T(x, t) = 0$:

$$D_l = \{(x, t) \in \mathbb{R}^2 / x > r(t), t > 0\},$$

2. solid region at temperature $T(x, t) < 0$:

$$D_s = \{(x, t) \in \mathbb{R}^2 / 0 < x < s(t), t > 0\},$$

3. mushy region at temperature $T(x, t) = 0$:

$$D_p = \{(x, t) \in \mathbb{R}^2 / s(t) < x < r(t), t > 0\},$$

where $x = s(t)$ and $x = r(t)$ represent the free boundaries of the mushy zone and $T = T(x, t)$ represents the temperature of the material. The mushy zone is considered as isothermal and the following assumptions on its structure are made:

1. the material contains a fixed portion of the total latent heat per unit mass (see condition (4) below),
2. its width is inversely proportional to the gradient of temperature (see condition (5) below).

Finally, we note that all of the thermal coefficients involved in the solidification process are assumed to be constant, where the bulk temperature $-D_\infty < 0$ and the coefficients $q_0 > 0$ and $h_0 > 0$ that characterize the heat flux and the heat transfer at the fixed face, respectively, are assumed to be known.

In this paper, we consider that we also know the evolution in time of one of the two free boundaries of the mushy zone. More precisely, we assume

that the free boundary $x = s(t)$ is given by:

$$s(t) = 2\sigma\sqrt{t}, \quad t > 0, \quad (1)$$

where $\sigma > 0$ is a known coefficient. Thanks to this additional information on the physical phenomenon, we will be able to determine simultaneously the temperature $T = T(x, t)$ in the solid region, the free boundary $x = r(t)$ and two unknown thermal coefficients among the latent heat by unit mass $l > 0$, the thermal conductivity $k > 0$, the mass density $\rho > 0$, the specific heat $c > 0$ and the two coefficients $0 < \varepsilon < 1$ and $\gamma > 0$ that characterize the mushy zone, by solving the following overspecified free boundary value problem:

$$\rho c T_t(x, t) - k T_{xx}(x, t) = 0, \quad 0 < x < s(t), \quad t > 0, \quad (2)$$

$$T(s(t), t) = 0, \quad t > 0, \quad (3)$$

$$k T_x(s(t), t) = \rho l [\varepsilon \dot{s}(t) + (1 - \varepsilon) \dot{r}(t)], \quad t > 0, \quad (4)$$

$$T_x(s(t), t)(r(t) - s(t)) = \gamma, \quad t > 0, \quad (5)$$

$$r(0) = 0, \quad (6)$$

$$k T_x(0, t) = \frac{q_0}{\sqrt{t}}, \quad t > 0, \quad (7)$$

$$k T_x(0, t) = \frac{h_0}{\sqrt{t}} (T(0, t) + D_\infty), \quad t > 0. \quad (8)$$

Since inverse Stefan problems are usually ill-posed problems, it is expected that restrictions on data have to be set in order to obtain solutions to problem (2)-(8). The goal of this paper is to obtain necessary and sufficient conditions on data, under which solutions can be obtained, for the fifteen phase-change problems (2)-(8) that arise depending on the choice of the unknown thermal coefficients. Moreover, we also expect to obtain those solutions explicitly.

The organization of the paper is as follows: first (Section 2) we proved a preliminary result in which necessary and sufficient conditions on data for

the phase-change process (2)-(8) are given in order to obtain the temperature $T = T(x, t)$ and the unknown free boundary $x = r(t)$. Then (Section 3), based on this preliminary result, we presented and solved the fifteen different cases for the phase-change process (2)-(8) corresponding to each possible choice of the two unknown thermal coefficients among $l, k, \rho, c, \varepsilon$ and γ . Under certain restrictions on data (Ri 's inequalities), we proved that there are twelve cases in which it is possible to find a unique explicit solution which depends on a dimensionless parameter defined as the unique solution of a certain equation (Ei 's equations) and that there are infinite explicit solutions for the remaining three cases.

2. Explicit Solution to the Phase-change Process

The following theorem represents the base on which we will prove the subsequent results.

