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Similarity solution for a two-phase one-dimensional Stefan problem with a convective boundary condition and a mushy zone model

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Abstract A two-phase solidification process for a one-dimensional semi-infinite material is considered. It is assumed that it is ensued from a constant bulk temperature present in the vicinity of the fixed boundary, which it is modelled through a convective condition (Robin condition). The interface between the two phases is idealized as a mushy region and it is represented following the model of Solomon, Wilson, and Alexiades. An exact similarity solution is obtained when a restriction on data is verified, and it is analysed the relation between the problem considered here and the problem with a temperature condition at the fixed boundary. Moreover, it is proved that the solution to the problem with the convective boundary condition converges to the solution to a problem with a temperature condition when the heat transfer coefficient at the fixed boundary goes to infinity, and it is given an estimation of the difference between these two solutions. Results in this article complete and improve the ones obtained in Tarzia (Comput Appl Math 9:201–211, 1990).

Keywords Stefan problems · Explicit solutions · Similarity solutions · Convective condition · Phase-change process

Mathematics Subject Classification 26A33 · 35C05 · 35R35 · 80A22

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1 Introduction

Phase-change processes involving solidification or melting are present in a large number of phenomena related to physics, engineering, chemistry, etc., and they have widely been studied since several decades. Some reference books in the subject are Alexiades and Solomon (1993), Cannon (1984), Crank (1984), Fasano (2005), Gupta (2003), Lunardini (1991), Rubinstein (1971), and a review of a long bibliography on moving and free boundary value problems for the heat equation can be consulted in Tarzia (2000). Sometimes, liquid in solidification processes is cooled until the phase-change temperature without becoming solid. This implies the presence of a region in the phase-change process containing the material at a special solid–liquid state, which is known as *mushy region* (Alexiades and Solomon 1993; Crank 1984; Gupta 2003). In this article, we consider a one-dimensional semi-infinite homogeneous material undergoing a two-phase solidification process with a mushy zone. This sort of problems was studied in Tarzia (1990) for boundary conditions of Dirichlet or heat flux type. We follow it, which is inspired by the model given for Solomon, Wilson and Alexiades in Solomon et al. (1982) for the one-phase case, to represent the mushy region. Encouraged by the recent relation between the classical (absence of mushy zone) two-phase Stefan problems with temperature and convective boundary conditions (Tarzia 2017), we consider here the following free boundary value problem:

$$\alpha_1\theta_{1_{xx}}(x,t)=\theta_{1_t}(x,t) \qquad 0<x<s(t),\quad t>0, \tag{1a}$$
$$\alpha_2\theta_{2_{xx}}(x,t)=\theta_{2_t}(x,t) \qquad x>r(t),\quad t>0, \tag{1b}$$
$$s(0)=r(0)=0, \tag{1c}$$
$$\theta_1(s(t),t)=\theta_2(r(t),t)=0 \qquad t>0, \tag{1d}$$
$$\theta_2(x,0)=\theta_2(+\infty,t)=\theta_0 \qquad x>0,\quad t>0, \tag{1e}$$
$$k_1\theta_{1_x}(s(t),t)-k_2\theta_{2_x}(r(t),t)=\rho l[\epsilon\dot{s}(t)-(1-\epsilon)\dot{r}(t)] \qquad t>0, \tag{1f}$$
$$\theta_{1_x}(s(t),t)(r(t)-s(t))=\gamma \qquad t>0, \tag{1g}$$
$$k_1\theta_{1_x}(0,t)=\frac{h_0}{\sqrt{t}}(\theta_1(0,t)+D_\infty) \qquad t>0, \tag{1h}$$

where the unknowns are:

θ_1 : temperature of the solid region (°C)
θ_2 : temperature of the liquid region (°C)
s : free boundary separating the mushy zone and the solid phase (m)
r : free boundary separating the mushy zone and the liquid phase (m)

and the physical parameters involved in the model are:

$\rho>0$: mass density (kg/m^3)
$k>0$: thermal conductivity [W/(m°C)]
$c>0$: specific heat [J/(kg°C)]
$l>0$: latent heat per unit mass (J/kg)
$0<\epsilon<1$: coefficient characterizing the amount of latent heat contained in the mushy region (dimensionless)
$\gamma>0$: coefficient characterizing the width of the mushy region (°C)
$\theta_0>0$: initial temperature of the material (°C)
$-D_\infty<0$: external bulk temperature at the boundary $x=0$ (°C)
$h_0>0$: coefficient characterizing the heat transfer at the boundary $x=0$ [kg/(°C s$^{5/2}$)]

$\alpha = \frac{k}{\rho c} > 0$: thermal diffusivity ($m^2\, s^{-1}$)

and the subscripts 1 and 2 refer to solid and liquid phases, respectively.

We note that we are making the following assumptions on the mushy region (Tarzia 1990, 2015b; Solomon et al. 1982):

1. It is isothermal at the phase-change temperature, which we are considering equal to 0 °C.
2. It contains a fixed portion of the total latent heat per unit mass [see condition (1f)].
3. Its width is inversely proportional to the gradient of temperature [see condition (1g)].

We also observe that, by considering the convective boundary condition (1h), we are thinking of a solidification process ensued due to the constant temperature $-D_\infty$ present in the vicinity of the fixed boundary $x = 0$ of the material, which is often represented through physically less appropriate boundary conditions of Dirichlet type (Carslaw and Jaeger 1959). Convective boundary conditions have been also used in the context of phase-change processes in, for example, Zubair and Chaudhry (1994), Beckett (1991), Cadwell and Kwan (2009), Foss (1978), Grzymkowski et al. (2013), Huang and Shil (1975), Lu (2000), Roday and Kazmiercza (2009), Sadoun et al. (2009), Singh et al. (2011), Wu and Wang (1994), Briozzo and Tarzia (1998), Boadbridge (1990). Especially, a heat transfer coefficient inversely proportional to the square root of time time, it was also considered in Zubair and Chaudhry (1994). Finally, we note that we are assuming that the temperature varies moderately. This allows us to consider (piecewise) constant thermophysical properties (Alexiades and Solomon 1993; Carslaw and Jaeger 1959). Even though when the latter adjusts well with most materials, more realistic approaches could be considered. For example, a density jump through the interface or a temperature-dependent thermal conductivity might be introduced in the model. As examples of these sort of generalizations we mention here Ceretani and Tarzia (2014) and Solomon et al. (1982).

