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About the convergence of a family of initial boundary value problems for a fractional diffusion equation of robin type

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ABSTRACT

We consider a family of initial boundary value problems governed by a fractional diffusion equation with Caputo derivative in time, where the parameter is the Newton heat transfer coefficient linked to the Robin condition on the boundary. For each problem we prove existence and uniqueness of solution by a Fourier approach. This will enable us to also prove the convergence of the family of solutions to the solution of the limit problem, which is obtained by replacing the Robin boundary condition with a Dirichlet boundary condition.

1. Introduction

Initial Boundary Value Problems (IBVPs) for the Fractional Diffusion Equation (FDE) have been profusely studied in the last years [1–6]. It is well known that the FDEs describe subdiffusion processes. That is, diffusion phenomena where the mean squared displacement of the particle is proportional to $t^{\alpha/2}$ instead of being proportional to $t^{1/2}$, as occurs when diffusion processes take place (see [7,8] and references therein).

Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$ and let $T>0$. Let Δ denote the usual Laplacian operator on $\mathbb{R}^d$. In this work we will study the IBVP associated to the FDE with the Caputo Derivative given by

$$D^{C\alpha}_{0t} u(\mathrm{x},t) = \Delta u(\mathrm{x},t),\ (\mathrm{x},t)\in \Omega\times(0,T), \tag{1}$$

where $\alpha\in(0,1)$ and ${}_0^C D_t^\alpha$ denotes the Caputo fractional derivative of order α in the t variable. This is defined for every absolutely continuous function $f\in AC[0,T]$ by

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} f'(\tau)\mathrm{d}\tau. \tag{2}$$

We will address the problem with a convective condition, that is, a boundary condition where the incoming flux is proportional to the temperature difference between the surface of the material and an imposed ambient temperature. This

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condition is also called Robin condition, which involves a linear combination of Dirichlet and Neumann conditions:

$$-\frac{\partial u}{\partial n}(\mathrm{x},t) = \beta[u(\mathrm{x},t) - u_\infty], \mathrm{x} \in \partial\Omega,\ 0 < t < T, \tag{3}$$

where $\frac{\partial}{\partial n}$ denotes the unitary exterior normal derivative on $\partial\Omega$, u_∞ is the external imposed temperature and the parameter β is the Newton coefficient of heat transfer, which is a positive constant that will play an important role in later analysis.

More precisely, the IBVP for FDE in time and Robin condition with parameter $\beta \in \mathbb{R}^+$ and initial data $u_0 \in L^2(\Omega)$ that we will consider is the following

$$\begin{array}{llll} (i) & D^{C\alpha}_{0t} u(\mathrm{x},t) = \Delta u(\mathrm{x},t), & \mathrm{x} \in \Omega,\ 0 < t < T, \\ (ii) & u(\mathrm{x},0) = u_0(\mathrm{x}), & \mathrm{x} \in \Omega \\ (iii) & \frac{\partial u}{\partial n}(\mathrm{x},t) + \beta u(\mathrm{x},t) = 0 & \mathrm{x} \in \partial\Omega,\ 0 < t < T. \end{array} \tag{4}$$

Notice that all the physical parameters involved in heat transfer, such as density, thermal conductivity or specific heat are considered constant and equal one, for the sake of simplicity, and the imposed temperature $u_\infty = 0$. We will refer to problem (4) as the $\beta-$IBVP.

Second-order parabolic equations in multidimensional domains are usually treated by using variational calculus techniques (see for example [9–11] among several others), and one of the key tools used in the proofs (for example, proofs for existence and uniqueness) is the application of the following property

$$\frac{d}{dt}|f(t)|^2 = 2\left(\frac{d}{dt}f(t), f(t)\right) \tag{5}$$

where the notation on the right hand side denotes an inner product, and the left hand side has the derivative of the squared of the norm, in an appopiated Hilbert space. The validity of property (5) follows from the Chain Rule. However, when working with fractional derivatives in time, we cannot deduce the same rule. Indeed, an analogous version of expression (5) for Caputo derivatives is the one given in [12, Theorem 2.1] but in contrast to the simplicity of (5), the left hand side contains, besides the fractional derivative term, other terms involving integrals depending on time with singular kernels. We will, however, use some variational techniques throughout the process related to the spatial variable. Let us also recall that in [13] a unique existence result of strong solution is obtained for the same problem with Robin conditions. This solution is given as the linear combination of the single-layer potential, the volume potential, and the Poisson integral. We will also not choose this path.

In our work, we will obtain existence and uniqueness of solution for each $\beta-$IBVP by a Fourier approach. Luchko in [2] and Sakamoto and Yamamoto in [5] took this approach for a more general operator than the Laplacian. They established the existence of a unique weak solution of an IBVP for the FDE (1) with Dirichlet boundary conditions on a bounded domain in $\mathbb{R}^d$. We will also be considering this Dirichlet problem, namely:

$$\begin{array}{lll} (i) & D^{C\alpha}_{0t} u(\mathrm{x},t) = \Delta u(\mathrm{x},t), & \mathrm{x} \in \Omega,\ 0 < t < T, \\ (ii) & u(\mathrm{x},0) = u_0(\mathrm{x}), & \mathrm{x} \in \Omega \\ (iii) & u(\mathrm{x},t) = 0 & \mathrm{x} \in \partial\Omega,\ 0 < t < T; \end{array} \tag{6}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega$. We will refer this problem as the $D-$IBVP.

We will obtain the solutions to the $\beta-$IBVP through a limit of functions, which are classical solutions to more regular problems. These kind of solutions are known as SOLA (Solutions Obtained as Limits of Approximations) solutions.

As a consequence of this approach, since the proofs are based on the eigenfunction expansions, we will be able to study the asymptotic behavior of the solutions of the $\beta-$IBVP when β increases to infinity and compare them to the solution of the $D-$IBVP. In doing this, we will make use of the variational techniques mentioned before, and recall the works of Filinovsky in [14] and [15]. The main theorems we will prove are the following:

Theorem 1. *Let $\beta > 0$. If $u_0 \in L^2(\Omega)$ then there exists a unique SOLA solution $u_\beta \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^1(\Omega))$ to the β-IBVP (4) such that ${}_0^C D_t^\alpha u_\beta \in C((0,T]; L^2(\Omega))$. Moreover,*

$$||u_\beta||_{C([0,T];L^2(\Omega))} \leq ||u_0||_{L^2(\Omega)}, \tag{7}$$

and we have

$$u_\beta(\mathrm{x},t) = \sum_{n=1}^{\infty} (u_0, \psi_n(\beta;\cdot))E_\alpha(-\lambda_n(\beta)t^\alpha)\psi_n(\beta;\mathrm{x}). \tag{8}$$

Theorem 2. *The family of solutions $\{u_\beta\}$ of the $\beta - IBVP$ (4), converges in the $L^2(\Omega)-$norm to the solution u_D to the $D - IBVP$ when $\beta \to \infty$, for every $t \in (0,T)$.*

The rest of this manuscript is structured as follows. In Section 2 some basics on fractional calculus and Sturm-Liouville theory related to the temporal and spatial variables respectively are given, as well as the definition of SOLA solution related to the Fourier approach. In Section 3 we study the $\beta-$IBVP by using the Fourier approach and Theorem 1 is proved. Finally, the convergence of the eigenvalues and the eigenfunctions of the u_β solutions are given in Section 4, leading to convergence in $L^2(\Omega)$, for every $t \in (0,T)$, of the u_β solutions to the u_D solution when $\beta \to \infty$ as stated in Theorem 2. Finally, in order to illustrate the convergence result given in Section 4, we study the one-dimensional case and give some examples using SageMath software in Section 5.