Theorem 2.1. *The solution to problem (2)-(8) is given by:*

$$T(x, t) = -\frac{q_0 \sqrt{\alpha t}}{k} \operatorname{erf}\left(\frac{\sigma}{\sqrt{\alpha}}\right) \left[1 - \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)}{\operatorname{erf}\left(\frac{\sigma}{\sqrt{\alpha}}\right)} \right], \quad 0 < x < s(t), \quad t > 0, \quad (9)$$

$$r(t) = \left[\frac{\gamma k \exp(\sigma^2/\alpha)}{q_0} + 2\sigma \right] \sqrt{t}, \quad t > 0 \quad (10)$$

if and only if the physical parameters satisfy the following two equations:

$$\frac{q_0}{\rho l} = \left[\sigma + \frac{\gamma k (1 - \varepsilon) \exp(\sigma^2/\alpha)}{2q_0} \right] \exp(\sigma^2/\alpha), \quad (11)$$

$$\operatorname{erf}\left(\frac{\sigma}{\sqrt{\alpha}}\right) = \frac{k D_\infty}{q_0 \sqrt{\alpha \pi}} \left(1 - \frac{q_0}{h_0 D_\infty} \right), \quad (12)$$

where the coefficient α , defined by:

$$\alpha = \frac{k}{\rho c}, \quad (13)$$

is the thermal diffusivity.

Proof. The free boundary value problem (2)-(8) has the solution [16, 19, 20]:

$$T(x, t) = A + B \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right), \quad 0 < x < s(t), \quad t > 0, \quad (14)$$

$$r(t) = 2\mu\sqrt{\alpha t}, \quad t > 0, \quad (15)$$

where coefficients A , B and μ have to be found.

By imposing conditions (3)-(5), (7) and (8) on (14)-(15), we have that:

$$A = -\frac{q_0\sqrt{\alpha\pi}}{k} \operatorname{erf}\left(\frac{\sigma}{\sqrt{\alpha}}\right), \quad B = \frac{q_0\sqrt{\alpha\pi}}{k} \quad \text{and} \quad \mu = \frac{\gamma k \exp(\sigma^2/\alpha)}{2q_0\sqrt{\alpha}} + \frac{\sigma}{\sqrt{\alpha}} \quad (16)$$

which corresponds to solution (9)-(10) and that the physical parameters must satisfy equations (11) and (12). \square

Hence, from Theorem 2.1, we have that there is an equivalence between solving the free boundary value problem (2)-(8) with two unknown thermal coefficients or solving the system of equations (11)-(12) for the same two unknown coefficients.

3. Explicit Formulae for the Unknown Thermal Coefficients

In this section, we are presenting and solving the fifteen different cases for the phase-change process (2)-(8) that arise depending on the choice of the two unknown thermal coefficients. With the aim of organizing our work, we have identified each problem by making reference to the coefficients which is necessary to know in order to solve it (see Theorem 2.1):

Case 1. Determination of ε and γ .

Case 2. Determination of ε and l .

Case 3. Determination of γ and l .

Case 4. Determination of ε and k .

Case 5. Determination of ε and ρ .

Case 6. Determination of ε and c .

Case 7. Determination of γ and k .

Case 8. Determination γ and ρ .

Case 9. Determination of γ and c .

Case 10. Determination of l and k .

Case 11. Determination of l and ρ .

Case 12. Determination of l and c .

Case 13. Determination of k and ρ .

Case 14. Determination of k and c

and

Case 15. Determination of ρ and c .

Moreover, we have introduced several functions and parameters which were labeled with an index according to the number of the cases where they arise for the first time.

Theorem 3.1 (Case 1: determination of ε and γ). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients ε and γ , then it has infinite solutions given by (9)-(10) with:*

$$\gamma = \frac{2q_0\sigma}{k(1-\varepsilon)} \left(\frac{q_0}{\rho l \sigma} \exp(-\sigma^2/\alpha) - 1 \right) \exp(-\sigma^2/\alpha) \quad (17)$$

and any $\varepsilon \in (0, 1)$ if and only if the remaining physical parameters satisfy condition (12) and the following inequality:

$$0 < \frac{q_0}{\rho l \sigma} \exp(-\sigma^2/\alpha) - 1. \quad (R1)$$

Proof. Owing to Theorem 2.1, we have that the phase-change process (2)-(8) has the solution given by (9)-(10) if and only if ε and γ satisfy

equation (11) and the remaining physical parameters satisfy condition (12). Then we have from equation (11) that γ must be given by (17) for any $\varepsilon \in (0, 1)$. To finish the proof, it only remains to observe that the coefficient γ given by (17) is positive if and only if inequality (R1) holds. \square

Theorem 3.2 (Case 2: determination of ε and l). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients ε and l , then it has infinite solutions given by (9)-(10) with:*

$$l = \frac{q_0 \exp(-\sigma^2/\alpha)}{\rho\sigma \left[1 + \frac{\gamma k(1-\varepsilon)}{2q_0\sigma} \exp(\sigma^2/\alpha) \right]} \quad (18)$$

and any $\varepsilon \in (0, 1)$ if and only if the physical parameters h_0 , q_0 , D_∞ , σ , ρ , c and k satisfy condition (12).