In the following (Sect. 2), we give a characterization for the existence and uniqueness of an explicit similarity solution to problem (1) in terms of the existence and uniqueness of a positive solution to a transcendental equation. We then prove that it has only one solution if and only if data verify a certain condition. Then, (Sect. 3), we analyse the relation of problem (1) with the problem ($1^\star$) given by (1a)–(1g) and the following temperature boundary condition:

$$\theta_1(0,t) = -D_0, \quad t > 0 \quad (D_0 > 0), \tag{1h*}$$

and we establish when both problems are equivalent. Finally, (Sect. 4), we prove that the solution to problem (1) converges to the solution to problem $(1^\star)_\infty$, that is the special case of problem ($1^\star$) in which the temperature boundary condition is given by:

$$\theta_1(0,t) = -D_\infty, \quad t > 0, \quad (1\mathrm{h}^\star)_\infty$$

when the heat transfer coefficient goes to infinity. Moreover, we obtain that the difference between the two solutions is $\mathcal{O}\left(\frac{1}{h_0}\right)$ when $h_0 \to \infty$.

2 Existence and uniqueness of solution

In this section, we will look for a similarity solution to problem (1). By following the classical method of Neumann (Weber 1912), that is, by introducing the similarity variables:

$$\eta_1 = \frac{x}{2\sqrt{\alpha_1 t}} \quad \text{and} \quad \eta_2 = \frac{x}{2\sqrt{\alpha_2 t}},$$

and proposing a solution defined by:

$$\begin{aligned} \theta_1(x,t) &= \theta_1(\eta_1) & 0 < x < s(t), \quad t > 0, \\ \theta_2(x,t) &= \theta_2(\eta_2) & x > r(t), \quad t > 0, \\ s(t) &= 2\xi\sqrt{\alpha_1 t} & t > 0, \\ r(t) &= 2\mu\sqrt{\alpha_2 t} & t > 0, \end{aligned}$$

with ξ and μ positive numbers to be determined, we obtain that θ_1 and θ_2 must be given by:

$$\begin{aligned} \theta_1(x,t) &= A_1 + B_1 \operatorname{erf}(\eta_1) & 0 < x < s(t), \quad t > 0, \\ \theta_2(x,t) &= A_2 + B_2 \operatorname{erf}(\eta_2) & x > r(t), \quad t > 0, \end{aligned}$$

where A_1, A_2, B_1, and B_2 are real numbers that must be specified from conditions (1d)–(1h), and erf is the *error function* defined by:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-y^2)\, dy, \quad x > 0.$$

Through conditions (1d) and (1h), we obtain that:

$$A_1 = -\frac{D_\infty \operatorname{erf}(\xi)}{\operatorname{erf}(\xi) + \frac{k_1}{h_0\sqrt{\alpha_1 \pi}}} \quad \text{and} \quad B_1 = \frac{D_\infty}{\operatorname{erf}(\xi) + \frac{k_1}{h_0\sqrt{\pi\alpha_1}}},$$

and from conditions (1d) and (1e) that:

$$A_2 = -\frac{\theta_0 \operatorname{erf}(\mu)}{\operatorname{erfc}(\mu)} \quad \text{and} \quad B_2 = \frac{\theta_0}{1 - \operatorname{erfc}(\mu)},$$

where erfc is the *complementary error function* defined by:

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x), \quad x > 0.$$

Exploiting condition (1g) we have that the parameters ξ and μ, which characterize the two free boundaries of the mushy region, are related as:

$$\mu = \sqrt{\alpha_{12}}\, W(\xi), \tag{2}$$

where α_{12} is the number defined by:

$$\alpha_{12} = \frac{\alpha_1}{\alpha_2} > 0,$$

and W is the function defined by:

$$W(x) = x + \frac{\gamma\sqrt{\pi}}{2D_\infty} \exp(x^2) \left(\operatorname{erf}(x) + \frac{k_1}{h_0\sqrt{\alpha_1 \pi}} \right), \quad x > 0. \tag{3}$$

Finally, through condition (1f), we have that ξ must be such that:

$$F(\xi) = \frac{l\sqrt{\pi}}{D_\infty c_1} G(\xi),$$

where F and G are the functions defined by:

$$F(x) = \frac{\exp(-x^2)}{\operatorname{erf}(x) + \frac{k_1}{h_0\sqrt{\alpha_1\pi}}} - \frac{\theta_0\sqrt{k_2 c_2}}{D_\infty\sqrt{k_1 c_1}} \frac{\exp\left(-\alpha_{12} W^2(x)\right)}{\operatorname{erfc}\left(\sqrt{\alpha_{12}}\, W(x)\right)} \qquad x > 0, \tag{4a}$$

$$G(x) = x + \frac{(1-\epsilon)\gamma\sqrt{\pi}}{2D_\infty}\exp(x^2)\left(\operatorname{erf}(x) + \frac{k_1}{h_0\sqrt{\alpha_1\pi}}\right) \qquad x > 0. \tag{4b}$$

Then, we have the following result:

Theorem 2.1 *The Stefan problem (1) has the similarity solution θ_1, θ_2, s, and r given by:*

$$\theta_1(x,t) = -\frac{D_\infty \operatorname{erf}(\xi)}{\operatorname{erf}(\xi) + \frac{k_1}{h_0\sqrt{\pi\alpha_1}}}\left(1 - \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_1 t}}\right)}{\operatorname{erf}(\xi)}\right) \qquad 0 < x < s(t), \quad t > 0, \tag{5a}$$

$$\theta_2(x,t) = \frac{\theta_0 \operatorname{erf}(\mu)}{\operatorname{erfc}(\mu)}\left(\frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_2 t}}\right)}{\operatorname{erf}(\mu)} - 1\right) \qquad x > r(t), \quad t > 0, \tag{5b}$$

$$s(t) = 2\xi\sqrt{\alpha_1 t} \qquad t > 0, \tag{5c}$$

$$r(t) = 2\mu\sqrt{\alpha_2 t} \qquad t > 0, \tag{5d}$$

with μ given by (2), if and only if ξ is a solution to the equation:

$$F(x) = \frac{l\sqrt{\pi}}{D_\infty c_1} G(x), \quad x > 0, \tag{6}$$

where F and G are the functions defined in (4).