2. Preliminaries

2.1. Temporal variable: Fractional calculus and functions involved

Definition 1. Let $0 < \alpha < 1$ be fixed. For $f \in L^1(0,T)$, we define the *fractional Riemann–Liouville integral of order* α as

$$ {}_0I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau. $$

Note 1. The Caputo derivative of order α defined in (2) can be expressed in terms of the fractional integral of Riemann-Liouville of order $1-\alpha$ by

$$ {}_a^CD^\alpha f(t) = \left[{}_0I^{1-\alpha}(f') \right](t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) \mathrm{d}\tau, $$

for every $t \in (0,T)$.

When dealing with fractional derivatives, it is widely known the importance of the Mittag-Leffler functions and its properties. We recall them now.

Definition 2. For every $t \in \mathbb{R}_0^+$ and $\alpha > 0$, the *Mittag-Leffler* function is defined by

$$ E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}. \tag{9} $$

Proposition 1. *For $\alpha \in (0,1)$ and $\lambda \in \mathbb{R}$ the next assertions follow:*

1. *If $\lambda > 0$, then $E_\alpha(-\lambda t^\alpha)$ is a positive decreasing function for $t \in \mathbb{R}_0^+$ such that*
$$ E_\alpha(-\lambda t^\alpha) \le 1. \tag{10} $$
Moreover, there exists a constant $C > 0$ such that for every $t \in \mathbb{R}_0^+$,
$$ E_\alpha(-\lambda t^\alpha) \le \frac{C}{1+\lambda t^\alpha}. \tag{11} $$
2. *For every $t \in \mathbb{R}_0^+$ and $\lambda > 0$,*
$$ {}_0^CD^\alpha E_\alpha(-\lambda t^\alpha) = -\lambda E_\alpha(-\lambda t^\alpha). \tag{12} $$

Proof. Item 1 is a consequence from the fact that the Mittag-Leffler function $t \mapsto E_\alpha(-t)$ is a completely monotonic function (see [16]). Estimate (11) can be found in [17, Corollary 3.7] and see [18, Theorem 4.3] for item 2. □

2.2. Space variable: Sobolev spaces, variational formulation

To be precise let us fix the set Ω. We are considering Ω to be an open subset of $\mathbb{R}^n$ of class C^2 with bounded boundary.

Let us denote the usual $L^2(\Omega)$ inner product and the usual associated norm by (u,v) and $||u||_{L^2(\Omega)}$, respectively. For the standard Sobolev spaces $H^m(\Omega)$ we consider the inner product

$$ (u,v)_{H^m(\Omega)} = \sum_{0 \le |\sigma| \le m} (D^\sigma u, D^\sigma v) $$

and the respective associated norm

$$ ||u||_{H^m(\Omega)} = \left(\sum_{0 \le |\sigma| \le m} ||D^\sigma u||^2_{L^2(\Omega)} \right)^{1/2}. $$

Also, the space $H_0^1(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$.

We will work with some other bilinear forms, inner products and norms. Let us describe them below and give some references.

In $H_0^1(\Omega)$ we have the bilinear form defined by

$$\begin{aligned} a: \quad H_0^1(\Omega)\times H_0^1(\Omega) &\to \mathbb{R} \\ (\mathrm{u},\mathrm{v}) &\to a(u,v)=\int_\Omega \nabla u\nabla v\mathrm{d}x \end{aligned} \tag{13}$$

and in $H^1(\Omega)$ we have the bilinear form defined by

$$\begin{aligned} a_\beta: \quad H^1(\Omega)\times H^1(\Omega) &\to \mathbb{R} \\ (u,v) &\to a_\beta(u,v)=\int_\Omega \nabla u\nabla v\mathrm{d}x+\beta\int_{\partial\Omega} uv\mathrm{d}\gamma. \end{aligned} \tag{14}$$

Note that in $H^1(\Omega)$ we have that $||v||_{H^1(\Omega)}=[a(v,v)+(v,v)]^{1/2}$.
The next Lemma was proved in [19,20] and it is a very useful tool.

Lemma 1. *There exists a constant $\eta_1>0$ such that*

$$a(v,v)+\int_{\partial\Omega} v^2\mathrm{d}\gamma \geq \eta_1||v||^2_{H^1(\Omega)} \qquad \forall\, v\in H^1(\Omega). \tag{15}$$

Moreover, the induced norm $\sqrt{a_\beta(\cdot,\cdot)}$ in $H^1(\Omega)$ is equivalent to the classical norm in $H^1(\Omega)$ and there exists a constant $\eta_\beta=\eta_1\cdot\min\{1,\beta\}$ such that

$$a_\beta(v,v)\geq \eta_\beta||v||^2_{H^1(\Omega)} \qquad \forall\, v\in H^1(\Omega). \tag{16}$$

The next theorem is well known (see [9, Theorem 9.31]), and we present it for the benefit of the reader. Recall that a Hilbert basis on a separable Hilbert space H means a sequence of vectors $\{e_n\}_{n\in\mathbb{N}}$ such that $(e_n,e_n)=1$, $(e_n,e_m)=0$ for $n\neq m$, and its linear span is dense in H.

Theorem 3. *There is a Hilbert basis $\{\varphi_n(\cdot)\}_{n\geq 1}$ of $L^2(\Omega)$ and a sequence of real positive numbers $\{\lambda_n^D\}_{n\geq 1}$, $0<\lambda_1^D<\lambda_2^D\leq\ldots$ enumerated according to their multiplicities, with $\lambda_n^D\to\infty$ such that*

(i) $\varphi_n(\cdot)\in H_0^1(\Omega)\cap C^\infty(\Omega)$ *and*
(ii) $\Delta\varphi_n(\mathrm{x})+\lambda_n^D\varphi_n(\mathrm{x})=0$, $\boldsymbol{x}\in\Omega$.

The λ_n^D's are the eigenvalues of $-\Delta$ with Dirichlet boundary condition, and the φ_n's are the associated eigenfunctions.

