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RESEARCH PAPER

DETERMINATION OF TWO UNKNOWN THERMAL COEFFICIENTS THROUGH AN INVERSE ONE-PHASE FRACTIONAL STEFAN PROBLEM

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Abstract

We consider a semi-infinite one-dimensional phase-change material with two unknown constant thermal coefficients among the latent heat per unit mass, the specific heat, the mass density and the thermal conductivity. Aiming at the determination of them, we consider an inverse one-phase Stefan problem with an over-specified condition at the fixed boundary and a known evolution for the moving boundary. We assume that it is given by a sharp front and we consider a time fractional derivative of order α ($0 < \alpha < 1$) in the Caputo sense to represent the temporal evolution of the temperature as well as the moving boundary. This might be interpreted as the consideration of latent-heat memory effects in the development of the phase-change process. According to the choice of the unknown thermal coefficients, six inverse fractional Stefan problems arise. For each of them, we determine necessary and sufficient conditions on data to obtain the existence and uniqueness of a solution of similarity type. Moreover, we present explicit expressions for the temperature and the unknown thermal coefficients. Finally, we show that the results for the classical statement of this problem, associated with $\alpha = 1$, are obtained through the fractional model when $\alpha \to 1^-$.

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1. Introduction

Determination of thermal coefficients for phase-change materials through inverse Stefan problems has been widely studied during the last decades [6, 7, 16, 29, 30, 31]. Especially, phase-change processes involving solidification or melting have been extensively studied because of their scientific and technological applications [1, 3, 5, 8, 10, 12, 20, 28, 33]. A review of a long bibliography on moving and free boundary value problems for the heat equation can be consulted in [32]. Recently, a new sort of Stefan problems including time-fractional derivatives have begun to be studied [2, 15, 17, 25, 26, 27, 35, 36, 37]. Some references in fractional derivatives can be found at [11, 18, 19, 21, 22, 24] and a survey on fractional calculus applications can be consulted in [9]. In [15, 25, 26, 27] there are considered free boundary value problems which are obtained by replacing the time derivative in a one-phase Stefan problem by a fractional derivative of order α $(0 < \alpha < 1)$ in the Caputo sense [4], and explicit solutions of similarity type are given for the resultant *fractional Stefan problems.* A physical interpretation of the problems considered in [15, 25, 26, 27] is given in [37]. In that article, authors derive fractional Stefan problems for phase-change processes by substituting the *local* expression of the heat flux given by the Fourier law for a new *non-local* definition. They consider a heat flux given as a weighted sum of local fluxes back in time, which they express in terms of the Riemann-Liouville integral of order α $(0 < \alpha < 1)$ of the local flux given by the Fourier law. They also explain how this change implies that the new model takes into consideration latent-heat memory effects in the evolution of the phase-change process and give to the parameter α the physical meaning of being the *strength of memory retention.* This fractional model reduces to the classical Stefan problem when $\alpha = 1$. That is, to the case of no memory retention. The same occurs with the solutions of similarity type given in [25, 26, 27] in the sense that they converge to the similarity solutions of the classical Stefan problems with which they are related to, when $\alpha \to 1^-$. To the authors knowledge, the first use of inverse fractional Stefan problems for the determination of thermal coefficients has been done recently in [34]. In that article the author studies the determination of one unknown thermal coefficient for a semi-infinite material through a fractional one-phase Stefan problem with an over-specified condition at the fixed boundary. Necessary and sufficient conditions on data to obtain the existence and uniqueness of solutions of similarity type are established, and explicit expressions for the the temperature of the material, the free boundary and the unknown thermal coefficient are given. Moreover, it has shown that results through the fractional model reduce to the results previously obtained in [29] for the determination of one unknown thermal coefficient using a classical inverse Stefan problem. Encouraged by [30, 34], we consider here the problem of determining two unknown thermal coefficients through an inverse fractional one-phase Stefan problem for which it is known the evolution of the free boundary. In order to have dimensional coherence in the time fractional heat equation as well as in the fractional Stefan condition, we have included two extra parameters μ_α, $\nu_\alpha \in (0,1]$ in the model, which are such that:

$$\mu_\alpha \to 1 \qquad \text{when} \quad \alpha \to 1^- \tag{1.1a}$$

$$\nu_\alpha \to 1 \qquad \text{when} \quad \alpha \to 1^-. \tag{1.1b}$$

In particular, μ_α and ν_α can be considered equal to 1 with the corresponding physical dimension (see below).

More precisely, we consider the following inverse problem for a one-phase melting process:

$$D^\alpha T(x,t) = \mu_\alpha \lambda^2 T_{xx}(x,t) \qquad 0 < x < s(t),\ t > 0 \tag{1.2a}$$

$$T(s(t),t) = T_m \qquad t > 0 \tag{1.2b}$$

$$-kT_x(s(t),t) = \nu_\alpha \rho l D^\alpha s(t) \qquad t > 0 \tag{1.2c}$$

$$T(0,t) = T_0 \qquad t > 0 \tag{1.2d}$$

$$kT_x(0,t) = -\frac{q_0}{t^{\alpha/2}} \qquad t > 0, \tag{1.2e}$$

where the unknowns are the temperature T [°C] of the liquid phase and two thermal coefficients among:

$k > 0$:	thermal conductivity	$[W\, m^{-1}\, (°C)^{-1}]$
$\rho > 0$:	mass density	$[kg\, m^3]$
$l > 0$:	latent heat per unit mass	$[J\, kg^{-1}]$
$c > 0$:	specific heat	$[J\, kg^{-1}\, (°C)^{-1}]$.