Therefore, finding a similarity solution to problem (1) reduces to studying Eq. (6). We begin this by introducing some functions related to Eq. (6) and some properties of them. Let F_1 and F_2 be the functions defined by:

$$F_1(x) = \frac{\exp(-x^2)}{\operatorname{erf}(x) + \frac{k_1}{h_0\sqrt{\alpha_1\pi}}}, \qquad x > 0, \tag{7}$$

$$F_2(x) = \frac{\exp(-x^2)}{\operatorname{erfc}(x)}, \qquad x > 0. \tag{8}$$

Then, (4a) can be rewritten as:

$$F(x) = F_1(x) - \frac{\theta_0\sqrt{k_2 c_2}}{D_\infty\sqrt{k_1 c_1}} F_2\left(\sqrt{\alpha_{12}} W(x)\right), \quad x > 0. \tag{9}$$

Lemma 2.1 *1. The functions W, F_1, and F_2 defined by (3), (7), (8), respectively, verify:*

$$W(0^+) = \frac{\gamma k_1}{2D_\infty h_0\sqrt{\alpha_1}}, \qquad W(+\infty) = +\infty, \qquad W'(x) > 0 \ \ \forall x > 0, \tag{10a}$$

$$F_1(0^+) = \frac{h_0\sqrt{\alpha_1\pi}}{k_1} > 0, \qquad F_1(+\infty) = 0, \qquad F_1'(x) < 0 \ \ \forall x > 0, \tag{10b}$$

$$F_2(0^+) = 1, \qquad F_2(+\infty) = +\infty, \qquad F_2'(x) > 0 \ \ \forall x > 0. \tag{10c}$$

2. The function F defined by (4a) verifies:

$$F(0^+) = \frac{h_0\sqrt{\alpha_1\pi}}{k_1} - \frac{\theta_0\sqrt{k_2c_2}}{D_\infty\sqrt{k_1c_1}} F_2\left(\frac{\gamma k_1}{2D_\infty h_0\sqrt{\alpha_2}}\right),$$
$$F(+\infty) = -\infty, \quad F'(x) < 0 \ \ \forall x > 0. \tag{11}$$

3. *The function G defined by (4b) verifies:*

$$G(0^+) = \frac{(1-\epsilon)\gamma k_1}{2D_\infty h_0\sqrt{\alpha_1}}, \quad G(+\infty) = +\infty, \quad G'(x) > 0 \;\; \forall x > 0. \tag{12}$$

Proof It follows from elementary computations. □

Then, we have:

Theorem 2.2 *Equation (6) has an only one positive solution if and only if the coefficient h_0 verifies the following inequality:*

$$h_0 > h_0^\star, \tag{13}$$

where $h_0^\star$ is defined by:

$$h_0^\star = \frac{\gamma k_1}{2D_\infty \eta\sqrt{\alpha_2}}, \tag{14}$$

with $\eta = \eta\left(\frac{\gamma k_1}{\theta_0 k_2}, \frac{(1-\epsilon)l}{\theta_0 c_2}\right)$, the only one solution to the equation:

$$F_3(x) = 0, \quad x > 0, \tag{15}$$

and the function F_3 is defined by:

$$F_3(x) = F_2(x) - \frac{\gamma k_1\sqrt{\pi}}{2\theta_0 k_2}\frac{1}{x} + \frac{(1-\epsilon)l\sqrt{\pi}}{\theta_0 c_2}x, \quad x > 0. \tag{16}$$

Proof It follows from the properties of the functions F, G given in Lemma 2.1 that Eq. (6) admits an only one positive solution if and only if:

$$F(0^+) > \frac{l\sqrt{\pi}}{D_\infty c_1}G(0^+). \tag{17}$$

Let us observe that, using the function F_3 given by (16), (17) can be rewritten as:

$$F_3\left(\frac{\gamma k_1}{2D_\infty h_0\sqrt{\alpha_2}}\right) < 0. \tag{18}$$

Let F_4 be the function defined by:

$$F_4(x) = \frac{\gamma k_1\sqrt{\pi}}{2\theta_0 k_2}\frac{1}{x} - \frac{(1-\epsilon)l\sqrt{\pi}}{\theta_0 c_2}x, \quad x > 0.$$

Since

$$F_3(x) = F_2(x) - F_4(x), \quad x > 0,$$

it follows from the properties of the function F_2 given in Lemma 2.1 and the fact that F_4 verifies:

$$F_4(0^+) = -\infty, \quad F_4(+\infty) = +\infty, \quad F_4'(x) < 0 \quad x > 0,$$

that F_3 is such that:

$$F_3(0^+) = -\infty, \quad F_3(+\infty) = +\infty, \quad F_3'(x) > 0 \quad x > 0.$$

Therefore, (18) holds if and only if:

$$0 < \frac{\gamma k_1}{2D_\infty h_0\sqrt{\alpha_2}} < \eta, \tag{19}$$

where $\eta = \eta\left(\frac{\gamma k_1}{\theta_0 k_2}, \frac{(1-\epsilon)l}{\theta_0 c_2}\right)$ is the only one positive solution to Eq. (15). Only remains to observe that inequality (19) is equivalent to (13). □

From Theorems 2.1 and 2.2, we can establish now the main result of this section:

Corollary 2.1 *The Stefan problem (1) has the similarity solution given by (5) if and only if the coefficient h_0 that characterizes the heat transfer coefficient at the boundary $x = 0$ is large enough so much as to verifies inequality (13).*

Remark 1 In Tarzia (2015b), it was obtained an explicit similarity solution for a one-phase solidification process with a mushy zone according to the model of Solomon et al. (1982). We note that Theorem 2.1 reduces to Theorem 1 in Tarzia (2015b), in which the explicit solution is established when it is considered an initial temperature for the liquid phase equal to the phase-change temperature. That is, when $\theta_0 = 0$, we have that the solution presented in this article coincides with the solution given in Tarzia (2015b) and that the hypothesis on the heat transfer coefficient under which we have the solution is equivalent to the one given in Tarzia (2015b).

In Tarzia (2017), it was obtained an explicit similarity solution for a two-phase solidification process without any mushy region. We also have that Theorem 2.1 reduces to Theorem 2 in Tarzia (2017), in which the explicit solution is obtained, if we think of a mushy region of zero thickness. In other words, when $\gamma = 0$, we have that the solution obtained here coincides with the solution given in Tarzia (2017) and that the condition for the heat transfer coefficient is equivalent to the one given there.

3 Relation between the problems with convective and temperature boundary conditions

As we have mentioned before, convective boundary conditions are physically more appropriate to represent a temperature imposed at the boundary of a material (actually, in the vicinity of) than conditions of Dirichlet type (Carslaw and Jaeger 1959). Nevertheless, Dirichlet conditions are frequently encountered in the literature modelling this sort of situations. Thus, we are interested in analysing the relationship between the problems with the two types of conditions. In other words, in how problems (1) and ($1^\star$) are related.