Following the same variational techniques and compact operators arguments, it is straightforward to derive the next result which is the analogous to Theorem 3 for the $\beta-IBVP$:

Theorem 4. *For every $\beta>0$, there is a Hilbert basis $\{\psi_n(\beta,\cdot)\}_{n\geq 1}$ of $L^2(\Omega)$ and a sequence of real positive numbers $\{\lambda_n(\beta)\}_{n\geq 1}$, $0<\lambda_1(\beta)<\lambda_2(\beta)\leq\ldots$ enumerated according to their multiplicities, with $\lambda_n(\beta)\to\infty$ such that*

(i) $\psi_n(\beta,\cdot)\in H^1(\Omega)\cap C^\infty(\Omega)$,
(ii) $\Delta\psi_n(\beta,\mathrm{x})+\lambda_n(\beta)\psi_n(\beta,\mathrm{x})=0$, $\boldsymbol{x}\in\Omega$, *and*
(iii) $\frac{\partial}{\partial n}\psi_n(\beta,\mathrm{x})+\beta\psi_n(\beta,\mathrm{x})=0$ *a. e. in* $\partial\Omega$.

The $\lambda_n(\beta)$'s are the eigenvalues of $-\Delta$ with Robin boundary condition of parameter β, and the $\psi_n(\beta,\cdot)$'s are the associated eigenfunctions.

Proof. Let us consider the operator T^β defined as

$$\begin{aligned} T^\beta: L^2(\Omega)&\to L^2(\Omega) \\ f&\to T^\beta f=u, \end{aligned}$$

where u is the unique solution to the variational problem

$$a_\beta(u,v)=f(v), \quad \forall\, v\in H^1(\Omega) \tag{17}$$

for a_β defined in (14) and $f(v)=\int_\Omega fv\mathrm{d}x$. Note that T^β is well defined because problem (17) admits a unique solution for every fixed β (see [21, Lemma 3] or [11, Ch. II § 5])

If we prove that T^β is a self adjoint compact operator, then we can assure the existence of a Hilbert basis from the spectral theory. Let $f,g\in L^2(\Omega)$ and let $u=T^\beta f$ and $v=T^\beta g$ be their respective images. From the definition of T^β and the symmetry of the bilinear form we get

$$(f,T^\beta g)_{L^2(\Omega)}=a_\beta(u,v)=a_\beta(v,u)=(g,T^\beta f)_{L^2(\Omega)}$$

which leads us to conclude that T^β is self adjoint. The continuity of T^β follows from the continuity of the bilinear form a_β and (16). Indeed,

$$||T^\beta f||^2_{L^2(\Omega)} = ||u||^2_{L^2(\Omega)} \le ||u||^2_{H^1(\Omega)} \le \frac{1}{\eta_\beta} a_\beta(u,u) \le \frac{M_\beta}{\eta_\beta} ||u||^2_{H^1(\Omega)}.$$

Also, T^β is injective and positive (meaning that the quadratic form $(T^\beta f, f)_{L^2(\Omega)}$ is non negative): $(T^\beta f, f)_{L^2(\Omega)} = f(u) = a_\beta(u,u) \ge 0$ due to (16). Finally, from the fact that the injection $i : H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact (see e.g. [9, Theorem 9.16]) we obtain that T^β is a compact operator from $L^2(\Omega)$ into itself. From the previous analysis follows existence of a Hilbert basis for $L^2(\Omega)$ of eigenfunctions of the operator T^β, say $\{\psi(\beta)_n\}$ and a set of associated eigenvalues $\{\mu_n(\beta)\}$, such that every $\mu_n(\beta) > 0$ and $\mu_n(\beta) \to 0$. Hence $T^\beta \psi_n(\beta) = \mu_n(\beta)\psi_n(\beta)$ for every $n \in \mathbb{N}$, which means that $\mu_n(\beta)\psi_n(\beta)$ is the unique solution to the variational problem (17) for $f = \psi_n(\beta)$. Setting $\lambda_n(\beta) = \frac{1}{\mu_n(\beta)}$, we have that

$$\int_\Omega \nabla\psi_n(\beta)\nabla v + \beta \int_{\partial\Omega} \psi_n(\beta) v \mathrm{d}\gamma = \lambda_n(\beta) \int_\Omega \psi_n(\beta) v, \quad \text{for all } v \in H^1(\Omega). \tag{18}$$

Recalling that (18) is the variational formulation associated to problem $(ii)-(iii)$ in the statement, we also deduce that $\psi_n(\beta)$ is a weak solution to problem $(ii)-(iii)$. From the regularity result in [10, Th. 1, Ch. 6.3] we know that $\psi_n(\beta) \in H^2_{loc}(\Omega)$ and from [9, Ch. XI, Remark 26] we have that $\psi_n(\beta) \in H^4_{loc}(\Omega)$. By repeating the former argument we deduce that for every compact set $\omega \subset \Omega$, $\psi_n(\beta) \in \cap_{m\ge1} H^m(\omega)$. Thus $\psi_n(\beta) \in C^\infty(\Omega)$. For the boundary condition we use the Green's formula for functions in $H^2(\Omega)$ which says that

$$-\int_\Omega \Delta w v \mathrm{d}x = \int_\Omega \nabla w \nabla v \mathrm{d}x - \int_{\partial\Omega} \frac{\partial w}{\partial n} v \mathrm{d}\gamma, \quad \forall w, v \in H^2(\Omega). \tag{19}$$

Thus, by applying (19) to (18) and considering an arbitrary function $v \in C^2_c(\Omega)$ we get that

$$-\int_\Omega \Delta\psi_n(\beta)\, v \mathrm{d}x = \lambda_n(\beta) \int_\Omega \psi_n(\beta) v, \quad \text{for all } v \in C^2_c(\Omega),$$

which leads to conclude that

$$-\Delta\psi_n(\beta) = \lambda_n(\beta)\psi_n(\beta) \quad \text{a.e. in } \Omega \tag{20}$$

and (ii) holds.

For the boundary condition we note that from the regularity assumed on the boundary, $\partial\Omega \in C^2$, it holds that $\psi_n(\beta) \in H^2(\Omega)$ ([10, Ch. 6, Th.4]), which implies that the function $\phi := \nabla\psi_n(\beta) \cdot \mathbf{n} + \beta\psi_n(\beta) = \frac{\partial}{\partial \mathbf{n}}\psi_n(\beta) + \beta\psi_n(\beta)$ is in $H^{1/2}(\partial\Omega)$, and from [22, Th. 8.3] we deduce that there exists a function $\tilde{v} \in H^1(\Omega)$ such that $\tilde{v}|_{\partial\Omega} = \phi$. Then, by using $\tilde{v}$ as a test function in (18) and using (20) we obtain that

$$\int_{\partial\Omega} \phi^2 \mathrm{d}\gamma = 0 \tag{21}$$

and then (ii) holds a.e. in $\partial\Omega$. □

2.3. SOLA Solutions.

In classical Fourier analysis the solutions obtained are classical solutions since, under certain hypothesis (such as uniform convergence on compact subsets, for example), it is possible to interchange derivatives with summation. In our setting, the eigenfunctions provide us with an $L^2(\Omega)$ basis. Moreover, we also know that these eigenfunctions are $C^\infty(\Omega)$, but there is no information on any possible uniform convergence on compact sets nor any extra information that allows us to perform the necessary interchange of derivations and summation. In the work of Sakamoto and Yamamoto [5] from 2011 the solution obtained is called a *weak solution* but such solution is not in the sense of any Sobolev space. Recently the concept of SOLA solutions (meaning Solutions Obtained as Limit of Approximations) has been recalled. It was introduced in 1996 by Dall'Aglio in [23] for solutions to elliptic and parabolic problems with L^1 data and, although our approximations are more regular than those considered in that article, we consider that this concept of limit approximation adjusts well for our case.

