Applied Mathematics Letters 70 (2017) 14–17

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Existence and uniqueness of the modified error function

Andrea N. Ceretani[a,b,*], Natalia N. Salva[c,d], Domingo A. Tarzia[a]

[a] *CONICET - Depto. de Matemática, Facultad de Ciencias Empresariales, Univ. Austral, Paraguay 1950, S2000FZF Rosario, Argentina*
[b] *Depto. de Matemática, Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Univ. Nacional de Rosario, Pellegrini 250, S2000BTP Rosario, Argentina*
[c] *CONICET - CNEA, Depto. de Mecánica Computacional, Centro Atómico Bariloche, Av. Bustillo 9500, 8400 Bariloche, Argentina*
[d] *Depto. de Matemática, Centro Regional Bariloche, Univ. Nacional del Comahue, Quintral 250, 8400 Bariloche, Argentina*

A R T I C L E   I N F O

Article history:
Received 2 January 2017
Received in revised form 21 February 2017
Accepted 21 February 2017
Available online 1 March 2017

Keywords:
Modified error function
Error function
Phase change problem
Temperature-dependent thermal conductivity
Nonlinear second order ordinary differential equation

A B S T R A C T

This article is devoted to the proof of the existence and uniqueness of the modified error function introduced in Cho and Sunderland (1974). This function is defined as the solution to a nonlinear second order differential problem depending on a real parameter. We prove here that this problem has a unique non-negative analytic solution when the parameter assumes small positive values.

1. Introduction

In 1974, Cho and Sunderland [1] studied a solidification process with temperature-dependent thermal conductivity and obtained an explicit similarity solution in terms of what they called a *modified error function*. This function is defined as the solution to the following nonlinear differential problem:

$$[(1+\delta y(x))y'(x)]' + 2xy'(x) = 0 \quad 0 < x < +\infty \tag{1a}$$

$$y(0) = 0 \tag{1b}$$

$$y(+\infty) = 1 \tag{1c}$$

* Corresponding author at: CONICET - Depto. de Matemática, Facultad de Ciencias Empresariales, Univ. Austral, Paraguay 1950, S2000FZF Rosario, Argentina.
E-mail address: aceretani@austral.edu.ar (A.N. Ceretani).

http://dx.doi.org/10.1016/j.aml.2017.02.017
0893-9659/

where $\delta \geq -1$ is given. Graphics for numerical solutions of (1) for different values of δ can be found in [1]. The classical error function is defined by:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2)dz, \quad x > 0, \tag{2}$$

and it is a solution to (1) when $\delta = 0$. This makes meaningful the denomination *modified error function* given for the solution to problem (1).

The modified error function has also appeared in the context of diffusion problems before 1974 [2,3]. It was also used later in several opportunities to find similarity solutions to phase-change processes, e.g. see [4,5]. It was cited in [6], where several nonlinear ordinary differential problems arise from a wide variety of fields are presented. Closed analytical solutions for Stefan problems with variable diffusivity is given in [7]. Temperature-dependent thermal coefficients are very important in thermal analysis, e.g. see [8]. Nevertheless, to the knowledge of the authors, the existence and uniqueness of the solution to problem (1) has not been yet proved. In general, the existence theorems for boundary value problems for second order ordinary differential equations include certain continuity and bounded derivatives that are not guaranteed for problem (1), even when it is reduced to a bounded domain (see, for example, [9–12]). This article is devoted to prove it for small $\delta > 0$ using a fixed point strategy.

2. Existence and uniqueness of solution to problem (1)

The main idea developed in this Section is to study problem (1) through the linear problem given by the differential equation:

$$[(1 + \delta\Psi_h(x))y'(x)]' + 2xy'(x) = 0, \quad 0 < x < +\infty, \tag{1a*}$$

and conditions (1b), (1c). The function Ψ_h in (1a*) is defined by:

$$\Psi_h(x) = 1 + \delta h(x), \quad x > 0, \tag{3}$$

where $\delta > 0$, $h \in K \subset X$ is given and:

$$X = \left\{h : \mathbb{R}_0^+ \to \mathbb{R} \,/\, h \text{ is an analytic function}, \, \|h\|_\infty < \infty\right\} \tag{4a}$$

$$K = \{h \in X \,/\, \|h\|_\infty \leq 1, \, 0 \leq h, \, h(0) = 0, \, h(+\infty) = 1\}. \tag{4b}$$

Hereinafter, we will refer to the problem given by (1a*), (1b) and (1c) as problem (1*). Let us observe that K is non-empty closed subset of the Banach space X.