Proof. It is similar to the proof of Theorem 3.1. \square

Theorem 3.3 (Case 3: determination of γ and l). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients γ and l , then it has infinite solutions given by (9)-(10) with any $\gamma > 0$ and l given by (18) if and only if the parameters h_0 , q_0 , D_∞ , σ , ρ , c and k satisfy condition (12).*

Proof. It is similar to the proof of Theorem 3.1. \square

Remark 1. Let us observe that it follows from the previous three theorems that, under certain conditions for the data of the problem, the phase-change process (2)-(8) corresponding to Cases 1, 2 and 3 has an infinite number of solutions.

Nevertheless, as we will see in the following, for the remaining cases it is possible to obtain necessary and sufficient conditions on data in order to obtain existence and uniqueness of solution. In each case, the solution depends on a dimensionless coefficient ξ which is defined as the unique positive solution to a certain equation. According to the notation used in this paper, equations for the coefficient ξ were labeled by making reference to the

case in which they arise for the first time (*Ei*'s equations). For this reason equations for ξ are not labeled in consecutive manner.

Theorem 3.4 (Case 4: determination of ε and k). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients ε and k , then it has the solution given by (9)-(10) with:*

$$\varepsilon = 1 - f_4(\xi), \quad (19)$$

$$k = \rho c \left(\frac{\sigma}{\xi} \right)^2, \quad (20)$$

where ξ is the only positive solution to the equation:

$$g_4(x) = \frac{\sigma \rho c D_\infty}{q_0 \sqrt{\pi}} \left(1 - \frac{q_0}{h_0 D_\infty} \right) \quad (E4)$$

and the real functions f_4 and g_4 are defined by:

$$f_4(x) = \frac{2q_0}{\gamma \rho c \sigma} \left(\frac{q_0}{\rho l \sigma} \exp(-x^2) - 1 \right) x^2 \exp(-x^2) \text{ and} \\ g_4(x) = x \operatorname{erf}(x), \quad x > 0 \quad (21)$$

if and only if the remaining physical parameters satisfy the next three inequalities:

$$0 < 1 - \frac{q_0}{h_0 D_\infty}, \quad (R2)$$

$$0 < 1 - \frac{q_0}{\rho l \sigma}, \quad (R3)$$

$$1 - \frac{q_0}{h_0 D_\infty} < \frac{q_0 \sqrt{\pi}}{\sigma \rho c D_\infty} g_4 \left(\sqrt{\ln \left(\frac{q_0}{\rho l \sigma} \right)} \right) \quad (R4)$$

and any of the following three groups of conditions:

Group 1.

$$f_4(\eta) > 1, \quad (R5)$$

$$1 - \frac{q_0}{h_0 D_\infty} < \frac{q_0 \sqrt{\pi}}{\sigma \rho c D_\infty} g_4(\zeta_1) \quad \text{or} \quad 1 - \frac{q_0}{h_0 D_\infty} > \frac{q_0 \sqrt{\pi}}{\sigma \rho c D_\infty} g_4(\zeta_2), \quad (\text{R6})$$

where ζ_1 and ζ_2 are the only two positive solutions to the equation:

$$f_4(x) = 1 \quad (22)$$

and η is the only positive solution to the equation:

$$\frac{q_0}{\rho l \sigma} (1 - 2x^2) = (1 - x^2) \exp(x^2). \quad (23)$$

Group 2.

$$f_4(\eta) = 1, \quad (\text{R7})$$

$$1 - \frac{q_0}{h_0 D_\infty} \neq \frac{q_0 \sqrt{\pi}}{\sigma \rho c D_\infty} g_4(\eta), \quad (\text{R8})$$

where η is the only positive solution to equation (23).

Group 3.

$$f_4(\eta) < 1, \quad (\text{R9})$$

where η is the only positive solution to equation (23).

Proof. We have from Theorem 2.1 that the phase-change process (2)-(8) has the solution given by (9)-(10) if and only if ε and k satisfy equations (11) and (12). By introducing the following dimensionless parameter:

$$\xi = \frac{\sigma}{\sqrt{\alpha}} = \sigma \sqrt{\frac{\rho c}{k}}, \quad (24)$$

we have that the solution of the system of equations (11)-(12) is given by (19)-(20) if and only if ξ is a solution to equation (E4). Then we need to prove that the restrictions on data given in the statement of the theorem are necessary and sufficient conditions for the existence of a positive solution to equation (E4) and for obtaining that the coefficient ξ given in (19) is a number between 0 and 1.

We first note that equation (E4) admits a positive solution if and only if inequality (R2) holds, because g_4 is an increasing function from 0 to $+\infty$ in \mathbb{R}^+ . Henceforth, we will assume that (R2) holds.

