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DETERMINATION OF TWO UNKNOWN THERMAL COEFFICIENTS THROUGH A MUSHY ZONE WITH A CONVECTIVE OVERSPECIFIED BOUNDARY CONDITION

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Abstract. We consider a semi-infinite material characterized by $x > 0$ which is initially assumed to be liquid at its melting temperature. At time $t = 0$ a heat flux condition is imposed at the fixed face $x = 0$, and a solidification process with a mushy zone begins. We impose an overspecified convective condition at $x = 0$ with the aim of the simultaneous determination of two thermal coefficients among l (latent heat by unit mass), k (thermal conductivity), ρ (mass density), c (specific heat), ϵ and γ (coefficients that characterize the mushy zone), when $q_0 > 0$, $h_0 > 0$ (coefficients that characterize the heat flux and heat transfer at $x = 0$, respectively), D_∞ (external temperature at $x = 0$) and one of the two boundaries of the mushy zone are determined experimentally. This lead us to the study of 15 different cases. We present the study of some of them, besides explicit formulae for the unknown thermal coefficients.

1 Introduction

Heat transfer problems with a phase-change such as melting and freezing have been studied in the last century due to their wide scientific and technological applications. Some previous books in the subject are [1–8, 14].

In this paper we consider a semi-infinite material characterized by $x > 0$ that is initially assumed to be liquid at its melting temperature, which without loss of generality we assume equal to 0°C. At time $t = 0$ a flux condition is imposed at the fixed face $x = 0$, and a solidification process begins where the following three regions can be distinguished [9,11]:

1. liquid region at temperature $T(x,t) = 0$: $D_l = \{(x,t) \in \mathbb{R}^2 / \, x > r(t), \, t > 0\}$,
2. solid region at temperature $T(x,t) < 0$: $D_s = \{(x,t) \in \mathbb{R}^2 / \, 0 < x < s(t), \, t > 0\}$,

3. mushy region at temperature $T(x,t)=0$: $D_p = \{(x,t) \in \mathbb{R}^2 / \, s(t) < x < r(t), \, t > 0\}$,

being $s = s(t)$ and $r = r(t)$ the functions that characterize the free boundaries of the mushy zone. We make the following assumptions on the structure of the mushy zone, which we consider isothermal:

1. the material contains a fixed portion ϵl of the total latent heat per unit mass $l > 0$, with $0 < \epsilon < 1$, that is:
$$kT_x(s(t),t) = \rho l[\epsilon \dot{s}(t) + (1-\epsilon)\dot{r}(t)], \quad t > 0,$$
where $k > 0$ is the thermal conductivity and $\rho > 0$ is the density mass of the material,

2. its width is inversely proportional to the gradient of temperature, that is:
$$T_x(s(t),t)(r(t) - s(t)) = \gamma, \quad t > 0$$
where $\gamma > 0$.

Encouraged by the recent works [12,13] and with the aim of the simultaneous determination of the temperature $T = T(x,t)$, the free boundary $x = r(t)$ and two of the thermal coefficients among l (latent heat by unit mass), k (thermal conductivity) , ρ (mass density), c (specific heat), ϵ and γ (coefficients which characterize the mushy zone), we impose an overspecified convective condition at $x = 0$ (see condition (7) below), which leads us to the following free boundary problem:

$$\rho c T_t(x,t) - kT_{xx}(x,t) = 0 \qquad 0 < x < s(t), t > 0 \qquad (1)$$
$$T(s(t),t) = 0 \qquad t > 0 \qquad (2)$$
$$kT_x(s(t),t) = \rho l[\epsilon \dot{s}(t) + (1-\epsilon)\dot{r}(t)] \qquad t > 0 \qquad (3)$$
$$T_x(s(t),t)(r(t) - s(t)) = \gamma \qquad t > 0 \qquad (4)$$
$$r(0) = s(0) = 0 \qquad (5)$$
$$kT_x(0,t) = \frac{q_0}{\sqrt{t}} \qquad t > 0 \qquad (6)$$
$$kT_x(0,t) = \frac{h_0}{\sqrt{t}}(T(0,t) + D_\infty) \qquad t > 0 \qquad (7)$$

where $q_0 > 0$ and $h_0 > 0$ are the coefficients that characterize the heat flux and the heat transfer at $x = 0$, respectively, and $-D_\infty < 0$ is the external temperature at $x = 0$.

We assume the free boundary $s(t)$ is given by:
$$s(t) = 2\sigma\sqrt{t}, \quad t > 0 \qquad (\sigma > 0) \qquad (8)$$
and the coefficients q_0, h_0, D_∞ and σ are determined experimentally.

The determination of the two unknown thermal coefficients for the one-phase Lamé-Clapeyron-Stefan problem without a mushy zone was done in [10]. The goal of this paper is to obtain the explicit solution to problem (1)-(7) with two unknown thermal coefficients and the necessary and sufficient conditions on data in order to obtain the explicit formulae for the two unknown thermal coefficients.