Let us start by considering problem (1) with h_0 satisfying condition (13). We know from Corollary 2.1 that it has the similarity solution given by (5), where ξ is the only one positive solution to Eq. (6). Since

$$\theta_1(0,t) = -\frac{D_\infty \operatorname{erf}(\xi)}{\operatorname{erf}(\xi) + \frac{k_1}{h_0\sqrt{\pi\alpha_1}}},$$

we will consider problem ($1^\star$) with D_0 defined as:

$$D_0 = \frac{D_\infty \operatorname{erf}(\xi)}{\operatorname{erf}(\xi) + \frac{k_1}{h_0\sqrt{\pi\alpha_1}}} > 0. \tag{20}$$

We know from Tarzia (1990) that this problem has the similarity solution given by:

$$\theta_1^\star(x,t) = -D_0\left(1 - \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_1 t}}\right)}{\operatorname{erf}(\xi^\star)}\right) \qquad 0 < x < s^\star(t), \quad t > 0, \tag{21a}$$

$$\theta_2^\star(x,t) = \frac{\theta_0 \operatorname{erf}(\mu^\star)}{\operatorname{erfc}(\mu^\star)}\left(\frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_2 t}}\right)}{\operatorname{erf}(\mu^\star)} - 1\right) \qquad x > r^\star(t), \quad t > 0, \tag{21b}$$

$$s^\star(t) = 2\xi^\star\sqrt{\alpha_1 t} \qquad t > 0, \tag{21c}$$
$$r^\star(t) = 2\mu^*\sqrt{\alpha_2 t} \qquad t > 0, \tag{21d}$$

where μ^* is given by:

$$\mu^* = \sqrt{\alpha_{12}}W_0(\xi^\star), \tag{22}$$

$\xi^\star$ is the only one solution to the equation:

$$F_0(x) = \frac{l\sqrt{\pi}}{D_0 c_1}G_0(x), \quad x > 0, \tag{23}$$

and W_0, F_0, and G_0 are the functions defined by:

$$W_0(x) = x + \frac{\gamma\sqrt{\pi}}{2D_0}\exp(x^2)\operatorname{erf}(x) \qquad x > 0, \tag{24a}$$
$$F_0(x) = \frac{\exp(-x^2)}{\operatorname{erf}(x)} - \frac{\theta_0\sqrt{k_2c_2}}{D_0\sqrt{k_1c_1}}\frac{\exp\left(-\alpha_{12}{W_0}^2(x)\right)}{\operatorname{erfc}\left(\sqrt{\alpha_{12}}W_0(x)\right)} \qquad x > 0, \tag{24b}$$
$$G_0(x) = x + \frac{(1-\epsilon)\gamma\sqrt{\pi}}{2D_0}\exp(x^2)\operatorname{erf}(x) \qquad x > 0. \tag{24c}$$

Exploiting the fact that ξ satisfies (6), it follows that it is also a solution to Eq. (23). In fact, when D_0 is given by (20), we have that:

$$\begin{aligned}
F_0(\xi) &= \frac{\exp(-\xi^2)}{\operatorname{erf}(\xi)} - \frac{\theta_0\sqrt{k_2c_2}}{D_\infty\sqrt{k_1c_1}}\frac{\operatorname{erf}(\xi)+\frac{k_1}{h_0\sqrt{\pi\alpha_1}}}{\operatorname{erf}(\xi)}\\
&\quad\times F_2\left(\sqrt{\alpha_{12}}\left(\xi + \frac{\gamma\sqrt{\pi}}{2D_\infty}\exp(\xi^2)\left(\operatorname{erf}(\xi)+\frac{k_1}{h_0\sqrt{\pi\alpha_1}}\right)\right)\right)\\
&= \frac{\operatorname{erf}(\xi)+\frac{k_1}{h_0\sqrt{\pi\alpha_1}}}{\operatorname{erf}(\xi)}\left[F_1(\xi) - \frac{\theta_0\sqrt{k_2c_2}}{D_\infty\sqrt{k_1c_1}}F_2\left(\sqrt{\alpha_{12}}W(\xi)\right)\right]\\
&= \frac{\operatorname{erf}(\xi)+\frac{k_1}{h_0\sqrt{\pi\alpha_1}}}{\operatorname{erf}(\xi)}F(\xi) = \frac{\operatorname{erf}(\xi)+\frac{k_1}{h_0\sqrt{\pi\alpha_1}}}{\operatorname{erf}(\xi)}\left[\frac{l\sqrt{\pi}}{D_\infty c_1}G(\xi)\right]\\
&= \frac{\operatorname{erf}(\xi)+\frac{k_1}{h_0\sqrt{\pi\alpha_1}}}{\operatorname{erf}(\xi)}\left[\frac{l\sqrt{\pi}}{D_0c_1}\frac{\operatorname{erf}(\xi)}{\operatorname{erf}(\xi)+\frac{k_1}{h_0\sqrt{\pi\alpha_1}}}\left(\xi + \frac{(1-\epsilon)\gamma\sqrt{\pi}}{2D_0}\exp(\xi^2)\operatorname{erf}(\xi)\right)\right]\\
&= \frac{l\sqrt{\pi}}{D_0c_1}G_0(\xi).
\end{aligned}$$

Therefore, $\xi = \xi^\star$. From this, it is easy to see that $\mu = \mu^\star$, $\theta_1 = \theta_1^\star$, and $\theta_2 = \theta_2^\star$.

Then, we have the following theorem:

Theorem 3.1 *If h_0 satisfies condition (13), then the similarity solution (5) to problem (1) coincides with the similarity solution (21) to problem (1*) when D_0 is given by (20).*

Let us consider now the problem (1*). It follows from Tarzia (1990) that it has the similarity solution given by (21), where $\xi^\star$ is the only one positive solution to Eq. (23). Let $D_\infty > D_0$ and let $h_0 > 0$. Since

$$k_1{\theta_1}_x^\star(0,t) = \frac{h_0}{\sqrt{t}}(\theta_1^\star(0,t) + D_\infty)$$

if and only if:

$$h_0 = \frac{k_1 D_0}{\sqrt{\pi\alpha_1}(D_\infty - D_0)\operatorname{erf}(\xi^\star)} > 0, \tag{25}$$

we will consider problem (1) with $D_\infty > D_0$ and h_0 given by (25). As before, by considering that $\xi^\star$ satisfies equation (23), it can be shown that $\xi^\star$ is a solution to equation (6). Then, we have from Theorem 2.1 that problem (1) admits the similarity solution given by (5) with $\xi = \xi^\star$. Moreover, Corollary 2.1 implies that h_0 satisfies (13), which, in this case, can be written as:

$$\operatorname{erf}(\xi^\star) < \frac{2D_\infty D_0 \eta}{\gamma(D_\infty - D_0)\sqrt{\pi\alpha_{12}}}. \tag{26}$$

Then, we have the following theorem:

Theorem 3.2 *The similarity solution* (21) *to problem* (1*) *coincides with the similarity solution (5) to problem* (1) *when $D_\infty > D_0$ and h_0 is given by* (25). *Moreover, the parameter $\xi^\star$ that characterizes the free boundary separating the solid phase and the mushy region verifies the following inequality:*

$$\operatorname{erf}(\xi^\star) < \min\left\{1, \frac{2D_\infty D_0 \eta}{\gamma(D_\infty - D_0)\sqrt{\pi\alpha_{12}}}\right\}, \tag{27}$$

where η is the only one solution to Eq. (15).

Therefore, in the sense established by Theorems 3.1 and 3.2, we have that problems (1) and (1*) are equivalent.