According to [13, 14], the coefficient $\mu_\alpha \lambda^2$ $[m^2\, s^{-\alpha}]$ in equation (1.2a) is a sort of *fractional diffusion coefficient*, λ^2 $[m^2\, s^{-1}]$ being the thermal diffusivity given by:

$$\lambda^2 = \frac{k}{\rho c} \qquad (\lambda > 0).$$

We assume that the remaining coefficients:

$T_m > 0$:	phase-change temperature	$[^\circ C]$
$T_0 > T_m$:	temperature at the boundary $x = 0$	$[^\circ C]$
$q_0 > 0$:	coefficient characterizing the heat flux at $x = 0$	$[W\, m^2\, (^\circ C)^{-1}]$
$0 < \alpha < 1$:	strength of the memory retention	(dimensionless)
$0 < \mu_\alpha \le 1$:	parameter required to have	
$0 < \nu_\alpha \le 1$:	parameter required to have dimensional coherence in condition (1.2c)	$[s^{\alpha-1}]$,

involved in Problem (1.2), are all known (e.g. through a phase-change experiment). Aiming at the simultaneous determination of two unknown thermal coefficients, we consider that the time evolution of the sharp interface s is also known. More precisely, we follow [15, 25, 26, 27] in assuming that it is given by:

$$s(t) = \sigma t^{\alpha/2}, \quad t > 0, \tag{1.3}$$

with $\sigma > 0$ $[m\, s^{-\alpha/2}]$. The operator D^α in (1.2a) and (1.2c) represents the fractional time derivative of order α in the Caputo sense, which is defined by [4]:

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau & \text{if} \quad 0 < \alpha < 1 \\ f'(t) & \text{if} \quad \alpha = 1 \end{cases}, \quad t > 0 \tag{1.4}$$

for any $f \in W^1(\mathbb{R}^+) = \left\{ f \in C^1(\mathbb{R}^+) / f' \in L^1(\mathbb{R}^+) \right\}$, where Γ is the Gamma function defined by:

$$\Gamma(x) = \int_0^{+\infty} s^{x-1} \exp(-s) ds, \quad x > 0.$$

We finally observe that, since we are considering a one-phase melting process, the temperature θ of the whole material is given by:

$$\theta(x,t) = \begin{cases} T(x,t) & \text{if} \quad 0 < x < s(t),\ t > 0 \\ T_m & \text{if} \quad s(t) \le x,\ t > 0 \end{cases}.$$

Some comments on the formulation of the fractional heat equation (1.2a) must be done before go any further. Since $x = 0$ is the starting point in the definition of the Caputo derivative given by (1.4), any function T which satisfies the fractional equation (1.2a) must be defined in a larger domain than the one of interest (see Definition 2.1. in the next section for further details). This formulation will enable us to find solutions of similarity type to problem (1.2) based on the similarity solutions found in [15, 25, 26, 27] to direct fractional Stefan problems. This article should be then interpreted as a mathematical game with fractional Stefan problems instead of one in which a physical theory for phase-change processes with memory is developed. In spite of this, we will use some terminology coming from the physical interpretation of classical Stefan problems in the context of phase-change problems, such as “temperature” or “thermal coefficients”. Through this sort of games, more precisely, through the explicit solutions and mathematical strategies presented here, we hope to contribute in the understanding of fractional Stefan problems and its relation with the modeling of phase-change processes under the effects of memory in their evolution.

2. Solutions of similarity type

We begin with a definition of being a solution to the inverse fractional Stefan problem (1.2), which is based in the definition established in [25, 26] for the direct case. Aiming at making it more readable, we first introduce the sets Ω, $\partial_p\Omega$ and function spaces $C^2_x(\mathbb{R}^+), W^1_t(\mathbb{R}^+)$ involved in the definition:

$$\begin{aligned}
&\Omega = \{(x,t)\,/\,0 < x < s(t),\, t > 0\},\\
&\partial_p\Omega = \{(0,t)/t > 0\} \cup \{(s(t),t)/t > 0\},\\
&C^2_x(\mathbb{R}^+) = \big\{f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}/\, f(\cdot,t) \in C^2(\mathbb{R}^+)\,\forall\, t > 0\big\},\\
&W^1_t(\mathbb{R}^+) = \big\{f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}/\, f(x,\cdot) \in W^1(\mathbb{R}^+)\,\forall\, x > 0\big\},
\end{aligned}$$

$W^1(\mathbb{R}^+)$ being the Sobolev space of all functions in the Lebesgue space $L^2(\mathbb{R}^+)$ which have a weak derivative in $L^2(\mathbb{R}^+)$.

Definition **2.1**. The triplet given by the temperature T and two unknown thermal coefficients among k, ρ, l and c is a solution to the inverse fractional one-phase Stefan problem (1.2) if:

(1) T is defined in $\mathbb{R}^+_0 \times \mathbb{R}^+_0$;
(2) $T \in C(\Omega) \cap C^2_x(\mathbb{R}^+) \cap W^1_t(\mathbb{R}^+)$;
(3) T is a continuous function over $\Omega \cup \partial_p\Omega$ and:

$$0 \leq \liminf_{(x,t)\to(0,0)} T(x,t) \leq \limsup_{(x,t)\to(0,0)} T(x,t) < \infty;$$

(4) For all $t > 0$, exists $\frac{\partial}{\partial x}T(s(t),t)$;
(5) The unknown thermal coefficients are positive real numbers;
(6) The temperature T and the two unknown thermal coefficients verify (1.2).

Remark **2.1**. We note that the function s given by (1.3) is consistent with the definition of being a solution to a (direct) fractional one-phase Stefan problem given in [25, 26], since it is a positive function belonging to $C(\mathbb{R}^+_0) \cap W^1(\mathbb{R}^+)$.

Encouraged by [15, 25, 26, 27, 34], in this section we are looking for a solution of similarity type to Problem (1.2). That is, we look for a temperature function T such that:

$$T(x,t) = A + B\left(1 - W\left(-\frac{x}{\sqrt{\mu_\alpha}\lambda t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right), \quad x > 0,\, t > 0, \quad (2.1)$$

where A and B are real numbers that must be determined, and W is the *Wright function* given by [38, 39]:

$$W(z,a,b) = \sum_{k=0}^{+\infty} \frac{z^k}{k!\,\Gamma(ak+b)}, \quad z \in \mathbb{C},\, a > -1,\, b \in \mathbb{C}. \quad (2.2)$$

According to [25, 26], we know that the function T given by (2.1) fulfills conditions 1 to 4 in Definition **2.1** and that it satisfies the fractional diffusion equation (1.2a). Noting that $W\left(0, -\frac{\alpha}{2}, 1\right) = 1$, it follows from condition (1.2d) that:

$$A = T_0. \quad (2.3)$$

From this, and condition (1.2b), we have that:

$$B = \frac{T_m - T_0}{1 - W\left(-\frac{\sigma}{\sqrt{\mu_\alpha}\lambda}, -\frac{\alpha}{2}, 1\right)}. \quad (2.4)$$

We observe that $W\left(-\frac{\sigma}{\sqrt{\mu_\alpha}\lambda}, -\frac{\alpha}{2}, 1\right) \neq 1$ because $W\left(-x, -\frac{\alpha}{2}, 1\right)$ is a strictly decreasing function in $\mathbb{R}^+$ (see [25]) and, as we have already noted, $W\left(0, -\frac{\alpha}{2}, 1\right) = 1$.