2 Solution of the Problem

We have:

Theorem 1. *The solution to problem (1)-(7) with $x = s(t)$ as in (8) is given by:*

$$T(x,t) = -\frac{q_0\sqrt{\alpha t}}{k}\operatorname{erf}\left(\frac{\sigma}{\sqrt{\alpha}}\right)\left[1 - \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)}{\operatorname{erf}\left(\frac{\sigma}{\sqrt{\alpha}}\right)}\right] \qquad 0 < x < s(t), t > 0 \qquad (9)$$

$$r(t) = \left(\frac{\gamma k \exp\left(\sigma^2/\alpha\right)}{q_0} + 2\sigma\right)\sqrt{t} \qquad t > 0 \qquad (10)$$

if and only if the parameters involved in the problem (1)-(7) satisfy the following two equations:

$$\frac{q_0}{\rho l} = \left[\sigma + \frac{\gamma k(1-\epsilon)\exp\left(\sigma^2/\alpha\right)}{2q_0}\right]\exp\left(\sigma^2/\alpha\right) \qquad (11)$$

$$\operatorname{erf}\left(\frac{\sigma}{\sqrt{\alpha}}\right) = \frac{kD_\infty}{q_0\sqrt{\alpha\pi}}\left(1 - \frac{q_0}{h_0 D_\infty}\right) \qquad (12)$$

Proof. Since the solution to problem (1)-(7) with $x = s(t)$ as in (8) has the form [9,11,13]:

$$T(x,t) = A + B\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) \qquad 0 < x < s(t), t > 0 \qquad (13)$$

$$r(t) = 2\mu\sqrt{\alpha t} \qquad t > 0 \qquad (14)$$

where $\alpha = \frac{k}{\rho c}$ (thermal diffusivity), by imposing conditions (2)-(7) we obtain that the coefficients A, B and μ must be given by:

$$A = -\frac{q_0\sqrt{\alpha\pi}}{k}\operatorname{erf}\left(\frac{\sigma}{\sqrt{\alpha}}\right), \qquad B = \frac{q_0\sqrt{\alpha\pi}}{k} \quad \text{and} \quad \mu = \frac{\gamma k\exp\left(\sigma^2/\alpha\right)}{2q_0\sqrt{\alpha}} + \frac{\sigma}{\sqrt{\alpha}} \qquad (15)$$

that is, the solution to problem (1)-(7) is given by (13)-(14), and that the parameters involved in the problem must satisfies equations (11) and (12). □

The problem of the determination of the temperature $T = T(x,t)$, the free boundary $x = r(t)$ and two coefficients among l, γ, ϵ, k, ρ y c, leads us to the study of 15 different cases, which we classify as:

Case 1: Determination of ϵ and γ, Caso 2: Determination of ϵ and l,

Caso 3: Determination of γ and l, Case 4: Determination of ϵ and k,

Caso 5: Determination of ϵ and ρ, Caso 6: Determination of ϵ and c, Case 7: Determination of γ and k, Caso 8: Determination γ and ρ,

Caso 9: Determination of γ and c, Case 10: Determination of l and k,

Caso 11: Determination of l and ρ, Caso 12: Determination of l and c,

Case 13: Determination of k and ρ, Caso 14: Determination of k and c,

and

Caso 15: Determination of ρ and c.

We will present the study of some of them in the following Section.

3 Results

We consider the cases 7 and 13.

Theorem 2 (Case 7: determination of γ and k)**.** *If in problem (1)-(7) we consider $x = s(t)$ as in (8) and the thermal parameters γ and k as unknowns, then its solution is given by (13)-(14) with γ and k given by:*

$$\gamma = \frac{2q_0\xi^2}{\sigma\rho c}\left(\frac{q_0}{\sigma\rho l}\exp\left(-\xi^2\right) - 1\right)\exp\left(-\xi^2\right) > 0 \tag{16}$$

$$k = \rho c\left(\frac{\sigma}{\xi}\right)^2 > 0 \tag{17}$$

being ξ the unique positive solution of the equation:

$$f(x) = \frac{\sigma\rho c D_\infty}{q_0\sqrt{\pi}}\left(1 - \frac{q_0}{h_0 D_\infty}\right), \quad x > 0, \tag{18}$$

where f is the function defined by:

$$f(x) = x\,\mathrm{erf}(x), \quad x > 0, \tag{19}$$

if and only if the parameters q_0, h_0, D_∞, σ, l, ρ and c satisfy the following three inequalities:

$$\frac{q_0}{\sigma\rho l} - 1 > 0 \tag{20}$$

$$0 < 1 - \frac{q_0}{h_0 D_\infty} < \frac{q_0\sqrt{\pi}}{\sigma\rho c D_\infty}\sqrt{\ln\left(\frac{q_0}{\sigma\rho l}\right)}\,\mathrm{erf}\left(\sqrt{\ln\left(\frac{q_0}{\sigma\rho l}\right)}\right) \tag{21}$$