Let us present the concept of SOLA solutions considered in this article, which will make sense in Section 3. Let us consider the space

$$W^1_t(\Omega \times [0,T]) = \{w \in C(\Omega \times (0,T)) : w(x,\cdot) \in AC[0,T]\}. \tag{22}$$

For the Fractional Diffusion Equation we have:

Definition 3. We say that a function $w : [0,T] \to L^2(\Omega)$ is a *SOLA solution* to the FDE (1) if there exists a sequence $(w_n)_{n\in\mathbb{N}}$ of smooth functions in $C(\bar{\Omega} \times [0,T]) \cap C^2_x(\Omega \times (0,T)) \cap W^1_t(\Omega \times [0,T])$ such that w_n verifies the FDE (1) in the classical sense for every $n \in \mathbb{N}$, and:

(i) $w_n \to w$ in $L^2(\Omega)$, for every $t \in (0,T)$.

(ii) There exists $v \in C\big((0,T);L^2(\Omega)\big)$ such that $\lim_{n\to\infty} \left\| {}_0^C D_t^\alpha w_n(\cdot,t) - v(\cdot.t)\right\|_{L^2(\Omega)} = 0$ for every $t \in (0,T)$. In such a case we will write $v = {}_0^C D_t^\alpha w$.

(iii) $\Delta w \in C((0,T);L^2(\Omega))$ and ${}_0^C D_t^\alpha w = \Delta w$ in $L^2(\Omega)$, for every $t \in (0,T)$.

Next, for the Dirichlet Initial Boundary Value Problem we have:

Definition 4. We say that u is a *SOLA solution* to problem (6) if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of functions in $C(\bar{\Omega}\times[0,T]) \cap C_x^2(\Omega\times(0,T)) \cap W_t^1(\Omega\times[0,T])$ such that

(i) There exist a sequence $(u_{n0})_n$ in $C(\bar{\Omega})$ such that $u_{n0} \to u_0 \in L^2(\Omega)$ and for every $n\in\mathbb{N}$, u_n is a classical solution to the approximate problem

$$\begin{array}{lll} (i) & D_{0t}^{C\alpha} u_n(x,t) = \Delta u_n(x,t), & x\in\Omega,\ 0<t<T, \\ (ii) & u_n(x,0) = u_{n0}(x), & x\in\Omega \\ (iii) & u_n(x,t) = 0 & x\in\partial\Omega,\ 0<t<T. \end{array} \tag{23}$$

(ii) ${}_0^C D_t^\alpha u$ exists in the sense defined in Definition 3 and u is a SOLA solution to (1).

(iii) $u \in C([0,T];L^2(\Omega))$ and $\lim_{t\to 0} \|u(\cdot,t) - u_0\|_{L^2(\Omega)} = 0$.

And finally, for the Robin Initial Boundary Value Problem we have:

Definition 5. We say that u is a *SOLA solution* to problem (4) if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of functions in $C(\bar{\Omega}\times[0,T]) \cap C_x^2(\Omega\times(0,T)) \cap W_t^1(\Omega\times[0,T])$ such that

(i) There exist a sequence $(u_{n0})_n$ in $C(\bar{\Omega})$ such that $u_{n0} \to u_0 \in L^2(\Omega)$ and for every $n\in\mathbb{N}$, u_n is a classical solution to the approximate problem

$$\begin{array}{lll} (i) & D_{0t}^{C\alpha} u_n(x,t) = \Delta u_n(x\ ,t), & x\in\Omega,\ 0<t<T, \\ (ii) & u_n(x\ ,0) = u_{n0}(x), & x\in\Omega \\ (iii) & \frac{\partial}{\partial\nu} u_n(x,t) + \beta u_n(x,t) = 0 & x\in\partial\Omega,\ 0<t<T. \end{array} \tag{24}$$

(ii) ${}_0^C D_t^\alpha u$ exists in the sense defined in Definition 3 and u is a SOLA solution to (1).

(iii) $u \in C([0,T];L^2(\Omega))$ and $\lim_{t\to 0} \|u(\cdot,t) - u_0\|_{L^2(\Omega)} = 0$.

Remark 1. Note that it is appropriated to consider the concept of SOLA solution given in Definition 3. In fact, when asking if the Caputo operator $A = {}_0^C D_t^\alpha$ is closed in the sense that, if a sequence of functions in $AC([0,T],C(\Omega))$, (w_n), verifies that $w_n \to w$ in $C((0,T);C(\Omega))$, and $Aw_n \to y$ in $C((0,T);C(\Omega))$ then $Aw = y$, the answer is no.

We can justify this assertion by considering the sequence of functions $(w_n)_n$ where $w_n(t) = t^{1/n} \in AC[0,T]$. It is straightforward that $w_n(t) \to w(t) = 1$ for every $t\in(0,T)$ uniformly in $[\varepsilon,T)$. Also, it is easy to see, by using classical formulas of fractional differentiation, that $Aw_n(t) = \frac{\Gamma(1/n+1)}{\Gamma((1/n+\alpha+1)} t^{1/n-\alpha}$ which converges to the function $y(t) = \frac{1}{\Gamma(1+\alpha)} t^{-\alpha}$ in $C(0,T)$. Then we have that $Aw \equiv 0$ and conclude that $Aw \neq y$.

3. The Fourier approach

Following the lines of the works [2] and [5], we look for a solution to problem (4) constructed analytically by using the Fourier method of variable separation. Thus, let us recall first Theorem 2.1 of [5].

Theorem 5. *If $u_0 \in L^2(\Omega)$ then there exists a unique SOLA solution $u_D \in C([0,T];L^2(\Omega)) \cap C((0,T];H^2(\Omega)\cap H_0^1(\Omega))$ to the D-IBVP (6) such that ${}_0^C D_t^\alpha u_D \in C((0,T];L^2(\Omega))$ in the sense of Definition 3. Moreover, there exists a constant $C>0$ such that*

$$||u_D||_{C([0,T];L^2(\Omega))} \le C||u_0||_{L^2(\Omega)}, \tag{25}$$

and the solution is given by the series

$$u_D(x,t) = \sum_{n=1}^{\infty} (u_0,\varphi_n) E_\alpha\left(-\lambda_n^D t^\alpha\right)\varphi_n(x) \tag{26}$$

where (λ_n^D, φ_n) are the eigenvalues and eigenfunctions given in Theorem 3.