Corollary 3.1 *The parameter $\xi^\star$ that characterizes the free boundary separating the solid and mushy regions in problem* (1*) *verifies the following inequality:*

$$\operatorname{erf}(\xi^\star) \le \min\left\{1, \frac{2D_0 \eta}{\gamma\sqrt{\pi\alpha_{12}}}\right\}, \tag{28}$$

where η is the only one solution to Eq. (15).

Proof It follows by making $D_\infty \to \infty$ into both sides of (26). □

Remark 2 Inequality (28), which is physically relevant when $\frac{2D_0\eta}{\gamma\sqrt{\pi\alpha_{12}}} < 1$, has already been obtained in Tarzia (1990) through the relationship between problem (1*) and the problem consisting in (1a)–(1g) and the following flux boundary condition:

$$k_1\theta_{1x}(0, t) = \frac{q_0}{\sqrt{t}}, \quad t > 0 \quad (q_0 > 0).$$

4 Asymptotic behaviour when $h_0 \to +\infty$

From a physical point of view, if we were able to consider an infinite heat transfer coefficient at $x = 0$, the convective boundary condition (1h) could be replaced by the temperature boundary condition (1h*)$_\infty$. Thus, it is reasonable to expect that the solution to problem (1) converges to the solution to problem (1*)$_\infty$ when the heat transfer coefficient increases its value. In this section, we will analyse this sort of convergence, which was already proved for some other Stefan problems in Ceretani and Tarzia (2014), Ceretani and Tarzia (2015), Ceretani and Tarzia (2016).

For each h_0 satisfying (13), we will consider problem (1) and we will denote its solution as θ_{1,h_0}, θ_{2,h_0}, s_{h_0}, and r_{h_0}. The solution to problem $(1^\star)_\infty$ will be referred to as $\theta^\star_{1,\infty}$, $\theta^\star_{2,\infty}$, $s^\star_\infty$, and $r^\star_\infty$.

The main result of this section is as follows:

Theorem 4.1 *The solution to problem* (1) *given by* (5) *punctually converges to the solution to problem* $(1^\star)_\infty$ *given by* (21)*, when* $h_0 \to \infty$*. Moreover, the following estimations holds when* $h_0 \to \infty$*:*

$$\theta_{1,h_0}(x,t) - \theta_{1,\infty}(x,t) = \mathcal{O}\left(\frac{1}{h_0}\right) \qquad \forall\, x > 0,\ t > 0, \tag{29a}$$

$$\theta_{2,h_0}(x,t) - \theta_{2,\infty}(x,t) = \mathcal{O}\left(\frac{1}{h_0}\right) \qquad \forall\, x > 0,\ t > 0, \tag{29b}$$

$$s_{h_0}(t) - s_\infty(t) = \mathcal{O}\left(\frac{1}{h_0}\right) \qquad t > 0, \tag{29c}$$

$$r_{h_0}(t) - r_\infty(t) = \mathcal{O}\left(\frac{1}{h_0}\right) \qquad t > 0. \tag{29d}$$

The key to prove Theorem 4.1 is the fact that $\xi_{h_0} - \xi_\infty = \mathcal{O}\left(\frac{1}{h_0}\right)$ when $h_0 \to \infty$. We will first prove it and then we will back and give the demonstration of Theorem 4.1.

Hereinafter, we will refer to the functions F, G, W, F_1 related to problem (1), as F_{h_0}, G_{h_0}, W_{h_0}, F_{1,h_0}, respectively. Analogously, we will refer to the functions F_0, G_0, W_0 associated with condition $(1\mathrm{h}^\star)_\infty$, as F_∞, G_∞, W_∞. That is, F_∞, G_∞, W_∞ will be the functions defined by:

$$F_\infty(x) = \frac{\exp(-x^2)}{\operatorname{erf}(x)} - \frac{\theta_0\sqrt{k_2c_2}}{D_\infty\sqrt{k_1c_1}}\frac{\exp\left(-\alpha_{12}W^2_\infty(x)\right)}{\operatorname{erfc}\left(\sqrt{\alpha_{12}}W_\infty(x)\right)} \quad x > 0, \tag{30a}$$

$$G_\infty(x) = x + \frac{(1-\epsilon)\gamma\sqrt{\pi}}{2D_\infty}\exp(x^2)\operatorname{erf}(x) \qquad x > 0, \tag{30b}$$

$$W_\infty(x) = x + \frac{\gamma\sqrt{\pi}}{2D_\infty}\exp(x^2)\operatorname{erf}(x) \qquad x > 0. \tag{30c}$$

Finally, let J_{h_0} J_∞ be the functions defined by:

$$J_{h_0}(x) = \frac{F_{h_0}(x)}{G_{h_0}(x)}, \qquad x > 0, \tag{31a}$$

$$J_\infty(x) = \frac{F_\infty(x)}{G_\infty(x)}, \qquad x > 0. \tag{31b}$$

Using the functions H_{h_0} and H_∞ defined by:

$$H_{h_0}(x) = \frac{G_{h_0}(x)}{F_{1,h_0}(x)}, \qquad x > 0, \tag{32a}$$

$$H_\infty(x) = \frac{G_\infty(x)}{F_{1,\infty(x)}}, \qquad x > 0, \tag{32b}$$

where the $F_{1,\infty}$ is the function given by:

$$F_{1,\infty}(x) = \frac{\exp(-x^2)}{\operatorname{erf}(x)}, \quad x > 0,$$

it follows that (31) can be written as:

$$J_{h_0}(x) = \frac{1}{H_{h_0}(x)} - \frac{\theta_0\sqrt{k_2c_2}}{D_\infty\sqrt{k_1c_1}} \frac{F_2\left(\sqrt{\alpha_{12}}W_{h_0}(x)\right)}{G_{h_0}(x)}, \qquad x > 0 \tag{33a}$$

$$J_\infty(x) = \frac{1}{H_\infty(x)} - \frac{\theta_0\sqrt{k_2c_2}}{D_\infty\sqrt{k_1c_1}} \frac{F_2\left(\sqrt{\alpha_{12}}W_\infty(x)\right)}{G_\infty(x)}, \qquad x > 0 \tag{33b}$$