The advantage in considering the linear equation (1a*) is that it can be easily solved through the substitution $v = y'$. Thus, we have the following result:

Theorem 2.1. *Let $h \in K$ and $\delta > 0$. The solution y to problem (1*) is given by:*

$$y(x) = C_h \int_0^x \frac{1}{\Psi_h(\eta)} \exp\left(-2\int_0^\eta \frac{\xi}{\Psi_h(\xi)}d\xi\right) d\eta \quad x \geq 0, \tag{5}$$

where the constant C_h is defined by:

$$C_h = \left(\int_0^{+\infty} \frac{1}{\Psi_h(\eta)} \exp\left(-2\int_0^\eta \frac{\xi}{\Psi_h(\xi)}d\xi\right) d\eta\right)^{-1}. \tag{6}$$

Proof. Let us first observe that the constant C_h given by (6) is well defined, that is, that $C_h \in \mathbb{R}$. In fact, we have:

$$|C_h^{-1}| \geq \frac{1}{1+\delta}\int_0^{+\infty} \exp(-\eta^2)d\eta = \frac{\sqrt{\pi}}{2(1+\delta)}. \tag{7}$$

Now the proof follows easily by checking that the function y given by (5) satisfies problem (1⋆). □

The following result is an immediate consequence of Theorem 2.1.

Corollary 2.1. *Let $y \in K$ and $\delta > 0$. Then y is a solution to problem (1) if and only if y is a fixed point of the operator τ from K to X defined by:*

$$\tau(h)(x) = C_h \int_0^x \frac{1}{\Psi_h(\eta)} \exp\left(-2\int_0^\eta \frac{\xi}{\Psi_h(\xi)}d\xi\right) d\eta \quad x > 0, \tag{8}$$

with C_h given by (6).

Observe that $\tau(K) \subset K$. We will now focus on analyzing when τ has only one fixed point. The estimations summarized next will be useful in the following.

Lemma 2.1. *Let $h, h_1, h_2 \in K$, $\delta > 0$ and $x \geq 0$. We have:*

(a) $\int_0^x \left| \frac{\exp\left(-2\int_0^\eta \frac{\xi}{\Psi_{h_1}(\xi)}d\xi\right)}{\Psi_{h_1}(\eta)} - \frac{\exp\left(-2\int_0^\eta \frac{\xi}{\Psi_{h_2}(\xi)}d\xi\right)}{\Psi_{h_2}(\eta)} \right| d\eta \leq \frac{\sqrt{\pi}}{4}\delta\sqrt{1+\delta}(3+\delta)\|h_1 - h_2\|_\infty$,

(b) $|C_{h_1} - C_{h_2}| \leq \frac{1}{\sqrt{\pi}}\delta\sqrt{1+\delta}(1+\delta)^2(3+\delta)\|h_1 - h_2\|_\infty$,

(c) $\int_0^x \frac{1}{\Psi_h(\eta)} \exp\left(-2\int_0^\eta \frac{\xi}{\Psi_h(\xi)}d\xi\right) d\eta \leq \frac{\sqrt{\pi(1+\delta)}}{2}$.

Proof. Let f be the real function defined on $\mathbb{R}_0^+$ by $f(x) = \exp(-2x)$. If $h_1 \leq h_2$, it follows from the Mean Value Theorem applied to function f that:

$$\begin{aligned}
&\left|\exp\left(-2\int_0^\eta \frac{\xi}{\Psi_{h_1}(\xi)}d\xi\right) - \exp\left(-2\int_0^\eta \frac{\xi}{\Psi_{h_2}(\xi)}d\xi\right)\right| \\
&= 2\exp\left(-2\int_0^\eta \frac{\xi}{\Psi_{h_3}(\xi)}d\xi\right)\left|\int_0^\eta \frac{\xi}{\Psi_{h_1}(\xi)}d\xi - \int_0^\eta \frac{\xi}{\Psi_{h_2}(\xi)}d\xi\right| \\
&\leq \delta\|h_2 - h_1\|_\infty \eta^2 \exp\left(\frac{-\eta^2}{1+\delta}\right),
\end{aligned} \tag{9}$$

where $h_1 \leq h_3 \leq h_2$. Now (a) follows from regular computations. When $h_1 \not\leq h_2$, as the LHS in (a) can be bounded for the same expression but applied to $h_m = \min\{h_1, h_2\}$ and $h_M = \max\{h_1, h_2\}$, the proof runs as before and it is completed having into consideration that $\|h_1 - h_2\|_\infty = \|h_M - h_m\|_\infty$.

The proof of (b) follows from (a), and (c) can be obtained from regular computations. □

Theorem 2.2. *Let $\delta_1 > 0$ be the only one positive solution to the equation:*

$$\frac{x}{2}(1+x)^{3/2}(3+x)[1+(1+x)^{3/2}] = 1. \tag{10}$$

If $0 < \delta < \delta_1$, then τ is a contraction.