As we said above, we look for a particular solution u of the equation $(4)-(i)$ of the form

$$u(x,t) = \psi(x)\eta(t), \tag{27}$$

then, by replacing (27) in (4)–i, we are lead to the following equations

$$\frac{D_{0t}^{C\alpha}\eta(t)}{\eta(t)} = \frac{\Delta\psi(x)}{\psi(x)} = -\lambda, \tag{28}$$

where λ is a constant which does not depend on $\boldsymbol{x}$ nor t. Thus we are left with two different problems.

One of them is regarding the spatial variable. It considers the spatial equation in (28) together with the Robin boundary condition, which derives in a classical Sturm-Liouville problem. The other one is for the temporal variable. It consists in an ordinary fractional differential equation. Both problems will be coupled later by using the initial condition (4) (ii). More precisely, we have the problems:

$$\begin{array}{lll} (i) & \Delta\psi(\mathrm{x})+\lambda\psi(\mathrm{x})=0, & \mathrm{x}\in\Omega, \\ (ii) & \frac{\partial\psi}{\partial n}(\mathrm{x})+\beta\psi(\mathrm{x})=0 & \mathrm{x}\in\partial\Omega; \end{array} \tag{29}$$

and

$$\begin{array}{lll} (i) & {}_0^C D_t^\alpha \eta(t)=-\lambda\eta(t), & t\in(0,T), \\ (ii) & \eta(0)=1. & \end{array} \tag{30}$$

According to Theorem 4 and Proposition 1 item 3, it is natural to consider function $u_\beta:[0,T]\to L^2(\Omega)$ defined as a series

$$u_\beta(\mathrm{x},t)=\sum_{n=1}^{\infty}(u_0,\psi_n(\beta;\cdot))E_\alpha(-\lambda_n(\beta)t^\alpha)\psi_n(\beta;\mathrm{x}) \tag{31}$$

as a desired solution to problem (4), where $(\psi_n(\beta),\lambda_n(\beta))$ are the eigenfunctions and eigenvalues given in Theorem 4.

It is easy to prove that u_β is well defined. Indeed, for each $t\in[0,T]$, $u_\beta(\cdot,t)$ belongs to $L^2(\Omega)$ since by the Bessel inequality and the properties of the Mittag-Leffler functions we have that

$$||u_\beta(\cdot,t)||^2_{L^2(\Omega)}\le\sum_{n=1}^{\infty}|(u_0,\psi_n(\beta;\cdot))|^2|E_\alpha(-\lambda_n(\beta)t^\alpha)|^2\le||u_0||^2_{L^2(\Omega)}.$$

Let us see that u_β is a SOLA solution to equation (1) in the sense of Definition 3. For that purpose let $w:(0,T]\to L^2(\Omega)$ be the function defined by

$$w(\mathrm{x},t)=\sum_{n=1}^{\infty}(u_0,\psi_n(\beta;\cdot))E_\alpha(-\lambda_n(\beta)t^\alpha)(-\lambda_n(\beta))\psi_n(\beta;\mathrm{x}) \tag{32}$$

which is well defined. In fact, by applying inequality (11) we get

$$\begin{aligned} ||w(\cdot,t)||^2_{L^2(\Omega)} &=\sum_{n=1}^{\infty}|(u_0,\psi_n(\beta;\cdot))|^2|E_\alpha(-\lambda_n(\beta)t^\alpha)|^2\lambda_n^2(\beta) \\ &\le C\sum_{n=1}^{\infty}|(u_0,\psi_n(\beta;\cdot))|^2\left(\frac{\lambda_n(\beta)}{1+\lambda_n(\beta)t^\alpha}\right)^2\le C||u_0||^2_{L^2(\Omega)}t^{-2\alpha}. \end{aligned} \tag{33}$$

On the other side, consider the sequence $(w_n)_{n\in\mathbb{N}}$, where for every $n\in\mathbb{N}$ the function $w_n:(\Omega\times[0,T])\to\mathbb{R}$ is given by the finite sum

$$w_n(\mathrm{x},t)=\sum_{k=1}^{n}(u_0,\psi_k(\beta))E_\alpha(-\lambda_k(\beta)t^\alpha)(-\lambda_k(\beta))\psi_k(\beta;\mathrm{x}).$$

Clearly, if we define the functions $u_{\beta n}=\sum_{k=1}^n(u_0,\psi_k(\beta))E_\alpha(-\lambda_k(\beta)t^\alpha)\psi_k(\beta;\mathrm{x})$ for every natural n, we have that ${}_0^C D_t^\alpha u_{\beta n}(x,t)=w_n$.

Now, for every $t>0$ we have

$$\begin{aligned} ||w(\cdot,t)-{}_0^C D_t^\alpha u_{\beta n}(\cdot,t)||^2_{L^2(\Omega)} &=\sum_{k=n+1}^{\infty}|(u_0,\psi_k(\beta;\cdot))|^2|E_\alpha(-\lambda_k(\beta)t^\alpha)|^2\lambda_k^2(\beta) \\ &\le Ct^{-2\alpha}\sum_{k=n+1}^{\infty}|(u_0,\psi_k(\cdot))|^2\to0,\ \text{if}\ n\to\infty \end{aligned}$$

from where we conclude that $w={}_0^C D_t^\alpha u_\beta$ in the sense of Definition 3.

Analogously, we deduce that $\Delta u_\beta=w$ and we conclude that

$${}_0^C D_t^\alpha u_\beta=\Delta u_\beta \quad \text{in } L^2(\Omega),t\in(0,T), \quad \text{for every } \beta>0,$$

that is, u_β is a SOLA solution to the FDE (1).

Remark 2. It is straightforward that (32) is a SOLA solution to problem (4) according to Definition 5. Note that the boundary conditions are verified from Theorem 4.

From the previous reasoning we state the following Lemma.

Lemma 2. *The function (31) belongs to $C([0,T];L^2(\Omega))$ and it is a SOLA solution to the FDE* (1) *in the sense of Definition 3.*

The proof of the next theorem is obtained by mimicking the steps of the proof of Theorem 5 given in [5] and we are lead to a similar result for the $\beta-$IBVP. The difference is on the Robin conditions and the spaces of functions involved. However, we include it for the sake of completeness. Thus, with the notation from Theorem 4 we have the following.

Theorem 6. *Let $\beta > 0$ be fixed. If $u_0 \in L^2(\Omega)$ then there exists a unique SOLA solution $u_\beta \in C([0,T];L^2(\Omega)) \cap C((0,T];H^1(\Omega))$ to the β-IBVP (4) such that ${}_0^C D_t^\alpha u_\beta \in C((0,T];L^2(\Omega))$. Moreover, u_β is given by the series (31) and*

$$||u_\beta||_{C([0,T];L^2(\Omega))} \leq ||u_0||_{L^2(\Omega)}, \tag{34}$$

Proof. We first prove that, if u is a formal solution to (4)(i), (ii) and (iii), then u is the SOLA solution given by (31).