Lemma 4.1 *1. The function J_{h_0} defined by (31a) verifies:*

$$J_{h_0}(0^+) > 0 \qquad \forall\, h_0 \geq h_1^\star, \tag{34a}$$

$$J'_{h_0}(x) < 0 \qquad \forall\, x \in (0, \nu_{h_0}),\ \forall\, h_0 \geq h_1^\star, \tag{34b}$$

where $h_1^\star$ is a positive number, such that:

$$\frac{1}{h_1^\star} F_2\left(\frac{\gamma k_1}{2D_\infty\sqrt{\alpha_2}} \frac{1}{h_1^\star}\right) < \zeta, \tag{35}$$

with:

$$\zeta = \frac{D_\infty\sqrt{\pi}}{\theta_0\sqrt{\rho k_2 c_2}}, \tag{36}$$

and ν_{h_0} is the only one solution to the equation:

$$J_{h_0}(x) = 0, \quad x > 0,\ h_0 \geq h_1^\star. \tag{37}$$

2. The function J_∞ defined by (31b) verifies:

$$J_\infty(0^+) = +\infty \tag{38a}$$

$$J'_\infty(x) < 0, \quad \forall\, x \in (0, \nu_\infty), \tag{38b}$$

where ν_∞ is the only one solution to the equation:

$$J_\infty(x) = 0, \quad x > 0. \tag{39}$$

Proof 1. We have from Lemma 2.1 that:

$$\frac{1}{H_{h_0}(0^+)} = \frac{2D_\infty\alpha_1\sqrt{\pi}}{(1-\epsilon)\gamma}\left(\frac{h_0}{k_1}\right)^2$$

$$\frac{F_2\left(\sqrt{\frac{\alpha_1}{\alpha_2}}W_{h_0}(0^+)\right)}{G_{h_0}(0^+)} = \frac{2D_\infty h_0\sqrt{\alpha_1}}{(1-\epsilon)\gamma k_1} F_2\left(\frac{\gamma k_1}{2D_\infty h_0\sqrt{\alpha_2}}\right).$$

Then:

$$J_{h_0}(0^+) = \frac{2D_\infty\alpha_1\sqrt{\pi}}{(1-\epsilon)\gamma}\left(\frac{h_0}{k_1}\right)^2\left(1 - \frac{1}{h_0\zeta}F_2\left(\frac{\gamma k_1}{2D_\infty h_0\sqrt{\alpha_2}}\right)\right), \tag{41}$$

where ζ is defined by (36). Therefore, $J_{h_0}(0^+) > 0$ if and only if:

$$\frac{1}{h_0} F_2\left(\frac{\gamma k_1}{2D_\infty\sqrt{\alpha_2}} \frac{1}{h_0}\right) < \zeta. \tag{42}$$

Let F_5 be the function defined by:

$$F_5(x) = \frac{1}{x} F_2\left(\frac{1}{x}\right), \quad x > 0.$$

Since F_5 verifies:

$$F_5(0^+) = +\infty, \quad F_5(+\infty) = 0, \quad F_5'(x) < 0 \quad \forall\, x > 0,$$

it follows that there exists a positive number $h_1^\star \geq h_0^\star$ which verifies (35). Moreover, as we know from Lemma 2.1 that F_2 is an increasing function, we have that (42) holds for any $h_0 \geq h_1^\star$.
It follows from (34a) and the properties of the function F_{h_0} given in Lemma 2.1, that there exists an only one solution ν_{h_0} to the Eq. (37) for any $h_0 \geq h_1^\star$. Moreover, since

$$F_{h_0}(x) > 0 \quad \forall\, x \in (0, \nu_{h_0}),\ \forall\, h_0 \geq h_1^\star, \tag{43}$$

it follows from the Leibnitz rule and the properties of the functions F_{h_0}', G_{h_0}' given in Lemma 2.1 that (34b) holds.
2. It is similar to the proof given for J_{h_0} in the previous item.

□

Lemma 4.2 *1. Let $h_1^\star$ be as in Lemma 4.1. The sequence of functions $\left\{J_{h_0}\right\}_{h_0 \geq h_1^\star}$ has the following properties:*

(a) $J_{h_0}(x) \to J_\infty(x)$ when $h_0 \to \infty$, for all $x \in \mathbb{R}^+$.
(b) If $h_1^\star \leq h_0^{(1)} < h_0^{(2)}$, then:

$$J_{h_0^{(1)}}(x) < J_{h_0^{(2)}}(x) \quad \forall\, x \in (0, \nu_{h_0^{(1)}}), \tag{44}$$

where $\nu_{h_0^{(1)}}$ is defined as in Lemma 4.1.

2. $\left\{\xi_{h_0}\right\}_{h_0 \geq h_1^\star}$ is an increasing sequence of numbers which converges to ξ_∞ when $h_0 \to \infty$.

Proof 1. Let $h_1^\star$ be as in Lemma 4.1.

(a) It follows immediately from the definitions of J_{h_0} and J_∞.
(b) Since

$$\frac{\partial F_{1,h_0}(x)}{\partial h_0} > 0 \quad \forall\, x > 0, \tag{45}$$

it follows that:

$$\frac{\partial W_{h_0}(x)}{\partial h_0} < 0 \quad \forall\, x > 0.$$

Then, as we also know from Lemma 2.1 that F_2 is an increasing function, we have that:

$$\frac{\partial}{\partial h_0}\left(F_2\left(\sqrt{\alpha_{12}}W_{h_0}(x)\right)\right) < 0 \quad \forall\, x > 0.$$

Therefore:

$$\frac{\partial F_{h_0}(x)}{\partial h_0} > 0 \quad \forall\, x > 0. \tag{46}$$

We also have from (45) that:

$$\frac{\partial G_{h_0}(x)}{\partial h_0} < 0 \quad \forall\, x > 0. \tag{47}$$

Then, it follows from (43), (46), (47), and the Leibnitz rule that:

$$\frac{\partial J_{h_0}(x)}{\partial h_0} > 0 \quad \forall\, x \in (0, \nu_{h_0}).$$

Therefore, $\left\{\nu_{h_0}\right\}_{h_0 \geq h_1^\star}$ is an increasing sequence of numbers and (44) holds.

2. It is a direct consequence of the previous item and the definitions of ξ_{h_0} and ξ_∞ as the only one solutions to the Eqs. (6) and (23), respectively.

Lemma 4.3 *Let $h_1^\star$ be as in Lemma 4.1. Then, there exist a positive function $\mathcal{J}$ and a number $h_0^{\star\star} \geq h_1^\star$, such that:*

$$\left|J_{h_0}(x) - J_\infty(x)\right| \leq \frac{\mathcal{J}(x)}{h_0} \quad \forall x \in [\xi_{h_0^{\star\star}}, \nu_{h_0}],\ \forall h_0 \geq h_0^{\star\star}, \tag{48}$$

where ν_{h_0} is defined as in Lemma 4.1.
Therefore, the following estimations holds when $h_0 \to \infty$:

$$\xi_{h_0} - \xi_\infty = \mathcal{O}\left(\frac{1}{h_0}\right) \tag{49a}$$

$$\mu_{h_0} - \mu_\infty = \mathcal{O}\left(\frac{1}{h_0}\right) \tag{49b}$$

Proof Let be $h_0 \geq h_1^\star$. We have from Lemma 4.2 that:

$$0 < J_\infty(x) - J_{h_0}(x) = \frac{H_{h_0}(x) - H_\infty(x)}{H_\infty(x) H_{h_0}(x)} + \frac{\theta_0 \sqrt{k_2 c_2}}{D_\infty \sqrt{k_1 c_1}} \left(\frac{F_2\left(\sqrt{\alpha_{12}} W_{h_0}(x)\right)}{G_{h_0}(x)} - \frac{F_2\left(\sqrt{\alpha_{12}} W_\infty(x)\right)}{G_\infty(x)} \right), \tag{50}$$

for all $x \in [\xi_{h_1^\star}, \nu_{h_0}]$.