By taking into consideration that the Wright function satisfies [38, 39]:

$$\frac{d}{dz}W(z,a,b) = W(z,a,a+b), \quad z \in \mathbb{C},\, a > -1,\, b \in \mathbb{C},$$

and the fractional Caputo derivative of a power function with positive exponent is given by [24]:

$$D^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, \quad t > 0,\, p > 0,$$

it follows from the fractional Stefan condition (1.2c) that:

$$\frac{\sqrt{\mu_\alpha} l \left[1 - W\left(-\frac{\sigma}{\sqrt{\mu_\alpha}\lambda}, -\frac{\alpha}{2}, 1\right)\right]}{\lambda c M_{\frac{\alpha}{2}}\left(\frac{\sigma}{\sqrt{\mu_\alpha}\lambda}\right)} = \frac{(T_0 - T_m)\Gamma\left(-\frac{\alpha}{2}+1\right)}{\nu_\alpha \sigma \Gamma\left(\frac{\alpha}{2}+1\right)}, \quad (2.5)$$

where M_ν is the *Mainardi function*, which has been defined by ([23]):

$$M_\nu(z) = W(-z, -\nu, 1-\nu), \quad z \in \mathbb{C},\, 0 < \nu < 1, \quad (2.6)$$

as special case of the Wright function, and satisfies $M_\nu(z) > 0$ for all $z \in \mathbb{R}^+$ ([25]).

Finally, when we consider B given by (2.4), the heat flux boundary condition (1.2e) implies that it must be satisfied the following equality:

$$\frac{\sqrt{\mu_\alpha} q_0 \lambda}{k}\left[1 - W\left(-\frac{\sigma}{\sqrt{\mu_\alpha}\lambda}, -\frac{\alpha}{2}, 1\right)\right] = \frac{T_0 - T_m}{\Gamma\left(-\frac{\alpha}{2}+1\right)}. \tag{2.7}$$

We have thus proved the following result:

THEOREM **2.1**. *If the moving boundary s is defined by (1.3), then the function T given by:*

$$T(x,t) = T_0 - \frac{T_0 - T_m}{1 - W\left(-\xi, -\frac{\alpha}{2}, 1\right)}\left[1 - W\left(-\frac{x}{\sqrt{\mu_\alpha}\lambda t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right], \quad x > 0,\ t > 0, \tag{2.8}$$

is a solution to Problem (1.2) with two unknown thermal coefficients among k, ρ, l and c, if and only if it is a solution to the following system of equations:

$$\frac{\xi\left[1 - W\left(-\xi, -\frac{\alpha}{2}, 1\right)\right]}{M_{\frac{\alpha}{2}}(\xi)} = \frac{c(T_0 - T_m)\Gamma\left(-\frac{\alpha}{2}+1\right)}{\mu_\alpha \nu_\alpha l \Gamma\left(\frac{\alpha}{2}+1\right)}, \tag{2.9a}$$

$$1 - W\left(-\xi, -\frac{\alpha}{2}, 1\right) = \frac{\sqrt{k\rho c}(T_0 - T_m)}{\sqrt{\mu_\alpha} q_0 \Gamma\left(-\frac{\alpha}{2}+1\right)}, \tag{2.9b}$$

where the dimensionless parameter ξ is defined by:

$$\xi = \frac{\sigma}{\sqrt{\mu_\alpha}\lambda}. \tag{2.10}$$

3. Existence and uniqueness of solutions of similarity type. Formulae for the two unknown thermal coefficients.

In this section we will look for necessary and sufficient conditions on data to have existence and uniqueness of solution to Problem (1.2) for each possible choice of the two unknown thermal coefficients, as well as explicit formulae for them. Thanks to Theorem **2.1**, it can be done through analysing and solving the system of equations (2.9) for each pair of unknown thermal coefficients. With the aim of organizing the main results of this section, we will write:

Case 1: Determination of l and c
Case 2: Determination of c and k
Case 3: Determination of l and k
Case 4: Determination of c and ρ
Case 5: Determination of l and ρ
Case 6: Determination of ρ and k.

For each $\alpha \in (0,1)$, we introduce the real functions F_α, G_α and H_α defined in $\mathbb{R}^+$ by:

$$F_\alpha(x) = \frac{f_\alpha(x)}{x}, \tag{3.1a}$$

$$G_\alpha(x) = x f_\alpha(x), \tag{3.1b}$$

$$H_\alpha(x) = \frac{x f_\alpha(x)}{M_{\frac{\alpha}{2}}(x)}, \tag{3.1c}$$

where

$$f_\alpha(x) = 1 - W\left(-x, -\frac{\alpha}{2}, 1\right). \tag{3.2}$$

The following result will be useful all throughout this section.

LEMMA **3.1**. *For any $\alpha \in (0,1)$, the real functions F_α, G_α and H_α defined in (3.1) satisfy the following conditions:*

$$F_\alpha(0^+) = \frac{1}{\Gamma\left(-\frac{\alpha}{2}+1\right)}, \quad F_\alpha(+\infty) = 0, \quad F'_\alpha(x) < 0 \quad \forall\, x > 0, \tag{3.3a}$$

$$G_\alpha(0^+) = 0, \quad G_\alpha(+\infty) = +\infty, \quad G'_\alpha(x) > 0 \quad \forall\, x > 0, \tag{3.3b}$$

$$H_\alpha(0^+) = 0, \quad H_\alpha(+\infty) = +\infty, \quad H'_\alpha(x) > 0 \quad \forall\, x > 0. \tag{3.3c}$$

P r o o f. The proof of (3.3a) was done in [34]. The demonstrations of (3.3b) and (3.3c) follow from elementary computations and the following facts:

(1) Since $0 < \alpha < 1$, f_α is a positive and strictly increasing function in $\mathbb{R}^+$, [25].

(2) Since $0 < \alpha < 1$, $M_{\frac{\alpha}{2}}$ is a positive and strictly decreasing function in $\mathbb{R}^+$, [25].