Proof. Let g be the real function defined by:

$$g(x) = \frac{x}{2}(1+x)^{3/2}(3+x)[1+(1+x)^{3/2}] \quad x \geq 0. \tag{11}$$

Since g is an increasing function from 0 to $+\infty$, we have that Eq. (10) admits only one positive solution δ_1.

Let now $h_1, h_2 \in K$ and $x \geq 0$. From Lemma 2.1, (7) and:

$$\begin{aligned}
&|\tau(h_1)(x) - \tau(h_2)(x)| \\
&\leq C_{h_1} \int_0^x \left| \frac{\exp\left(-2\int_0^\eta \frac{\xi}{\Psi_{h_1}(\xi)} d\xi\right)}{\Psi_{h_1}(\eta)} - \frac{\exp\left(-2\int_0^\eta \frac{\xi}{\Psi_{h_2}(\xi)} d\xi\right)}{\Psi_{h_2}(\eta)} \right| d\eta \\
&\quad + |C_{h_1} - C_{h_2}| \int_0^x \frac{1}{\Psi_{h_2}(\eta)} \exp\left(-2\int_0^\eta \frac{\xi}{\Psi_{h_2}(\xi)} d\xi\right) d\eta,
\end{aligned}$$

it follows that $\|\tau(h_1) - \tau(h_2)\|_\infty \leq \gamma \|h_1 - h_2\|_\infty$, where $\gamma = g(\delta)$. Recalling that g is an increasing function, it follows that τ is a contraction when $0 < \delta < \delta_1$. □

From a numerical computation it can be found that $0.203701 < \delta_1 < 0.203702$. We are now in the position to formulate our main result.

Corollary 2.2. *Let δ_1 be as in Theorem 2.2. If $0 < \delta < \delta_1$, then problem (1) has a unique non-negative analytic solution.*

Proof. It is a direct consequence of Corollary 2.1, Theorem 2.2 and the Banach Fixed Point Theorem. □

The modified error function Φ_δ arises when looking for similarity solutions for the Stefan problem considered in [1]. It corresponds to a phase-change process with a temperature-dependent linear thermal conductivity whose slope is related in this paper to the parameter δ. In particular, Corollary 2.2 imposes a restriction to the value of this slope to ensure the existence and uniqueness of an analytic solution to the Stefan problem in [1].

Acknowledgments

This paper has been partially sponsored by the Project PIP No. 0534 from CONICET-UA (Rosario, Argentina) and AFOSR-SOARD Grant FA 9550-14-0122. The authors would like to thank the two anonymous referees for the helpful comments.

References

[1] S.H. Cho, J.E. Sunderland, Phase-change problems with temperature-dependent thermal conductivity, J. Heat Trans. 96-2 (1974) 214–217.
[2] J. Crank, The Mathematics of Diffusion, Clarendon Press, Oxford, 1956.
[3] C. Wagner, Diffusion of lead chloride dissolved in solid silver chloride, J. Chem. Phys. 18 (1950) 1227–1230.
[4] D.L.R. Oliver, J.E. Sunderland, A phase-change problem with temperature-dependent thermal conductivity and specific heat, Int. J. Heat Mass Transfer 30 (1987) 2657–2661.
[5] N.N. Salva, D.A. Tarzia, A sensitivity analysis for the determination of unknown thermal coefficients through a phase-change process with temperature-dependent thermal conductivity, Int. Commun. Heat Mass Transfer 38 (2011) 418–424.
[6] M. Countryman, R. Kannan, Nonlinear boundary value problem on semi-infinite intervals, Comput. Appl. Math. Appl. 3 (1994) 59–75.
[7] V.R. Voller, F. Falcini, Two exact solutions of a Stefan problem with varying diffusivity, Int. J. Heat Mass Transfer 58 (2013) 80–85.
[8] M.G. Sobamowo, Thermal analysis of longitudinal fin with temperature-dependent properties and internal heat generation using Galerkin's method of weighted residual, Appl. Therm. Eng. 99 (2016) 1316–1330.
[9] L.H. Erbe, Existence of solutions to boundary value problems for second order differential equations, Nonlinear Analysis TMA 6 (11) (1982) 1155–1162.
[10] Ch. Fabry, P. Habets, Upper and lower solutions for second order boundary value problems with nonlinear boundary conditions, Nonlinear Analysis TMA 10 (10) (1986) 985–1007.
[11] J.V. Baxley, Global existence and uniqueness for second-order ordinary differential equations, J. Differential Equations 23 (1977) 315–334.
[12] D. O'Regan, Existence Theory for Nonlinear Ordinary Differential Equations, Kluwer Academic, Dordrecht, 1977.