On one hand, we know from Tarzia (2015b) that there exist a positive function $\mathcal{J}_1$ and a number $h_0^{\star\star} \geq h_1^\star$, such that:

$$0 < H_{h_0}(x) - H_\infty(x) \leq \frac{\mathcal{J}_1(x)}{h_0}, \quad \forall x \in [\xi_{h_0^{\star\star}}, \nu_{h_0}], \quad \forall h_0 \geq h_0^{\star\star}. \tag{51}$$

Then, since $\left\{H_{h_0}\right\}_{h_0 \geq h_0^{\star\star}}$ is a decreasing sequence of functions which punctually converges to H_∞ when $h_0 \to \infty$, it follows that:

$$0 < \frac{H_{h_0}(x) - H_\infty(x)}{H_\infty(x) H_{h_0}(x)} < \frac{\mathcal{J}_2(x)}{h_0}, \quad \forall x \in [\xi_{h_0^{\star\star}}, \nu_{h_0}], \quad \forall h_0 \geq h_0^{\star\star}, \tag{52}$$

where $\mathcal{J}_2$ is the function defined by:

$$\mathcal{J}_2(x) = \frac{\mathcal{J}_1(x)}{H_\infty^2(x)}, \quad x > 0. \tag{53}$$

On the other hand, since $\left\{W_{h_0}\right\}_{h_0 \geq h_0^{\star\star}}$ is a decreasing sequence of functions which converges to W_∞ when $h_0 \to \infty$ and F_2 is an increasing function, we have that:

$$0 < F_2\left(\sqrt{\alpha_{12}} W_{h_0}(x)\right) - F_2\left(\sqrt{\alpha_{12}} W_\infty(x)\right), \quad \forall x \in [\xi_{h_0^{\star\star}}, \nu_{h_0}], \quad \forall h_0 \geq h_0^{\star\star}. \tag{54}$$

Then, as $\left\{G_{h_0}\right\}_{h_0 \geq h_0^{\star\star}}$ is a decreasing sequence of functions which punctually converges to G_∞ when $h_0 \to \infty$, it follows that:

$$\begin{aligned} 0 &< \frac{F_2\left(\sqrt{\alpha_{12}} W_{h_0}(x)\right)}{G_{h_0}(x)} - \frac{F_2\left(\sqrt{\alpha_{12}} W_\infty(x)\right)}{G_\infty(x)} \\ &< \frac{1}{G_\infty(x)} \left(F_2\left(\sqrt{\alpha_{12}} W_{h_0}(x)\right) - F_2\left(\sqrt{\alpha_{12}} W_\infty(x)\right)\right) \\ &< \frac{\mathcal{J}_3(x)}{h_0}, \quad \forall x \in [\xi_{h_0^{\star\star}}, \nu_{h_0}], \quad \forall h_0 \geq h_0^{\star\star}. \end{aligned} \tag{55}$$

where $\mathcal{J}_3$ is the function defined by:

$$\mathcal{J}_3(x) = \frac{L_2 \gamma k_1}{2D_\infty \sqrt{\alpha_2}} \frac{\exp(x^2)}{G_\infty(x)}, \quad x > 0, \tag{56}$$

and L_2 is a Lipschitz constant for F_2 in $\left[W_\infty(\xi_{h_0^{\star\star}}), W_{h_0^{\star\star}}(\nu_\infty)\right]$. Henceforth, we have from (50), (52), and (55) that (48) holds when we consider the function $\mathcal{J}$ defined by:

$$\mathcal{J}(x) = \mathcal{J}_2(x) + \frac{\theta_0 \sqrt{k_2 c_2}}{D_\infty \sqrt{k_1 c_1}} \mathcal{J}_3(x), \quad x > 0. \tag{57}$$

To prove (49a), we will use some geometric arguments. Let T be the right triangle with vertices $P_1(\xi_{h_0}, J_{h_0}(\xi_{h_0}))$, $P_2(\xi_{h_0}, J_\infty(\xi_{h_0}))$, and $P_3(\xi_\infty, J_\infty(\xi_\infty))$. Then, we have that:

$$0 < \xi_\infty - \xi_{h_0} = \frac{J_\infty(\xi_{h_0}) - J_{h_0}(\xi_{h_0})}{\tan(\alpha_{h_0})}, \tag{58}$$

where α_{h_0} is the inner angle of T with vertex P_3. Let also be $\tan(\widetilde{\alpha}_{h_0})$, $\widetilde{\alpha}_{h_0} \in (0, \pi)$, the slope of the secant line to the graph of J_∞ which contains the points P_2 and P_3, and let be $\tan(\widetilde{\beta}, \beta \in (0, \pi)$, the slope of the tangent line at P_3 of the same graph. Since $\xi_{h_0} < \xi_\infty$ and J_∞ is a decreasing convex function in $[\xi_{h_0^{\star\star}}, \nu_\infty]$, we have that:

$$\widetilde{\alpha}_{h_0} < \beta \quad \text{and} \quad \widetilde{\alpha}_{h_0}, \beta \in \left(\frac{\pi}{2}, \pi\right).$$

Then:

$$\tan(\alpha_{h_0}) > \tan(-\beta) = -J'_\infty(\xi_\infty) > 0, \tag{59}$$

since $\alpha_{h_0} = \pi - \widetilde{\alpha}_{h_0}$. Therefore, it follows from (48), (58), and (59) that:

$$0 < \xi_\infty - \xi_{h_0} < \frac{\mathcal{J}(\xi_{h_0})}{-J'_\infty(\xi_\infty)} \frac{1}{h_0} \quad \forall\, h_0 \geq h_0^{\star\star}. \tag{60}$$

We know from Tarzia (2015a) that $\mathcal{J}_1$ can be considered as given by:

$$\mathcal{J}_1(x) = \frac{k}{\sqrt{\pi \alpha_1}} \frac{\exp(-x^2)}{\operatorname{erf}^2(x)} \left(x + \gamma(1-\epsilon) \frac{\sqrt{\pi}}{D_\infty} \frac{1}{F_{1,h_0^\star}(x)} \right) \frac{1}{F_{1,\infty}(x) F_{1,h_0^\star}(x)}.$$