Let us now focus on the fact that $\varepsilon \in (0, 1)$. On one hand, we have that ε is less than 1 if and only if (see (19)):

$$0 < \frac{q_0}{\rho l \sigma} \exp(-\xi^2) - 1$$

which is equivalent to inequality (R3) and:

$$\xi < \log\left(\frac{q_0}{\rho l \sigma}\right). \quad (25)$$

Since g_4 is an increasing function and ξ satisfies equation (E4), by applying function g_4 side by side of inequality (25), we have that it is equivalent to inequality (R4). Therefore, from now on, we will assume that inequalities (R3) and (R4) also hold.

On the other hand, we have that ε is positive if and only if (see (19)):

$$f_4(\xi) < 1. \quad (26)$$

It is easy to prove that f_4 has a finite maximum M in \mathbb{R}^+ and that $M = f(\eta) > 0$. In the following, we will study three different situations: $f(\eta) > 1$, $f(\eta) = 1$ and $0 < f(\eta) < 1$ which are related to the conditions given in Groups 1, 2 and 3, respectively.

If $f(\eta) > 1$, that is if inequality (R5) holds, then we have that ξ satisfies inequality (26) if and only if:

$$\xi < \zeta_1 \quad \text{or} \quad \xi > \zeta_2, \quad (27)$$

where ζ_1 and ζ_2 are the only two positive solutions to equation (22). By applying the increasing function g_4 side by side to both inequalities and taking into account that ξ satisfies equation (E4), it follows that (27) is equivalent to (R6).

If $f(\eta) = 1$, that is if (R7) holds, then we have that ξ satisfies inequality (26) if and only if $\xi \neq \eta$. We now proceed as in the previous situation and obtain that $\xi \neq \eta$ is equivalent to (R8).

Finally, if $0 < f(\eta) < 1$, that is if (R9) holds, then we have that inequality (26) holds immediately. \square

The previous Theorem 3.4 states necessary and sufficient conditions on data for problem (2)-(8) under which it is possible to find the temperature $T = T(x, t)$, the free boundary $x = r(t)$ and the two unknown thermal coefficients ε and k . There are also some sufficient conditions on data which are easier to check than the necessary and sufficient conditions given in Theorem 3.3 that enable us to find the solution to problem (2)-(8). Next, proposition is related to those sufficient conditions.

Proposition 3.1 (Sufficient conditions for Case 4). *Let us consider the phase-change process (2)-(8) with unknown thermal coefficients ε and k . If the remaining physical parameters satisfy inequality (R3) and the following three conditions:*

$$\frac{q_0 \sqrt{\pi}}{\sigma \rho c D_\infty} g_4\left(\frac{1}{v_4}\right) < 1 - \frac{q_0}{h_0 D_\infty} < \frac{q_0 \sqrt{\pi}}{\sigma \rho c D_\infty} g_4\left(\ln\left(\frac{q_0}{\rho l \sigma}\right)\right), \quad (\text{R10})$$

$$0 < \frac{2q_0}{\rho \gamma c \sigma} \ln\left(\frac{q_0}{\rho l \sigma}\right) \left(\frac{q_0}{\rho l \sigma} - 1\right) - 1, \quad (\text{R11})$$

where

$$v_4 = \frac{\rho l \sigma}{2q_0} \ln\left(\frac{q_0}{\rho l \sigma}\right) \left[1 + \sqrt{1 + \frac{2\gamma c}{l \ln\left(\frac{q_0}{\rho l \sigma}\right)}} \right], \quad (28)$$

then the solution to problem (2)-(8) is given by (9)-(10) with ε and k given by (19) and (20), being ξ the only positive solution to equation (E4).

Proof. Let us assume that inequalities (R3), (R10) and (R11) hold. We have seen in the proof of Theorem 3.4 that ε and k must be given by (19) and

(20) and that ξ must satisfy equation (E4). We have also seen that (R2) is a necessary and sufficient condition for the existence and uniqueness of the solution to equation (E4). Then, since the first inequality in (R10) implies (R2), we have that equation (E4) admits only one positive solution. Moreover, we have seen that ε given in (19) is less than 1 if and only if inequalities (R3) and (R4) hold. Since inequality (R10) implies inequalities (R3) and (R4), we have that the coefficient ε given by (19) is less than 1. Finally, we have also seen that the coefficient ε given by (19) is positive if and only if inequality (26) holds. The rest of the proof will be devoted to demonstrate that the first inequality in (R10) and (R11) imply inequality (26).