Let $t \in [0,T]$. If $u(\cdot,t) \in L^2(\Omega)$, then from Theorem 4 we have the following expansion. For $\mathrm{x} \in \Omega$,

$$u(\mathrm{x},t) = \sum_{n=1}^{\infty} (u(\cdot,t), \psi_n(\beta;\cdot))\psi_n(\beta;\mathrm{x}) = \sum_{n=1}^{\infty} u_n(\beta;t)\psi_n(\beta;\mathrm{x}), \tag{35}$$

where we defined

$$u_n(\beta;t) = (u(\cdot,t), \psi_n(\beta;\cdot)).$$

Since ${}_0^C D_t^\alpha u(x,t) = \Delta u(x,t)$ for $\boldsymbol{x} \in \Omega$, it follows that for each $n \in \mathbb{N}$,

$$({}_0^C D_t^\alpha u(\cdot,t), \psi_n(\beta;\cdot)) = (\Delta u(\cdot,t), \psi_n(\beta;\cdot)).$$

In order to extract some information on u from the above equation let us compute both sides and then compare them.

For the left side we recall the definition for the Caputo derivative, namely formula (2). By applying Fubini's Theorem and the derivation under the integral sign Theorem we end up with

$$({}_0^C D_t^\alpha u(\cdot,t), \psi_n(\beta;\cdot)) = {}_0^C D_t^\alpha u_n(\beta;t).$$

For the right side we apply Green's formula and recall the boundary conditions (4)-(iii) for u and conditions (ii) and (iii) in Theorem 4 for ψ_n (eigenvalue and boundary condition, respectively), to obtain

$$(\Delta u(\cdot,t), \psi_n(\beta;\cdot)) = -\lambda_n(\beta)u_n(\beta;t).$$

Thus for each $n \in \mathbb{N}$ we are left with an ordinary fractional differential equation together with the initial condition (4) - (ii), that is to say, the following initial value problem.

$$\begin{array}{rcll} D_{0t}^{C\alpha} u_n(\beta;t) & = & -\lambda_n(\beta)u_n(\beta;t), & \text{for} t > 0 \\ u_n(\beta;0) & = & (u_0, \psi_n(\beta;\cdot)). & \end{array} \tag{36}$$

The well known theory (see for example [24], or [25], among others) provides us with a unique solution for this problem by means of the Mittag-Leffler functions that we defined in Section 2. More precisely,

$$u_n(\beta;t) = (u_0, \psi_n(\beta;\cdot))E_\alpha(-\lambda_n(\beta)t^\alpha).$$

Replacing the above solution in (35) gives us the desired expression for u which coincide with the function given in (31) and then the formal solution constructed is a SOLA solution to problem (4) according to Lemma 2.

We obtained that $u_\beta \in L^2(\Omega)$ for every $t \in [0,T]$. And, moreover, we have that $||u_\beta||_{C([0,T];L^2(\Omega))} \leq ||u_0||_{L^2(\Omega)}$, which means that it is uniformly bounded in $[0,T]$. Hence, $u_\beta(\cdot,\cdot) \in C([0,T];L^2(\Omega))$.

Also, we have that $u_\beta \in C((0,T];H^1(\Omega))$. In order to prove this, let us note that the family $\{\psi_n(\beta;\cdot)\}_{n\in\mathbb{N}}$ is orthogonal with respect to the norm $\sqrt{a_\beta(\cdot,\cdot)}$, which is equivalent to the $||\cdot||_{H^1}$ norm as was stated in Lemma 1. Indeed, for every $k,l \in \mathbb{N}$

$$\begin{aligned} a_\beta(\psi_n(\beta;\cdot),\psi_l(\beta;\cdot)) &= \int_\Omega \nabla\psi_n(\beta)\nabla\psi_l(\beta)\mathrm{d}x + \beta\int_{\partial\Omega} \psi_n(\beta)\psi_l(\beta)\mathrm{d}\gamma \\ &= -\int_\Omega \Delta\psi_n(\beta)\psi_l(\beta)\mathrm{d}x + \int_{\partial\Omega} \frac{\partial\psi_n(\beta)}{\partial n}\psi_l(\beta)\mathrm{d}\gamma + \beta\int_{\partial\Omega} \psi_n(\beta)\psi_l(\beta)\mathrm{d}\gamma \\ &= \int_\Omega \lambda_n(\beta)\psi_n(\beta)\psi_l(\beta)\mathrm{d}x = \lambda_n(\beta)\delta_{nl}. \end{aligned}$$

Here, the δ_{kl} denotes the Kronecker's delta and we have applied Green's theorem (19), then the fact that every ψ_k verifies (ii) and (iii) in Theorem 4 and finally that $\{\psi(\beta)_k\}$ is an orthonormal basis in $L^2(\Omega)$.

Hence, for every $t > 0$ we have

$$a_\beta\big(u_\beta(\cdot,t),u_\beta(\cdot,t)\big)=\sum_{n=1}^{\infty}|(u_0,\psi_n(\beta;\cdot))|^2E_\alpha(-\lambda_n(\beta)t^\alpha)^2\lambda_n(\beta)$$
$$\leq\sum_{n=1}^{\infty}|(u_0,\psi_n(\beta;\cdot))|^2E_\alpha(-\lambda_n(\beta)t^\alpha)^2\frac{\lambda_n^2(\beta)}{\lambda_1(\beta)}\leq\frac{C}{\lambda_1(\beta)t^{2\alpha}}\|u_0\|^2_{L^2(\Omega)}$$

from where we conclude that

$$||\nabla u_\beta||_{L^2(\Omega)}\leq\frac{C}{\lambda_1(\beta)t^\alpha}||u_0||_{L^2(\Omega)}$$

and that $u_\beta\in C((0,T];H^1(\Omega))$.