(3) $\lim\limits_{x\to+\infty} f(x) = 1$ and $\lim\limits_{x\to+\infty} M_{\frac{\alpha}{2}}(x) = 0$, [11]. □

THEOREM **3.1** (Case 1: Determination of l and c). *If the moving boundary s is given by (1.3), then Problem (1.2) admits the solution T, l and c given by (2.8) and:*

$$c = \frac{1}{\rho k}\left[\frac{q_0\sqrt{\mu_\alpha}\,\Gamma\left(-\frac{\alpha}{2}+1\right)\left[1 - W\left(-\xi, -\frac{\alpha}{2}, 1\right)\right]}{T_0 - T_m}\right]^2, \tag{3.4a}$$

$$l = \frac{q_0^2\,\Gamma^3\left(-\frac{\alpha}{2}+1\right)\left[1 - W\left(-\xi, -\frac{\alpha}{2}, 1\right)\right] M_{\frac{\alpha}{2}}(\xi)}{\nu_\alpha \rho k (T_0 - T_m)\Gamma\left(\frac{\alpha}{2}+1\right)\xi}, \tag{3.4b}$$

respectively, ξ being the only one solution to the equation:

$$F_\alpha(x) = \frac{k(T_0 - T_m)}{\sigma q_0 \Gamma\left(-\frac{\alpha}{2}+1\right)}, \quad x > 0, \tag{3.5}$$

if and only if the following inequality holds:

$$\frac{k(T_0 - T_m)}{\sigma q_0} < 1. \tag{3.6}$$

P r o o f. Isolating c from equation (2.9b), we have that c is given by (3.4a). Now, by combining this with equation (2.9a), it can be obtained that l is given by (3.4b). It must be noted that the parameter ξ involved in both (3.4a) and (3.4b) depends on c. Nevertheless, it can be determined without making any reference to c as follows. By replacing (3.4a) in the definition of ξ given in (2.10), we have that ξ must be a solution to equation (3.5). It follows from (3.3a) that the equation (3.5) admits a solution if and only if its RHS is between 0 and $\frac{1}{\Gamma\left(-\frac{\alpha}{2}+1\right)}$. To complete the proof only remains to observe that this is equivalent to say that inequality (3.6) must hold and that, when this happens, equation (3.5) has an only one positive solution. □

THEOREM **3.2** (Case 2: Determination of c and k). *If the moving boundary s is given by (1.3), then Problem (1.2) admits the solution T, c and k given by (2.8) and:*

$$c = \frac{\mu_\alpha \nu_\alpha l \Gamma\left(\frac{\alpha}{2}+1\right)\xi\left[1 - W\left(-\xi, -\frac{\alpha}{2}, 1\right)\right]}{(T_0 - T_m)\Gamma\left(-\frac{\alpha}{2}+1\right) M_{\frac{\alpha}{2}}(\xi)}, \tag{3.7a}$$

$$k = \frac{q_0^2 \Gamma^3\left(-\frac{\alpha}{2}+1\right)\left[1 - W\left(-\xi, -\frac{\alpha}{2}, 1\right)\right] M_{\frac{\alpha}{2}}(\xi)}{\nu_\alpha \rho l (T_0 - T_m)\Gamma\left(\frac{\alpha}{2}+1\right)\xi}, \tag{3.7b}$$

respectively, ξ being the only one solution to the equation:

$$M_{\frac{\alpha}{2}}(x) = \frac{\nu_\alpha \sigma \rho l \Gamma\left(\frac{\alpha}{2}+1\right)}{q_0 \Gamma^2\left(-\frac{\alpha}{2}+1\right)}, \quad x > 0, \tag{3.8}$$

if and only if the following inequality holds:

$$\frac{\nu_\alpha \sigma \rho l \Gamma\left(\frac{\alpha}{2}+1\right)}{q_0 \Gamma\left(-\frac{\alpha}{2}+1\right)} < 1. \tag{3.9}$$

P r o o f. By following the same steps as in the demonstration of the Theorem **3.1**, it can be shown that c and k must be given by (3.7), where the parameter ξ should be a solution to equation (3.8). Since the Mainardi function $M_{\frac{\alpha}{2}}$ is a strictly decreasing function from $\frac{1}{\Gamma(-\frac{\alpha}{2}+1)}$ to 0 in $\mathbb{R}^+$ ([25]), we have that the equation (3.8) admits a solution if and only if its RHS is between 0 and $\frac{1}{\Gamma(-\frac{\alpha}{2}+1)}$. This is equivalent to say that inequality (3.9) must holds. Moreover, when data satisfy (3.9), equation (3.8) has an only one positive solution. □

THEOREM **3.3** (Case 3: Determination of l and k). *If the moving boundary s is given by (1.3), then Problem (1.2) admits the solution T, k and l given by (2.8) and:*

$$k = \frac{1}{\rho c}\left[\frac{q_0\sqrt{\mu_\alpha}\Gamma\left(-\frac{\alpha}{2}+1\right)\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}{T_0-T_m}\right]^2, \tag{3.10a}$$

$$l = \frac{c(T_0-T_m)\Gamma\left(-\frac{\alpha}{2}+1\right)M_{\frac{\alpha}{2}}(\xi)}{\mu_\alpha\nu_\alpha\Gamma\left(\frac{\alpha}{2}+1\right)\xi\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}, \tag{3.10b}$$

respectively, ξ being the only one solution to the equation:

$$G_\alpha(x) = \frac{\sigma\rho c(T_0-T_m)}{\mu_\alpha q_0\Gamma\left(-\frac{\alpha}{2}+1\right)}, \quad x>0. \tag{3.11}$$

P r o o f. By proceeding analogously to the proofs of the previous Theorems **3.1** and **3.2**, we have that k and l should be given by (3.10), ξ being a solution to equation (3.11). Since the RHS of the equation (3.11) is a positive number, it follows from (3.3b) that the equation (3.11) admits an only one solution for any set of data. □

The following Theorems **3.4**, **3.5**, corresponding to Cases 4, 5, can be proved in much the same way as Theorem **3.1**. Therefore, we do not include here their demonstrations.

THEOREM **3.4** (Case 4: Determination of c and ρ). *If the moving boundary s is given by (1.3), then Problem (1.2) admits the solution T, c and ρ given by (2.8), (3.7a) and:*

$$\rho = \frac{q_0^2\Gamma^3\left(-\frac{\alpha}{2}+1\right)\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]M_{\frac{\alpha}{2}}(\xi)}{\nu_\alpha kl(T_0-T_m)\Gamma\left(\frac{\alpha}{2}+1\right)\xi}, \tag{3.12a}$$

respectively, ξ being the only one solution to the equation (3.5), if and only if inequality (3.6) holds.