Since the coefficient ε given by (19) is positive, we have that $\frac{q_0}{\rho l \sigma} \exp(-\xi^2) - 1 > 0$, that is,

$$\xi^2 < \ln\left(\frac{q_0}{\rho l \sigma}\right).$$

Then we have (see (21)):

$$f_4(\xi) < \frac{2q_0}{\gamma \rho c \sigma} \left(\frac{q_0}{\rho l \sigma} \exp(-\xi^2) - 1 \right) \ln\left(\frac{q_0}{\rho l \sigma}\right) \exp(-\xi^2). \quad (29)$$

From the above analysis, it follows that it is enough to prove that the first inequality in (R10) and (R11) imply that the right hand side of (29) is less than 1. Let w_4 be the function defined by:

$$w_4(x) = a_4 x^2 - b_4 x - 1, \quad x > 0 \quad (30)$$

with a_4 and b_4 given by:

$$a_4 = \frac{2q_0^2}{\rho^2 \sigma^2 l \gamma c} \ln\left(\frac{q_0}{\rho l \sigma}\right) > 0 \quad \text{and} \quad b_4 = \frac{2q_0}{\rho \sigma \gamma c} \ln\left(\frac{q_0}{\rho l \sigma}\right) > 0. \quad (31)$$

We have that v_4 given in (28) is a positive root of w_4 . Moreover, we have that inequalities (R3) and (R11) imply $v_4 < 1$. Since the other root of w_4 is

negative, we have that the right hand side of (29) is less than 1 if and only if $\exp(-\xi^2) < v_4$, that is if and only if:

$$\xi > \sqrt{\ln\left(\frac{1}{v_4}\right)}.$$

Only remains to observe that this last inequality is equivalent to the first inequality in (R10) because g_4 is an increasing function and ξ satisfies equation (E4). \square

Theorem 3.5 (Case 5: determination of ε and ρ). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients ε and ρ , then it has the solution given by (9)-(10) with:*

$$\varepsilon = 1 - f_5(\xi), \quad (32)$$

$$\rho = \frac{k}{c} \left(\frac{\xi}{\sigma} \right)^2, \quad (33)$$

where ξ is the only positive solution to the equation:

$$g_5(x) = \frac{kD_\infty}{q_0\sigma\sqrt{\pi}} \left(1 - \frac{q_0}{h_0D_\infty} \right) \quad (E5)$$

and the real functions f_5 and g_5 are defined by:

$$f_5(x) = \frac{2q_0\sigma}{\gamma k} \left(\frac{q_0 c \sigma}{lk} \frac{\exp(-x^2)}{x^2} - 1 \right) \exp(-x^2) \text{ and}$$

$$g_5(x) = \frac{\operatorname{erf}(x)}{x}, \quad x > 0 \quad (34)$$

if and only if the remaining physical parameters satisfy the following two conditions:

$$\frac{q_0\sigma\sqrt{\pi}}{kD_\infty} g_5(\zeta_1) < 1 - \frac{q_0}{h_0D_\infty} < \min \left\{ \frac{2q_0\sigma}{kD_\infty}, \frac{q_0\sigma\sqrt{\pi}}{kD_\infty} g_5(\zeta_2) \right\}, \quad (R12)$$

where ζ_1 and ζ_2 are, respectively, the only positive solutions to equations:

$$\frac{q_0 \sigma c}{lk} \exp(-x^2) = x^2, \quad (35)$$

$$\frac{q_0 \sigma c}{lk} \exp(-x^2) = \left[\frac{\gamma k}{2q_0 \sigma} \exp(x^2) + 1 \right] x^2. \quad (36)$$

Proof. We have from Theorem 2.1 that the phase-change process (2)-(8) has the solution given by (9)-(10) if and only if ε and p are given by (32) and (33), where the dimensionless parameter ξ (see (24)) is a solution to equation (E5). Then we need to prove that the two restrictions given by (R12) are necessary and sufficient conditions for the existence of a positive solution to equation (E5) and for obtaining that the coefficient ε given by (32) is a number between 0 and 1.

Since g_5 is a decreasing function from $\frac{2}{\sqrt{\pi}}$ to 0 in \mathbb{R}^+ , we have that equation (E5) admits positive solutions if and only if:

$$0 < 1 - \frac{q_0}{h_0 D_\infty} < \frac{2q_0 \sigma}{k D_\infty}. \quad (37)$$

Let us assume for a moment that (37) holds and focus on the fact that $\varepsilon \in (0, 1)$. It is easy to see that f_5 has a finite minimum m and that $m = f(\eta) < 0$. Then we have that the coefficient ε given by (32) is a number between 0 and 1 if and only if $\zeta_1 < \xi < \zeta_2$ with ζ_1 and ζ_2 the only two positive numbers such that $f_5(\zeta_1) = 0$ and $f_5(\zeta_2) = 1$, that is the only positive solutions to equations (35) and (36), respectively. Since g_4 is a decreasing function and ξ satisfies equation (E5), we have that $\zeta_1 < \xi < \zeta_2$ is equivalent to:

$$\frac{q_0 \sigma \sqrt{\pi}}{k D_\infty} g_5(\zeta_1) < 1 - \frac{q_0}{h_0 D_\infty} < \frac{q_0 \sigma \sqrt{\pi}}{k D_\infty} g_5(\zeta_2). \quad (38)$$