Now, for the series

$$\Delta u_\beta(x,t)=\sum_{n=1}^{\infty}(u_0,\psi_n(\beta;\cdot))E_\alpha(-\lambda_nt^\alpha)(-\lambda_n(\beta))\psi_n(\beta;x,t)$$

it holds that

$$||\Delta u_\beta(\cdot,t)||^2_{L^2(\Omega)}=\sum_{n=1}^{\infty}|(u_0,\psi_n(\beta;\cdot))|^2|E_\alpha(-\lambda_n(\beta)t^\alpha)|^2\lambda_n^2(\beta)$$
$$\leq\frac{C}{t^{2\alpha}}||u_0||^2_{L^2(\Omega)},$$

which shows that the series is uniformly convergent in $t\in[\delta,T]$ for any given $\delta>0$ and thus, $\Delta u_\beta\in C((0,T];L^2(\Omega))$. Taking into account that, according to definition 3, ${}_0^CD_t^\alpha u_\beta=\Delta u_\beta$ in $L^2(\Omega)$ we also have that

$${}_0^CD_t^\alpha u_\beta\in C((0,T];L^2(\Omega)).$$

To finally see that we have constructed a SOLA solution, let us check that

$$\lim_{t\to 0}\left\|u_\beta(\cdot,t)-u_0\right\|_{L^2(\Omega)}=0. \tag{37}$$

Again we use the series expansions in $L^2(\Omega)$ and the properties of the Mittag-Leffler functions:

$$\left\|u_\beta(\cdot,t)-u_0\right\|^2_{L^2(\Omega)}\leq\sum_{n=1}^{\infty}|(u_0,\psi_n(\beta;\cdot))|^2|E_\alpha(-\lambda_n(\beta)t^\alpha)-1|^2$$
$$\leq\sum_{n=1}^{\infty}|(u_0,\psi_n(\beta;\cdot))|^2\left[\left(\frac{C}{1+\lambda_n(\beta)t^\alpha}\right)^2+1\right],$$

which is convergent for every $t\in[0,T]$. Hence, from the Lebesgue dominated convergence Theorem, observing that $\lim_{t\to 0}E_\alpha(-\lambda_nt^\alpha)=1$, we conclude that the desired limit (37) holds.

The last step is to prove uniqueness. Let us consider $u_0\equiv 0$, we will show that the only solution to the $\beta-$IBVP is the trivial one. Indeed, for each $n\in\mathbb{N}$ the problem (36) has unique trivial solution $u_n(\beta,t)=0$ for $t\in[0,T]$. Hence the series that defines u must be the zero function. □

4. Convergences

In this section we analyze the convergence of the solution u_β of the $\beta-$IBVP (4), to the solution u_D of the $D-$IBVP (6).

Let us now study the very interesting relation between the eigenvalues and the eigenfunctions given in Theorem 3 and Theorem 4. In a recent work of Filinovsky, namely [14], the following theorem is proved through variational techniques.

Theorem 7. *For $k\in\mathbb{N}$ and $\beta>0$ the eigenvalues given in Theorems 3 and 4, enumerated according to their multiplicities, satisfy the following estimate*

$$0\leq\lambda_k^D-\lambda_k(\beta)\leq C_1\beta^{-\frac{1}{2}}(\lambda_k^D)^2, \tag{38}$$

where the constant C_1 does not depend on k.

From (38) it is obvious that

$$\lim_{\beta\to\infty}\lambda_k(\beta)=\lambda_k^D,\text{ for every }\ k\in\mathbb{N}. \tag{39}$$

The next Theorem deals with the weak convergence related to the eigenfunctions.

Let us observe that a similar result is given in an even more recent work [15] of Filinovsky, but we present a different proof according to our problems.

Theorem 8. *Let $\{\varphi_k\}_k$ and $\left\{\lambda_k^D\right\}_k$ be the sequence of eigenfunctions and eigenvalues of the Dirichlet problem given in Theorem 3, where $\lambda_1^D < \lambda_2^D \leq \ldots \to \infty$. And let $\{\psi_k(\beta)\}_k$ and $\{\lambda_k(\beta)\}_k$ be the sequence of eigenfunctions and eigenvalues of the Robin eigenvalue problem given in (4), where $\lambda_1(\beta) < \lambda_2(\beta) \leq \ldots \to \infty$. Then for each $k = 1, 2, \ldots$*

$$\psi_k(\beta) \rightharpoonup \varphi_k \quad \text{weak in} \quad H^1(\Omega), \quad \text{when} \beta \to \infty. \tag{40}$$

Proof. For each fixed $k \in \mathbb{N}$ let $\lambda_k(\beta)$ and λ_k^D be the eigenvalues given in Theorems 4 and 3, respectively. Sience φ_k is a function that verifies (i) and (ii) of Theorem 3, and that $\psi_k(\beta)$ is a function that verifies (ii) and (iii) of Theorem 4, we can consider the bilinear forms given in (13) and (14) and affirm that: i) The eigenfunction φ_k is the unique solution to the variational problem: *Find the function* $u \in H_0^1(\Omega)$ such that

$$a(u, v) = \lambda_k^D (u, v) \quad \text{for every} \, v \in H_0^1(\Omega). \tag{41}$$

ii) The eigenfunction $\psi_k(\beta)$ is the unique solution to the variational problem: *Find the function* $u \in H^1(\Omega)$ such that

$$a_\beta(u, v) = \lambda_k(\beta)(u, v) \quad \text{for every} \;\; v \in H^1(\Omega). \tag{42}$$

Inserting $v = \psi_k(\beta) - \varphi_k \in H^1(\Omega)$ in (42) and subtracting $a(\varphi_k, \psi_k(\beta) - \varphi_k)$ to both members we get

$$a(\psi_k(\beta) - \varphi_k, \psi_k(\beta) - \varphi_k) + \int\limits_{\partial\Omega} (\psi_k(\beta) - \varphi_k)^2 \mathrm{d}\gamma + (\beta - 1) \int\limits_{\partial\Omega} (\psi_k(\beta) - \varphi_k)^2 \mathrm{d}\gamma$$

$$= \lambda_k(\beta)(\psi_k(\beta), \psi_k(\beta) - \varphi_k) - a(\varphi_k, \psi_k(\beta) - \varphi_k). \tag{43}$$

Note that in the above equality we have assumed that $\beta > 1$ and splitted $\beta = (\beta - 1) + 1$ as in [20]. Now, from the continuity of the bilinear form a, the inequality (15), the fact that $\varphi_k = 0$ in $\partial\Omega$ and applying Theorem 7 we obtain

$$\eta_1 ||\psi_k(\beta) - \varphi_k||^2_{H^1(\Omega)} + (\beta - 1) \int\limits_{\partial\Omega} (\psi_k(\beta))^2 \mathrm{d}\gamma \leq$$

$$\leq M_k ||\psi_k(\beta)||_{L^2(\Omega)} ||\psi_k(\beta) - \varphi_k||_{L^2(\Omega)} + C||\varphi_k||_{H^1(\Omega)} ||\psi_k(\beta) - \varphi_k||_{H^1(\Omega)}. \tag{44}$$

Naming $C_k := M_k + C||\varphi_k||_{H^1(\Omega)}$ we get

$$\eta_1 ||\psi_k(\beta) - \varphi_k||^2_{H^1(\Omega)} + (\beta - 1) \int\limits_{\partial\Omega} \psi_k(\beta)^2 \mathrm{d}\gamma \leq C_k ||\psi_k(\beta) - \varphi_k||_{H^1(\Omega)}. \tag{45}$$

From (45) we can state that

$$||\psi_k(\beta) - \varphi_k||_{H^1(\Omega)} \leq \frac{C_k}{\eta_1}, \tag{46}$$

and from (45) and (46)

$$(\beta - 1) \int\limits_{\partial\Omega} \psi_k(\beta)^2 \mathrm{d}\gamma \leq \frac{C_k^2}{\eta_1}. \tag{47}$$