THEOREM **3.5** (Case 5: Determination of l and ρ). *If the moving boundary s is given by (1.3), then Problem (1.2) admits the solution T, l and ρ given by (2.8), (3.10b) and:*

$$\rho = \frac{1}{kc}\left[\frac{q_0\sqrt{\mu_\alpha}\Gamma\left(-\frac{\alpha}{2}+1\right)\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}{T_0-T_m}\right]^2, \tag{3.13a}$$

respectively, ξ being the only one solution to the equation (3.5), if and only if inequality (3.6) holds.

THEOREM **3.6** (Case 6: Determination of ρ and k). *If the moving boundary s is given by (1.3), then Problem (1.2) admits the solution T, ρ and k given by (2.8) and:*

$$\rho = \frac{q_0\mu_\alpha\Gamma\left(-\frac{\alpha}{2}+1\right)\xi\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}{\sigma c(T_0-T_m)}, \tag{3.14a}$$

$$k = \frac{\sigma q_0\Gamma\left(-\frac{\alpha}{2}+1\right)\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}{(T_0-T_m)\xi}, \tag{3.14b}$$

respectively, ξ being the only one solution to the equation:

$$H_\alpha(x) = \frac{c(T_0-T_m)\Gamma\left(-\frac{\alpha}{2}+1\right)}{\mu_\alpha\nu_\alpha l\Gamma\left(\frac{\alpha}{2}+1\right)}, \quad x>0. \tag{3.15}$$

P r o o f. By following the same ideas as in the proof of Theorem **3.1**, we have that ρ and k must be given by (3.14), where ξ should be a solution to equation (3.15). By noting that the RHS of this equation is a positive number, it follows from (3.3c) that the equation (3.15) admits an only one positive solution for any set of data. □

Table 1 summarizes the formulae obtained for the two unknown thermal coefficients and the condition that data must verify to obtain them, for each one of the six possible choices of the two unknown thermal coefficients among k, ρ, l and c in Problem (1.2) when the moving boundary s is defined by (1.3).

Case	Formulae for the unknown thermal coefficients	⋆ Equation for ξ ∗ Restriction for data
1	$c=\frac{1}{\rho k}\left[\frac{q_0\sqrt{\mu_\alpha}\Gamma\left(-\frac{\alpha}{2}+1\right)\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}{T_0-T_m}\right]^2$	$\star\, F_\alpha(x)=\frac{k(T_0-T_m)}{\sigma q_0\Gamma\left(-\frac{\alpha}{2}+1\right)},\ x>0$
	$l=\frac{q_0^2\Gamma^3\left(-\frac{\alpha}{2}+1\right)\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]M_{\frac{\alpha}{2}}(\xi)}{\nu_\alpha\rho k(T_0-T_m)\Gamma\left(\frac{\alpha}{2}+1\right)\xi}$	$\ast\,\frac{k(T_0-T_m)}{\sigma q_0}<1$
2	$c=\frac{\mu_\alpha\nu_\alpha l\Gamma\left(\frac{\alpha}{2}+1\right)\xi\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}{(T_0-T_m)\Gamma\left(-\frac{\alpha}{2}+1\right)M_{\alpha/2}(\xi)}$	$\star\, M_{\frac{\alpha}{2}}(x)=\frac{\nu_\alpha\sigma\rho l\Gamma\left(\frac{\alpha}{2}+1\right)}{q_0\Gamma^2\left(-\frac{\alpha}{2}+1\right)},\ x>0$
	$k=\frac{q_0^2\Gamma^3\left(-\frac{\alpha}{2}+1\right)\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]M_{\frac{\alpha}{2}}(\xi)}{\nu_\alpha\rho l(T_0-T_m)\Gamma\left(\frac{\alpha}{2}+1\right)\xi}$	$\ast\,\frac{\nu_\alpha\sigma\rho l\Gamma\left(\frac{\alpha}{2}+1\right)}{q_0\Gamma\left(-\frac{\alpha}{2}+1\right)}<1$
3	$k=\frac{1}{\rho c}\left[\frac{q_0\sqrt{\mu_\alpha}\Gamma\left(-\frac{\alpha}{2}+1\right)\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}{T_0-T_m}\right]^2$	$\star\, G_\alpha(x)=\frac{\sigma\rho c(T_0-T_m)}{\mu_\alpha q_0\Gamma\left(-\frac{\alpha}{2}+1\right)},\ x>0$
	$l=\frac{c(T_0-T_m)\Gamma\left(-\frac{\alpha}{2}+1\right)M_{\frac{\alpha}{2}}(\xi)}{\mu_\alpha\nu_\alpha\Gamma\left(\frac{\alpha}{2}+1\right)\xi\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}$	$\ast\,-$
4	$c=\frac{\mu_\alpha\nu_\alpha l\Gamma\left(\frac{\alpha}{2}+1\right)\xi\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}{(T_0-T_m)\Gamma\left(-\frac{\alpha}{2}+1\right)M_{\alpha/2}(\xi)}$	$\star\, F_\alpha(x)=\frac{k(T_0-T_m)}{\sigma q_0\Gamma\left(-\frac{\alpha}{2}+1\right)},\ x>0$
	$\rho=\frac{q_0^2\Gamma^3\left(-\frac{\alpha}{2}+1\right)\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]M_{\frac{\alpha}{2}}(\xi)}{\nu_\alpha kl(T_0-T_m)\Gamma\left(\frac{\alpha}{2}+1\right)\xi}$	$\ast\,\frac{k(T_0-T_m)}{\sigma q_0}<1$
5	$l=\frac{c(T_0-T_m)\Gamma\left(-\frac{\alpha}{2}+1\right)M_{\alpha/2}(\xi)}{\mu_\alpha\nu_\alpha\Gamma\left(\frac{\alpha}{2}+1\right)\xi\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}$	$\star\, F_\alpha(x)=\frac{k(T_0-T_m)}{\sigma q_0\Gamma\left(-\frac{\alpha}{2}+1\right)},\ x>0$
	$\rho=\frac{1}{kc}\left[\frac{q_0\sqrt{\mu_\alpha}\Gamma\left(-\frac{\alpha}{2}+1\right)\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}{T_0-T_m}\right]^2$	$\ast\,\frac{k(T_0-T_m)}{\sigma q_0}<1$
6	$\rho=\frac{\mu_\alpha q_0\Gamma\left(-\frac{\alpha}{2}+1\right)\xi\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}{\sigma c(T_0-T_m)}$	$\star\, H_\alpha(x)=\frac{c(T_0-T_m)\Gamma\left(-\frac{\alpha}{2}+1\right)}{\mu_\alpha\nu_\alpha\Gamma\left(\frac{\alpha}{2}+1\right)},\ x>0$
	$k=\frac{\sigma q_0\Gamma\left(-\frac{\alpha}{2}+1\right)\left[1-W\left(-\xi,-\frac{\alpha}{2},1\right)\right]}{(T_0-T_m)\xi}$	$\ast\,-$