Only remains to observe that inequalities (37) and (38) are equivalent to the inequalities given by (R12). \square

Theorem 3.6 (Case 6: determination of ε and c). *If we consider the*

phase-change process (2)-(8) with unknown thermal coefficients ε and c , then it has the solution given by (9)-(10) with:

$$\varepsilon = 1 - f_6(\xi), \quad (39)$$

$$c = \frac{k}{\rho} \left(\frac{\xi}{\sigma} \right)^2, \quad (40)$$

where ξ is the only positive solution to equation (E5) and f_6 is the real function defined by:

$$f_6(x) = \frac{2q_0\sigma}{\gamma k} \left(\frac{q_0}{\rho l \sigma} \exp(-x^2) \right) \exp(-x^2), \quad x > 0 \quad (41)$$

if and only if the remaining physical parameters satisfy the following condition:

$$\frac{q_0\sigma\sqrt{\pi}}{kD_\infty} f_5\left(\sqrt{\ln\left(\frac{q_0}{\rho l \sigma}\right)}\right) < 1 - \frac{q_0}{h_0 D_\infty} \quad (R13)$$

and any of the following two groups of conditions:

Group 1.

$$\frac{q_0}{\rho l \sigma} \geq \frac{\gamma k}{2q_0\sigma} + 1, \quad (R14)$$

$$1 - \frac{q_0}{h_0 D_\infty} < \min\left\{\frac{2q_0\sigma}{kD_\infty}, \frac{q_0\sigma\sqrt{\pi}}{kD_\infty} f_5\left(\sqrt{\ln\left(\frac{1}{v_6}\right)}\right)\right\}, \quad (R15)$$

where

$$v_6 = \frac{\rho l \sigma}{2q_0} \left[1 + \sqrt{1 + \frac{2\gamma k}{\sigma^2 \rho l}} \right]. \quad (42)$$

Group 2.

$$1 < \frac{q_0}{\rho l \sigma} < \frac{\gamma k}{2q_0\sigma} + 1, \quad (R16)$$

$$1 - \frac{q_0}{h_0 D_\infty} < \frac{2q_0\sigma}{kD_\infty}. \quad (R17)$$

Proof. It is similar to the proof of Theorem 3.4. \square

Theorem 3.7 (Case 7: determination of γ and k). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients γ and k , then it has the solution given by (9)-(10) with:*

$$\gamma = \frac{2q_0}{\sigma \rho c(1-\varepsilon)} \left(\frac{q_0}{\rho l \sigma} \exp(-\xi^2) - 1 \right) \xi^2 \exp(-\xi^2) \quad (43)$$

and k given by (20), where ξ is the only positive solution to equation (E4) if and only if the physical parameters h_0 , q_0 , D_∞ , σ , l , ρ and c satisfy conditions (R2), (R3) and (R4).

Proof. We have from Theorem 2.1 that γ and k must be given by (43) and (20), where the dimensionless parameter ξ (see (24)) is a solution to equation (E4). As we saw in the proof of Theorem 3.4, equation (E4) admits positive solutions if and only if inequality (R2) holds.

To complete the proof, it only remains to observe that the coefficient γ given by (43) is positive if and only if $0 < \frac{q_0}{\rho l \sigma} \exp(-\xi^2) - 1$ and as we also saw in the proof of Theorem 3.4 that this inequality is equivalent to inequalities (R3) and (R4). \square

Theorem 3.8 (Case 8: determination of γ and ρ). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients γ and ρ , then it has the solution given by (9)-(10) with:*

$$\gamma = \frac{2q_0\sigma}{k(1-\varepsilon)} \left(\frac{q_0 c \sigma}{lk} \frac{\exp(-\xi^2)}{\xi} - 1 \right) \exp(-\xi^2) \quad (44)$$

and ρ given by (33), where ξ is the only positive solution to equation (E5) if and only if the physical parameters h_0 , q_0 , D_∞ , σ , k and c satisfy conditions (R2), (R17) and:

$$g_5(\eta) < \frac{kD_\infty}{q_0\sigma\sqrt{\pi}} \left(1 - \frac{q_0}{h_0D_\infty} \right), \quad (R18)$$

where g_5 is defined in (34) and η is the only positive solution to the equation:

$$\frac{q_0 c \sigma}{lk} \frac{\exp(-x^2)}{x} = 1. \quad (45)$$

Proof. We have from Theorem 2.1 that γ and ρ must be given by (44) and (33), where the dimensionless parameter ξ (see (24)) is a solution to equation (E5). As we saw in the proof of Theorem 3.5, equation (E5) admits a positive solution if and only if inequality (37) holds, that is if and only if inequalities (R2) and (R17) hold.