Proof. We know from Theorem 1 that (13)-(14) is the solution to problem (1)-(7) with $x = s(t)$ as in (8) if and only if γ and k satisfy the system of equations (11)-(12). This system can be written as:

$$\gamma = \frac{2q_0\xi^2}{\sigma\rho c}\left(\frac{q_0}{\sigma\rho l}\exp(-\xi^2) - 1\right)\exp(-\xi^2) \tag{22}$$

$$f(\xi) = \frac{\sigma\rho c D_\infty}{q_0\sqrt{\pi}}\left(1 - \frac{q_0}{h_0 D_\infty}\right) \tag{23}$$

where

$$\xi = \frac{\sigma}{\sqrt{\alpha}} = \sigma\sqrt{\frac{\rho c}{k}}. \tag{24}$$

Since f is a strictly increasing function in $\mathbb{R}^+$ such that $f(0^+) = 0$ and $f(+\infty) = +\infty$, we have that equation (23) admits a unique positive solution if and only if the first inequality in (21) holds. In other words, that equation (12) admits a unique solution k, which is given by (17) (see (24)) where ξ is the unique positive solution of the equation (18), if and only if the first inequality in (21) holds. Finally, let us observe that the coefficient γ given in (22) is positive if and only if $\frac{q_0}{\sigma\rho l}\exp(-\xi^2) > 1$, that is, if and only if inequality (20) holds and

$$\xi < \sqrt{\ln\left(\frac{q_0}{\sigma\rho l}\right)}. \tag{25}$$

By applying the function f side by side of this last inequality and taking into account that f is strictly increasing in $\mathbb{R}^+$ and ξ is the unique solution of the equation (18), it follows that this last inequality is equivalent to the second inequality in (21). □

Theorem 3 (Case 13: determination of k and ρ)**.** *If in problem (1)-(7) we consider $x = s(t)$ as in (8) and the thermal parameters k and ρ as unknowns, then its solution is given by (13)-(14) with k and ρ given by:*

$$k = \frac{q_0\sigma\sqrt{\pi}}{D_\infty\left(1 - \frac{q_0}{h_0 D_\infty}\right)}\frac{\operatorname{erf}(\xi)}{\xi} > 0 \tag{26}$$

$$\rho = \frac{q_0\sqrt{\pi}}{c\sigma D_\infty}\xi\operatorname{erf}(\xi) \tag{27}$$

being ξ the unique positive solution of the equation:

$$g(x) = h(x), \quad x > 0, \tag{28}$$

where functions g and h are defined by:

$$g(x) = \frac{a}{\exp(x^2)\operatorname{erf}(x)} \qquad \text{and} \qquad h(x) = bx + c\exp(x^2)\operatorname{erf}(x), \quad x > 0 \tag{29}$$

with

$$a = \frac{2cD_\infty}{l\sqrt{\pi}}\left(1 - \frac{q_0}{h_0 D_\infty}\right)^2 > 0, \qquad b = 2\left(1 - \frac{q_0}{h_0 D_\infty}\right) > 0, \qquad c = \frac{\gamma\sqrt{\pi}(1-\epsilon)}{D_\infty} > 0, \tag{30}$$

if and only if the parameters q_0, h_0 and D_∞ satisfy the following inequality:

$$1 - \frac{q_0}{h_0 D_\infty} > 0 \tag{31}$$

Proof. We know from Theorem 1 that (13)-(14) is the solution to problem (1)-(7) with $x = s(t)$ as in (8) if and only if k and ρ satisfy the system of equations given by (11)-(12). This system can be written as:

$$\frac{q_0 c\sigma^2}{lk\xi^2} = \left[\sigma + \frac{\gamma k(1-\epsilon)}{2q_0}\exp\left(\xi^2\right)\right]\exp\left(\xi^2\right) \tag{32}$$

$$\operatorname{erf}(\xi) = \frac{kD_\infty \xi}{q_0\sigma\sqrt{\pi}}\left(1 - \frac{q_0}{h_0 D_\infty}\right) \tag{33}$$

where ξ is as in (24). Let us observe that inequality (31) is a necessary condition for the existence of a solution $k > 0$ and $\rho > 0$ to system (32)-(33). Then, henceforth, we assume that inequality (31) holds.

From equation (33), we have that k is given by (26). By replacing this expression for k in (32), we obtain that equation (32) is equivalent to $g(\xi) = h(\xi)$, that is ξ must satisfy equation (28).

Since g is a strictly decreasing function in $\mathbb{R}^+$ such that $g(0^+) = +\infty$ and $g(+\infty) = 0$, and h is a strictly increasing function such that $h(0^+) = 0$ and $h(+\infty) = +\infty$, it follows that there exists a unique positive solution to equation (28). Therefore, condition (31) is a necessary and sufficient condition for the existence and uniqueness of the solution to system (11)-(12), which is given by (26) and (27) (see (24)), where ξ is the unique positive solution to equation (28). □

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