TABLE 1. Formulae for the two unknown thermal coefficients and restriction on data for Problem (1.2)

4. Convergence to the classic case when $\alpha \to 1^-$

When $\alpha = 1$, the time fractional derivative of order α in the Caputo sense of a function coincides with its classical time derivative. Then, if we allow α to be equal to 1 in Problem (1.2) and we consider that case, we obtain that Problem (1.2) reduces to the classical inverse one-phase Stefan problem studied in [30]. This problem, which we will refer to as Problem (1.2*), is given by the classical diffusion equation:

$$T_t(x,t) = \lambda^2 T_{xx}(x,t), \quad 0 < x < s(t),\ t > 0, \tag{1.2a*}$$

the classical Stefan condition:

$$-k\, T_x(s(t),t) = \rho\, l\, \dot{s}\,(t), \quad t > 0, \tag{1.2c*}$$

and conditions (1.2b), (1.2d) and (1.2e). Of course, to obtain (1.2a*) and (1.2c*) we have also considered $\mu_\alpha = 1$ and $\nu_\alpha = 1$ in (1.2a) and (1.2c), respectively.

The determination of two unknown thermal coefficients through a classical inverse one-phase Stefan problem was done in [30]. In that article, necessary and sufficient conditions on data to obtain existence and uniqueness of solution to Problem (1.2*) are given, together with formulae for the unknown thermal coefficients. In several articles [15, 25, 26, 27, 34] it has been proved the convergence when $\alpha \to 1^-$ of the solution to a fractional Stefan problem with $0 < \alpha < 1$ to the solution to the associated classical problem obtained by considering $\alpha = 1$. Encouraged by those works, we are interested in this section in proving the convergence when $\alpha \to 1^-$ of the results obtained in Section **3** to the ones given in [30].

In order to emphasize the dependence on α of the formulae given in Theorems **3.1** to **3.6**, we will mention it here explicitly. For example, if we are analyzing the convergence of the solution to Problem (1.2) given in Theorem **3.1**, we will refer to it as $T(x,t,\alpha)$, $l(\alpha)$ and $c(\alpha)$. We will also write $\xi(\alpha)$ to represent the coefficient defined by (2.10).

We start our analysis by recalling the limit behaviour when $\alpha \to 1^-$ of the Wright and Mainardi functions involved in the results given in Section **3**. The proof of the following lemma can be consulted in [25].

Lemma **4.1**. *For each $x > 0$, the Wright and Mainardi functions are such that:*

$$1 - W\left(-x, -\frac{\alpha}{2}, 1\right) \to \operatorname{erf}\left(\frac{x}{2}\right), \qquad \textit{when} \quad \alpha \to 1^- \tag{4.1a}$$

$$M_{\alpha/2}(x) \to \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{4}\right), \qquad \textit{when} \quad \alpha \to 1^-, \tag{4.1b}$$

where erf *is the error function, which is defined by:*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(s^2) ds, \quad x > 0.$$

THEOREM **4.1** (Convergence related to Case 1). *If inequality (3.6) holds, then the solution $T(x,t,\alpha)$, $l(\alpha)$, $c(\alpha)$ to Problem (1.2) given in Theorem* **3.1** *converges to the solution obtained in* [30], *which is given by:*

$$T(x,t) = T_0 + \frac{T_0 - T_m}{\operatorname{erf}\left(\frac{\sigma^\star}{\lambda}\right)} \operatorname{erf}\left(\frac{x}{2\lambda\sqrt{t}}\right), \quad 0 < x < s(t),\ t > 0, \tag{4.2a}$$

$$c = \frac{k}{\rho} \left(\frac{\xi^\star}{\sigma^\star}\right)^2, \tag{4.2b}$$

$$l = \frac{q_0 \exp(-\xi^{\star 2})}{\rho \sigma^\star}, \tag{4.2c}$$

where $\sigma^\star$ is defined by:

$$\sigma^\star = \frac{\sigma}{2} \tag{4.3}$$

and $\xi^\star$ is the only one solution to the equation:

$$\frac{\operatorname{erf}(x)}{x} = \frac{k(T_0 - T_m)}{q_0 \sigma^\star \sqrt{\pi}}, \quad x > 0. \tag{4.4}$$

P r o o f. Taking the limit when $\alpha \to 1^-$ into both sides of the equation (3.5) and using (4.1b), we obtain the following equation:

$$\frac{\operatorname{erf}(x/2)}{x} = \frac{k(T_0 - T_m)}{q_0 \sigma \sqrt{\pi}}, \quad x > 0. \tag{4.5}$$

On one hand, we have that the LHS of the equation (4.5) defines a strictly decreasing function from $\frac{1}{\sqrt{\pi}}$ to 0 in $\mathbb{R}^+$. On the other hand, since inequality (3.6) holds, the RHS of equation (3.6) is between 0 and $\frac{1}{\sqrt{\pi}}$. Therefore, it follows that equation (4.5) has an only one positive solution. By introducing the parameter $\sigma^\star$ defined by (4.3), we can rewrite equation (4.5) as follows:

$$\frac{\operatorname{erf}(x/2)}{x/2} = \frac{k(T_0 - T_m)}{q_0 \sigma^\star \sqrt{\pi}}, \quad x > 0,$$

and see that the solution $\xi(\alpha)$ to the equation (3.5) is such that:

$$\xi(\alpha) \to 2\xi^\star, \qquad \text{when} \quad \alpha \to 1^-, \tag{4.6}$$

$\xi^\star$ being the only one solution to the equation (4.4). From (1.1), (4.1) and (4.6), it follows from elementary computations that:

$$c(\alpha) \to c, \qquad \text{when} \quad \alpha \to 1^-, \tag{4.7a}$$

$$l(\alpha) \to l, \qquad \text{when} \quad \alpha \to 1^-, \tag{4.7b}$$

where c and l are given by (4.2b) and (4.2c), respectively. Finally, it follows from (1.1a), (4.1), (4.6) and (4.7) that:

$$T(x,t,\alpha) \to T(x,t), \qquad \text{when} \quad \alpha \to 1^-, \tag{4.8}$$

for each pair (x,t) with $0 < x < s(t)$ and $t > 0$, where $T(x,t)$ is given by (4.2a). We then have proved that the solution to Problem (1.2) given in Theorem **3.1** converges to the solution to Problem (1.2*) given in [30] when $\alpha \to 1^-$. □

REMARK **4.1**. We note that inequality (3.6) can be written as:

$$\frac{k(T_0 - T_m)}{2\sigma^\star q_0} < 1, \tag{4.9}$$

$\sigma^\star$ being the parameter defined by (4.3), which is the condition established in [30] to ensure the existence and uniqueness of the solution to Problem (1.2*) given by (4.2).