We also have that the coefficient γ given by (44) is positive if and only if:

$$u_8(\xi) > 0, \quad (46)$$

where u_8 is the real function defined by:

$$u_8(x) = \frac{q_0 c \sigma}{lk} \frac{\exp(-x^2)}{x} - 1, \quad x > 0. \quad (47)$$

Since u_8 is a decreasing function from $+\infty$ to -1 in \mathbb{R}^+ , we have that equation (45) has only one positive solution η . Therefore, inequality (46) holds if and only if $\xi < \eta$. It only remains to observe that $\xi < \eta$ is equivalent to condition (R18) because g_5 is a decreasing function and ξ satisfies equation (E5). \square

Theorem 3.9 (Case 9: determination of γ and c). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients γ and c , then it has the solution given by (9)-(10) with:*

$$\gamma = \frac{2q_0\sigma}{k(1-\varepsilon)} \left(\frac{q_0}{\rho l \sigma} \exp(-\xi^2) - 1 \right) \exp(-\xi^2) \quad (48)$$

and c given by (40), where ξ is the only positive solution to equation (E5) if and only if the physical parameters h_0 , q_0 , D_∞ , σ , l , k and ρ satisfy conditions (R3), (R13) and (R17).

Proof. It is similar to the proof of Theorem 3.8. \square

Theorem 3.10 (Case 10: determination of l and k). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients l and k , then it has the solution given by (9)-(10) with:*

$$l = \frac{q_0}{\rho\sigma} \left[\frac{1}{1 + \frac{\rho c \sigma \gamma (1 - \varepsilon)}{2q_0}} \frac{\exp(\xi^2)}{\xi^2} \right] \exp(-\xi^2) \quad (49)$$

and k given by (20), where ξ is the only positive solution to equation (E4) if and only if the physical parameters h_0 , q_0 and D_∞ satisfy condition (R2).

Proof. It is similar to the proof of Theorem 3.7. \square

Theorem 3.11 (Case 11: determination of l and ρ). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients l and ρ , then it has the solution given by (9)-(10) with:*

$$l = \frac{q_0 c \sigma}{k} \left[\frac{1}{1 + \frac{\gamma k (1 - \varepsilon)}{2q_0 \sigma} \exp(\xi^2)} \right] \frac{\exp(-\xi^2)}{\xi} \quad (50)$$

and ρ given by (33), where ξ is the only positive solution to equation (E5) if and only if the physical parameters h_0 , q_0 , D_∞ and k satisfy conditions (R2) and (R17).

Proof. It is similar to the proof of Theorem 3.8. \square

Theorem 3.12 (Case 12: determination of l and c). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients l and c , then it has the solution given by (9)-(10) with:*

$$l = \frac{q_0}{\rho\sigma} \left[\frac{1}{1 + \frac{\gamma k (1 - \varepsilon)}{2q_0 \sigma} \exp(\xi^2)} \right] \exp(-\xi^2) \quad (51)$$

and c given by (40), where ξ is the only positive solution to equation (E5) if

and only if the physical parameters h_0 , q_0 , D_∞ and k satisfy conditions (R2) and (R17).

Proof. It is similar to the proof of Theorem 3.8. \square

Theorem 3.13 (Case 13: determination of k and ρ). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients k and ρ , then it has the solution given by (9)-(10) with:*

$$k = \frac{q_0 \sigma \sqrt{\pi}}{D_\infty \left(1 - \frac{q_0}{h_0 D_\infty}\right)} g_5(\xi), \quad (52)$$

$$\rho = \frac{q_0 \sqrt{\pi}}{c \sigma D_\infty} g_4(\xi), \quad (53)$$

where the real functions g_4 and g_5 are defined in (21) and (34), ξ is the only positive solution to the equation:

$$g_{13}(x) = \frac{a_{13}}{c_{13}} h_{13}(x) \quad (\text{E13})$$

and the real functions g_{13} and h_{13} are defined by:

$$g_{13}(x) = \frac{\exp(-x^2)}{\text{erf}(x)} \quad \text{and} \quad h_{13}(x) = x + b_{13} \exp(x^2) \text{erf}(x), \quad x > 0 \quad (54)$$

with

$$a_{13} = \frac{2cD_\infty}{l\sqrt{\pi}} \left(1 - \frac{q_0}{h_0 D_\infty}\right)^2, \quad b_{13} = \frac{\gamma\sqrt{\pi}(1-\varepsilon)}{2D_\infty \left(1 - \frac{q_0}{h_0 D_\infty}\right)},$$

$$c_{13} = 2 \left(1 - \frac{q_0}{h_0 D_\infty}\right) \quad (55)$$

if and only if the physical parameters h_0 , q_0 and D_∞ satisfy condition (R2).