Then:

$$\mathcal{J}_2(\xi_{h_0}) = \frac{F_{1,\infty}(\xi_{h_0})}{G_\infty^2(\xi_{h_0})} \frac{k}{\sqrt{\pi \alpha_1}} \frac{\exp(-\xi_{h_0}^2)}{\operatorname{erf}^2(\xi_{h_0})} \left(\xi_{h_0} + \gamma(1-\epsilon) \frac{\sqrt{\pi}}{D_\infty} \frac{1}{F_{1,h_0^\star}(\xi_{h_0})} \right) \frac{1}{F_{1,h_0^\star}(\xi_{h_0})} < \mathcal{M}_1, \tag{61}$$

where $\mathcal{M}_1$ is the number defined by:

$$\mathcal{M}_1 = \frac{k}{\sqrt{\pi \alpha_1}} \frac{F_{1,\infty}(\xi_{h_0^{\star\star}})}{G_\infty^2(\xi_{h_0^{\star\star}}) F_{1,h_0^\star}(\nu_\infty) \operatorname{erf}^2(\xi_{h_0^{\star\star}})} \left(\nu_\infty + \frac{\gamma(1-\epsilon)\sqrt{\pi}}{D_\infty} \frac{1}{F_{1,h_0^\star}(\nu_\infty)} \right) > 0.$$

We also have that:

$$\frac{\theta_0 \sqrt{k_2 c_2}}{D_\infty \sqrt{k_1 c_1}} \mathcal{J}_3(x) < \mathcal{M}_2, \tag{62}$$

where $\mathcal{M}_2$ is the number defined by:

$$\mathcal{M}_2 = \frac{\theta_0 L \gamma k_1 \sqrt{k_2 c_2}}{2 D_\infty^2 \sqrt{k_1 c_1 \alpha_2}} \frac{\exp(\nu_\infty^2)}{G_\infty(\xi_{h_0^{\star\star}})}.$$

Then, it follows from (60), (61), and (62) that:

$$0 < \xi_\infty - \xi_{h_0} < \frac{\mathcal{M}}{h_0} \quad \forall\, h_0 \geq h_0^{\star\star}, \tag{63}$$

where $\mathcal{M}$ is the number defined by:

$$\mathcal{M} = \frac{\mathcal{M}_1 + \mathcal{M}_2}{-J'_\infty(\xi_\infty)} > 0.$$

Then, (49a) holds.

Finally, we have that:

$$\begin{aligned}\left|\mu_{h_0} - \mu_\infty\right| &\leq \sqrt{\alpha_{12}}\left(\frac{\mathcal{M}}{h_0} + \frac{\gamma\sqrt{\pi}}{2D_\infty}\left(\exp(\xi_\infty^2)\operatorname{erf}(\xi_\infty) - \exp(\xi_{h_0}^2)\operatorname{erf}(\xi_{h_0})\right)\right.\\ &\quad\left. + \frac{\gamma k_1 \exp(\nu_\infty^2)}{2D_\infty\sqrt{\alpha_1}}\frac{1}{h_0}\right)\\ &\leq \frac{\mathcal{M}_3}{h_0} \quad \forall\, h_0 \geq h_0^{\star\star},\end{aligned}$$

where $\mathcal{M}_3$ is the number defined by:

$$\mathcal{M}_3 = \sqrt{\alpha_{12}}\left(\mathcal{M}\left(1 + \frac{\gamma\sqrt{\pi}L_6}{2D_\infty}\right) + \frac{\gamma k_1 \exp(\nu_\infty^2)}{2D_\infty\sqrt{\alpha_1}}\right) > 0$$

and L_6 is a Lipschitz constant in $\left[\xi_{h_0^{\star\star}}, \nu_\infty\right]$ for the function F_6 defined by:

$$F_6(x) = \exp(x^2)\operatorname{erf}(x), \quad x > 0.$$

□

We are now in a position to prove Theorem 4.1:

Proof of Theorem 4.1 Let be $x > 0$ and $t > 0$. We have that:

$$\begin{aligned}\left|\theta_{1,h_0}(x,t) - \theta_{1,\infty}(x,t)\right| &\leq \frac{D_\infty}{1 + \frac{\sqrt{\alpha_1\pi}}{k_1}\operatorname{erf}(\xi_{h_0^{\star\star}})}\frac{1}{h_0}\left[1 + \frac{1}{\operatorname{erf}(\xi_{h_0})}\left(\frac{h_0\sqrt{\alpha_1\pi}}{k_1}(\operatorname{erf}(\xi_\infty)\right.\right.\\ &\quad\left.\left. - \operatorname{erf}(\xi_{h_0})\right) + 1\right)\Bigg]\\ &\leq \frac{\mathcal{M}_{\theta_1}}{h_0} \quad \forall\, h_0 \geq h_0^{\star\star},\end{aligned}$$

where $\mathcal{M}_{\theta_1}$ is the number defined by:

$$\mathcal{M}_{\theta_1} = \frac{D_\infty}{1 + \frac{\sqrt{\alpha_1\pi}}{k_1}\operatorname{erf}(\xi_{h_0^{\star\star}})}\left[1 + \frac{1}{\operatorname{erf}(\xi_\infty)}\left(\frac{L\sqrt{\alpha_1\pi}}{k_1}\mathcal{M} + 1\right)\right] > 0$$

and L is a Lipschitz constant for the error function. Then, (29a) holds.

We also have that:

$$\left|\theta_{2,h_0}(x,t) - \theta_{2,\infty}(x,t)\right| \leq \frac{2\theta_0}{\operatorname{erfc}^2(\mu_\infty)}\left(\operatorname{erf}(\mu_\infty) - \operatorname{erf}(\mu_{h_0})\right) \leq \frac{\mathcal{M}_{\theta_2}}{h_0} \quad \forall\, h_0 \geq h_0^{\star\star},$$

where $\mathcal{M}_{\theta_2}$ is the number defined by:

$$\mathcal{M}_{\theta_2} = \frac{2\theta_0 L \mathcal{M}_3}{\operatorname{erfc}^2(\mu_\infty)} > 0.$$

Therefore, (29b) also holds.

The proofs of (29c) and (29d) follow straightforward from (49a) and (49b).

Conclusions

In this article, we have considered a two-phase solidification process for a one-dimensional semi-infinite material. We have assumed that the phase-change process starts from a constant bulk temperature imposed in the vicinity of the boundary and we have modelled it through a convective condition. Regarding the interface between solid and liquid phases, we have assumed the existence of a mushy zone and we have represented it by following the model of Solomon, Wilson, and Alexiades. Thermophysical properties were assumed to be (piecewise) constant, which is reasonable for most materials under moderate temperature variations. For this problem, we have obtained a similarity solution that depends on a dimensionless parameter, which is defined as the only one solution to a transcendental equation. Moreover, we have analysed the relationship between the problems with convective and temperature boundary conditions and we have established when both problems are equivalent. We have also proved that the solution to the problem with the temperature boundary condition can be obtained from the solution to a problem with a convective boundary condition when the heat transfer coefficient at the fixed boundary goes to infinity and we have given the order of that convergence.

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