THEOREM **4.2** (Convergence related to Case 2). *If inequality (3.9) holds for each $\alpha \in (0,1)$, then the solution $T(x,t,\alpha)$, $c(\alpha)$, $k(\alpha)$ to Problem (1.2) given in Theorem* **3.2** *converges to the solution obtained in* [30], *which is given by (4.2a) and:*

$$c = \frac{q_0\sqrt{\pi}\xi^\star \operatorname{erf}(\xi^\star)}{\rho\sigma^\star(T_0 - T_m)}, \tag{4.10a}$$

$$k = \frac{\sigma^\star q_0\sqrt{\pi} \operatorname{erf}(\xi^\star)}{(T_0 - T_m)\xi^\star}, \tag{4.10b}$$

where $\sigma^\star$ is given by (4.3) and $\xi^\star$ is the only one solution to the equation:

$$\exp(x^2) = \frac{q_0}{\rho l \sigma^\star}, \quad x > 0. \tag{4.11}$$

P r o o f. If we take the limit when $\alpha \to 1^-$ side by side of equation (3.8) and we have into consideration (1.1b) and (4.1b), the following equation is obtained:

$$\exp\left(\frac{x^2}{4}\right) = \frac{2q_0}{\sigma\,\rho\,l}, \quad x > 0. \tag{4.12}$$

Since inequality (3.9) holds for all $\alpha \in (0,1)$, we have that the following inequality also holds:

$$\frac{2q_0}{\sigma\rho l} < 1. \tag{4.13}$$

Therefore, equation (4.12) admits only one positive solution. We note that equation (4.12) can be rewritten as:

$$\exp\left(\left(\frac{x}{2}\right)^2\right) = \frac{q_0}{\sigma^\star \rho l}, \quad x > 0,$$

where $\sigma^\star$ is given by (4.3), from which we can see that the solution $\xi(\alpha)$ to equation (3.8) is such that:

$$\xi(\alpha) \to 2\xi^\star, \qquad \text{when} \quad \alpha \to 1^-, \tag{4.14}$$

$\xi^\star$ being the only one solution to equation (4.11). It follows now from (1.1) (4.1), (4.14) and elementary computations that:

$$c(\alpha) \to c, \tag{4.15a}$$

$$k(\alpha) \to k, \tag{4.15b}$$

where c and k are given by (3.7), respectively. Finally, we have from (1.1a), (4.1), (4.14) and (4.15) that $T(x,t,\alpha)$ satisfies (4.8). □

Remark **4.2**. By introducing the parameter $\sigma^\star$ defined by (4.3), we have that the inequality (4.13) can be rewritten as:

$$\frac{q_0}{\rho l \sigma^\star} > 1, \tag{4.16}$$

which is the condition established in [30] to ensure the existence and uniqueness of the solution to Problem ($1.2^\star$) given by (4.2a) and (4.10).

Theorem **4.3** (Convergence related to Case 3). *If inequality (3.6) holds, then the solution $T(x,t,\alpha)$, $l(\alpha)$, $k(\alpha)$ to Problem (1.2) given in Theorem* **3.3** *converges to the solution obtained in* [30], *which is given by (4.2a) and:*

$$l = \frac{q_0 \exp(-\xi^{\star 2})}{\rho\sigma^\star}, \tag{4.17a}$$

$$k = \rho c \left(\frac{\sigma^\star}{\xi^\star}\right)^2, \tag{4.17b}$$

where $\sigma^\star$ is given by (4.3) and $\xi^\star$ is the only one solution to the equation:

$$x \operatorname{erf}(x) = \frac{\rho\, c\, \sigma^\star (T_0 - T_m)}{q_0\sqrt{\pi}}, \quad x > 0. \tag{4.18}$$

P r o o f. Taking the limit when $\alpha \to 1^-$ into both sides of the equation (3.11) and having into consideration (1.1a) and (4.1a), we obtain the following equation:

$$x \text{ erf}\left(\frac{x}{2}\right) = \frac{\sigma\,\rho\,c\,(T_0 - T_m)}{q_0\sqrt{\pi}}, \quad x > 0. \tag{4.19}$$

Since the LHS of equation (4.19) defines a strictly increasing function from 0 to $+\infty$ in $\mathbb{R}^+$ and the RHS of equation (4.19) is a positive number, it follows that equation (4.19) has an only one positive solution. By introducing the parameter $\sigma^\star$ defined by (4.3), the equation (4.19) can be rewritten as:

$$\frac{x}{2} \text{ erf}\left(\frac{x}{2}\right) = \frac{\sigma^\star\,\rho\,c\,(T_0 - T_m)}{q_0\sqrt{\pi}}, \quad x > 0,$$

from which we can see that the solution $\xi(\alpha)$ to the equation (3.11) is such that:

$$\xi(\alpha) \to 2\xi^\star, \qquad \text{when} \quad \alpha \to 1^-, \tag{4.20}$$

$\xi^\star$ being the only one solution to equation (4.18). The rest of the proof runs as before. □

The following results can be proved in the same manner as the previous theorems in this section. Then, we prefer to omit their proofs.

THEOREM **4.4** (Convergence related to Case 4). *If inequality (3.6) holds, then the solution $T(x,t,\alpha)$, $c(\alpha)$, $\rho(\alpha)$ to Problem (1.2) given in Theorem* **3.4** *converges to the solution obtained in* [30], *which is given by (4.2a) and:*

$$c = \frac{kl\xi^{\star,} \exp(\xi^{\star 2})}{q_0\sqrt{\pi}}, \tag{4.21a}$$

$$\rho = \frac{q_0 \, \exp(-\xi^{\star 2})}{l\sigma^\star}, \tag{4.21b}$$

where $\sigma^\star$ is given by (4.3) and $\xi^\star$ is the only one solution to the equation (4.4).