Proof. On one hand, we have from Theorem 2.1 that the phase-change process (2)-(8) has the solution given by (9)-(10) if and only if k and ρ

satisfy equations (11) and (12). Since inequality (R2) is a necessary condition for the existence of a solution to equation (12), now and on, we will assume that inequality (R2) holds.

On the other hand, the solution to the system of equations (11)-(12) is given by (52) and (53), where the dimensionless parameter ξ is a solution to equation (E13). It only remains to observe that equation (E13) admits a positive solution since g_{13} is a decreasing function from $+\infty$ to 0 in \mathbb{R}^+ and h_{13} is an increasing function from 0 to $+\infty$ in \mathbb{R}^+ . \square

Theorem 3.14 (Case 14: determination of k and c). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients k and c , then it has the solution given by (9)-(10) with k given by (52) and:*

$$c = \frac{q_0 \sqrt{\pi}}{\sigma \rho D_\infty \left(1 - \frac{q_0}{h_0 D_\infty}\right)} g_4(\xi), \quad (56)$$

where the real function g_4 is defined in (21), ξ is the only positive solution to the equation:

$$a_{14} g_{14}(x) = h_{14}(x) \quad (E14)$$

and the real functions g_{14} and h_{14} are defined by:

$$g_{14}(x) = \left(\frac{q_0}{\rho l \sigma} \exp(-x^2) - 1 \right) x \quad \text{and} \quad h_{14}(x) = \operatorname{erf}(x) \exp(x^2), \quad x > 0 \quad (57)$$

with

$$a_{14} = \frac{2D_\infty}{\gamma \sqrt{\pi} (1 - \varepsilon)} \left(1 - \frac{q_0}{h_0 D_\infty} \right) \quad (58)$$

if and only if the remaining physical parameters satisfy conditions (R2), (R3) and

$$g_{14}(\eta) > h_{14}(\eta), \quad (R19)$$

where η is the only positive solution to the equation:

$$\frac{q_0}{\rho l \sigma} (1 - 2x^2) = \exp(x^2). \quad (59)$$

Proof. It is similar to the proof of Theorem 3.13. \square

Theorem 3.15 (Case 15: determination of ρ and c). *If we consider the phase-change process (2)-(8) with unknown thermal coefficients ρ and c , then it has the solution given by (9)-(10) with*

$$\rho = \frac{q_0}{l\sigma} \frac{\exp(-\xi^2)}{1 + \frac{\gamma k(1-\varepsilon)}{2q_0\sigma} \exp(\xi^2)}, \quad (60)$$

$$c = \frac{kl}{\sigma q_0} \left[1 + \frac{\gamma k(1-\varepsilon)}{2q_0\sigma} \exp(\xi^2) \right] \xi^2 \exp(\xi^2), \quad (61)$$

where ξ is the only positive solution to equation (E5) if and only if the physical parameters h_0 , q_0 , D_∞ , σ and k satisfy conditions (R2) and (R17).

Proof. It is similar to the proof of Theorem 3.13. \square

4. Conclusions

In this paper, we have studied the simultaneous determination of two unknown thermal coefficients for a semi-infinite material which was under a solidification process with a mushy zone according to the model of Solomon, Wilson and Alexiades. The unknown thermal coefficients were chosen among the latent heat by unit mass, the thermal conductivity, the mass density, the specific heat and the two coefficients that characterize the mushy zone. We have assumed that the evolution in time of one of the two free boundaries of the mushy zone, the bulk temperature and the coefficients that characterize the heat flux and the heat transfer at the fixed face were known. We have assumed that the solidification process ensued from a heat flux imposed at the fixed boundary and we have overspecified the associated free boundary value problem aiming at the simultaneous determination of the temperature in the solid region, the unknown free boundary and two

unknown thermal coefficients. We have first proved a preliminary result where necessary and sufficient conditions on data for the phase-change process were given in order to obtain the temperature and the unknown free boundary. Then, based on this preliminary result, we have presented and solved the fifteen different cases for the phase-change process corresponding to each possible choice of the two unknown thermal coefficients. We have proved that, under certain restrictions on data, there are twelve cases in which it is possible to find a unique explicit solution and that there are infinite explicit solutions for the remaining three cases. For each case, we have given formulae for the temperature, the unknown free boundary and the two unknown thermal coefficients with the necessary and sufficient conditions on data in order to obtain them.

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