THEOREM **4.5** (Convergence related to Case 5). *The solution $T(x,t,\alpha)$, $l(\alpha)$, $\rho(\alpha)$ to Problem (1.2) given in Theorem* **3.5** *converges to the solution obtained in* [30], *which is given by (4.2a) and:*

$$l = \frac{q_0 c\sigma^\star \, \exp(-\xi^{\star 2})}{k\xi^{\star 2}}, \tag{4.22a}$$

$$\rho = \frac{k}{c}\left(\frac{\xi^\star}{\sigma^\star}\right)^2, \tag{4.22b}$$

where $\sigma^\star$ is given by (4.3) and $\xi^\star$ is the only one solution to the equation (4.4).

Theorem **4.6** (Convergence related to Case 6). *The solution $T(x,t,\alpha)$, $\rho(\alpha)$, $k(\alpha)$ to Problem (1.2) given in Theorem* **3.6** *converges to the solution obtained in* [30], *which is given by (4.2a) and:*

$$\rho = \frac{q_0 \exp(-\xi^{\star 2})}{l\sigma^\star} \tag{4.23a}$$

$$k = \frac{q_0 c \sigma^\star \exp(-\xi^{\star 2})}{l\xi^{\star 2}}, \tag{4.23b}$$

where $\sigma^\star$ is given by (4.3) and $\xi^\star$ is the only one solution to the equation:

$$x \operatorname{erf}(x) \exp(x^2) = \frac{c(T_0 - T_m)}{l\sqrt{\pi}}, \quad x > 0. \tag{4.24}$$

Table 2 summarizes the formulae obtained for the two unknown thermal coefficients and the condition that data must verify to obtain them, for each one of the six possible choices for the two unknown thermal coefficients among k, ρ, l and c in Problem (1.2) when $\alpha \to 1^-$.

Conclusions

In this article we have considered a semi-infinite one-dimensional phase-change material with two unknown constant thermal coefficients. These were assumed to be among the latent heat per unit mass, the specific heat, the mass density and the thermal conductivity. The determination of them have been done through an inverse one-phase fractional Stefan problem with an overspecified condition at the fixed boundary of the material and a known evolution of the moving boundary. It was considered that this problem corresponds to a melting process with latent-heat memory effects, which we have represented by replacing the classical time derivative involved in the diffusion equation and the Stefan condition, by a time fractional derivative of order α $(0 < \alpha < 1)$ in the Caputo sense. Solutions of similarity type were looked for and necessary and sufficient conditions on data to have their existence and uniqueness were given for each of the six inverse fractional Stefan problems that arise according to the choice of the two unknown thermal coefficients. We have also obtained explicit expressions for the temperature and the two unknown thermal coefficients. Finally, we have compared our results with those obtained for the determination of two coefficients through the classical statement $(\alpha = 1)$ of the inverse Stefan problem and we have proved the convergence of our results to those obtained by the classic case.

Case	Formulae for the unknown thermal coefficients	$\star$ Equation for $\xi^\star$ $*$ Restriction for data
1	$c = \frac{k}{\rho}\left(\frac{\xi^\star}{\sigma^\star}\right)^2$	$\star\ \frac{\operatorname{erf}(x)}{x} = \frac{k(T_0-T_m)}{q_0\sigma^\star\sqrt{\pi}},\ x>0$
	$l = \frac{q_0\exp(-\xi^{\star 2})}{\rho\sigma^\star}$	$*\ \frac{k(T_0-T_m)}{2\sigma^\star q_0} < 1$
2	$c = \frac{q_0\sqrt{\pi\xi^\star\operatorname{erf}(\xi^\star)}}{\rho\sigma^\star(T_0-T_m)}$	$\star\ \exp(x^2) = \frac{q_0}{\rho l\sigma^\star},\ x>0$
	$k = \frac{\sigma^\star q_0\sqrt{\pi\operatorname{erf}(\xi^2)}}{(T_0-T_m)\xi^\star}$	$*\ \frac{q_0}{\rho l\sigma^\star}$
3	$k = \rho c\left(\frac{\sigma^\star}{\xi^\star}\right)^2$	$\star\, x\operatorname{erf}(x) = \frac{\rho c\sigma^\star(T_0-T_m)}{q_0\sqrt{\pi}},\ x>0$
	$l = \frac{q_0\exp(-\xi^{\star 2})}{\rho\sigma^\star}$	$*\, -$
4	$c = \frac{kl\xi^{\star 2}\exp(\xi^{\star 2})}{q_0\sigma^\star}$	$\star\ \frac{\operatorname{erf}(x)}{x} = \frac{k(T_0-T_m)}{q_0\sigma^\star\sqrt{\pi}},\ x>0$
	$\rho = \frac{q_0\exp(-\xi^{\star 2})}{l\sigma^\star}$	$*\ \frac{k(T_0-T_m)}{2\sigma^\star q_0} < 1$
5	$l = \frac{q_0 c\sigma^\star\exp(-\xi^{\star 2})}{k\xi^{\star 2}}$	$\star\ \frac{\operatorname{erf}(x)}{x} = \frac{k(T_0-T_m)}{q_0\sigma^\star\sqrt{\pi}},\ x>0$
	$\rho = \frac{k}{\rho}\left(\frac{\xi^\star}{\sigma^\star}\right)^2$	$*\ \frac{k(T_0-T_m)}{2\sigma^\star q_0} < 1$
6	$\rho = \frac{q_0\exp(-\xi^{\star 2})}{l\sigma^\star}$	$\star\, x\operatorname{erf}(x)\exp(x^2) = \frac{c(T_0-T_m)}{l\sqrt{\pi}},\ x>0$
	$k = \frac{q_0 c\sigma^\star\exp(-\xi^{\star 2})}{l\xi^{\star 2}}$	$*\, -$

TABLE 2. Formulae for the two unknown thermal coefficients and restriction on data for Problem (1.2) when $\alpha \to 1^-$

Time fractional diffusive equations have been used to model sub-diffusive processes and phenomena evolving under the effects of memory. This, together with the emerging and strong mathematical development on fractional calculus, motivates scientists to reach a deeper understanding of time fractional Stefan problems for the diffusive equation. In this regard, there are several open problems to be answered. The main of them it might be the one related to the mathematical derivation of fractional Stefan problems from physical assumptions of a system evolving under the effects of memory. Inwards the mathematical world, there are many other problems to be solved. Numerical treatment of fractional Stefan problems, for example, is faced with the problem that diffusive models seems not to converge to sharp front models (see [37]). This, an several other problems such us the study of two-phase fractional Stefan problems or the determination of thermal coefficients through them, make the study of time fractional Stefan problems an interesting area to look at.

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