Inequality (46) implies that $\{\psi_k(\beta)\}_\beta$ is bounded in $H^1(\Omega)$, then there exists $\xi_k \in H^1(\Omega)$ such that

$$\psi_k(\beta) \rightharpoonup \xi_k \quad \text{in} \; H^1(\Omega) \, \text{weak} \;, \quad \text{when} \; \beta \to \infty. \tag{48}$$

By taking the limit when $\beta \to \infty$ in (47) and by using first the compactness of the trace operator (see e.g. [26, Th. 6.1–7]) and then the lower semicontinuity of $v \to \int\limits_{\partial\Omega} v^2 \mathrm{d}\gamma$ and, it holds that $\xi_k|_{\partial\Omega} = 0$. Note that $\xi \neq 0$ since is the $H^1(\Omega)$ limit of functions with $L^2(\Omega)$ norm equal to 1 for each k. Then

$$\xi \; \in \; H_0^1(\Omega). \tag{49}$$

Finally, note that from (42) we have that

$$a(\psi_k(\beta), w) = \lambda_k(\beta)(\psi_k(\beta), w) \quad \text{for every} \;\; w \in H_0^1(\Omega), \tag{50}$$

Taking the limit when $\beta \to \infty$ in (50) and using (49) and Theorem 7 we obtain

$$a(\xi_k, w) = \lambda_k^D (\xi_k, w) \quad \text{for every} \;\; w \in H_0^1(\Omega), \tag{51}$$

Then, from (51) and the uniqueness of the solution to the variational problem (41) we conclude that $\xi_k = \varphi_k$ and the thesis holds. □

Theorem 9. *The family of solutions* $\{u_\beta\}$ *of the* $\beta - IBVP$ *(4), converges to the solution* u_D *to the* $D - IBVP$ *in* $L^2(\Omega)$ *when* $\beta \to \infty$*, for every* $t \in (0, T)$*.*

Proof. Fix $t > 0$. From (25) and (34) we can choose a natural number N such that

$$\sum_{k=N+1}^{\infty} |(u_0, \varphi_k)|^2 |E_\alpha(-\lambda_k^D t^\alpha)|^2 < \varepsilon^2 \tag{52}$$

and

$$\sum_{k=N+1}^{\infty} |(u_0, \psi_k(\beta))|^2 |E_\alpha(-\lambda_k(\beta) t^\alpha)|^2 < \varepsilon^2. \tag{53}$$

Therefore, we have:

$$\begin{aligned} \left\| u_D(\cdot, t) - u_\beta(\cdot, t) \right\|_{L^2(\Omega)} &\le \left(\|A_N\|_{L^2(\Omega)} + \|B_N\|_{L^2(\Omega)} + \|C_N\|_{L^2(\Omega)} \right) \\ &\le \|A_N\|_{L^2(\Omega)} + 2\varepsilon, \end{aligned} \tag{54}$$

where

$$A_N := \sum_{k=1}^{N} \left[(u_0, \varphi_k) E_\alpha(-\lambda_k^D t^\alpha) \varphi_k - (u_0, \psi_k(\beta)) E_\alpha(-\lambda_k(\beta) t^\alpha) \psi_k(\beta) \right], \tag{55}$$

$$B_N := \sum_{k=N+1}^{\infty} (u_0, \varphi_k) E_\alpha(-\lambda_k^D t^\alpha) \varphi_k, \tag{56}$$

$$C_N := \sum_{k=N+1}^{\infty} (u_0, \psi_k(\beta)) E_\alpha(-\lambda_k(\beta) t^\alpha) \psi_k(\beta), \tag{57}$$

and we have applied inequalities (52) and (53). Now

$$\begin{aligned} \|A_N\|_{L^2(\Omega)} \le \sum_{k=1}^{N} \Big[&\left\| (u_0, \varphi_k) E_\alpha(-\lambda_k^D t^\alpha) \varphi_k - (u_0, \psi_k(\beta)) E_\alpha(-\lambda_k^D t^\alpha) \varphi_k \right\|_{L^2(\Omega)} \\ &+ \left\| (u_0, \psi_k(\beta)) E_\alpha(-\lambda_k^D t^\alpha) \varphi_k - (u_0, \psi_k(\beta)) E_\alpha(-\lambda_k(\beta) t^\alpha) \varphi_k \right\|_{L^2(\Omega)} \\ &+ \| (u_0, \psi_k(\beta)) E_\alpha(-\lambda_k(\beta) t^\alpha) \varphi_k - (u_0, \psi_k(\beta)) E_\alpha(-\lambda_k(\beta) t^\alpha) \psi_k(\beta) \|_{L^2(\Omega)} \Big]. \end{aligned} \tag{58}$$

Now, for each $k \in \{1, \dots, N\}$ we have:

(I) From Theorem 8, there exists $\beta_{I,N}^k > 0$ such that if $\beta > \beta_{I,N}^k$, then

$$|(u_0, \varphi_k) - (u_0, \psi_k(\beta))| < \frac{\varepsilon}{N}.$$

(II) Observe that for every $\beta > 0$, $|(u_0, \psi_k(\beta))| \le ||u_0||_{L^2(\Omega)}$.

(III) From the continuity of the Mittag-Leffler functions and Theorem 7, there exists $\beta_{III,N}^k > 0$ such that if $\beta > \beta_{III,N}^k$, then for each $t \in (0, T)$,

$$|E_\alpha(-\lambda_k^D t^\alpha) - E_\alpha(-\lambda_k(\beta) t^\alpha)| < \frac{\varepsilon}{N}.$$

(IV) From Theorem 8, since weak convergence in H^1 gives strong convergence in L^2, there exists $\beta_{IV,N}^k > 0$ such that if $\beta > \beta_{IV,N}^k$, then

$$\|\varphi_k - \psi_k(\beta)\|_{L^2(\Omega)} < \frac{\varepsilon}{N}.$$

Finally, taking $\bar\beta = \max\left\{\beta_{l,N}^k ; l = I, III, IV;\ k = 1, \dots, N\right\}$ we deduce that for $\beta > \bar\beta$,

$$\|A_N\|_2 < (2M + 1)\varepsilon, \ \text{ for every } t \in (0, T). \tag{59}$$

From (54) and (59) we conclude that, for every $t \in (0, T)$ it holds that $\lim_{\beta \to \infty} \left\| u_D(\cdot, t) - u_\beta(\cdot, t) \right\|_{L^2(\Omega)} = 0$, as desired. □

5. The one-dimensional case

In order to illustrate the convergence result by the aid of some software, we set ourselves in the following one-dimensional setting: let the domain be the real unit interval $\Omega = [0, 1]$, and let $T > 0$ to be fixed, say $T = 1$. Here, we have that the boundary of the domain consists of two points, namely $\partial\Omega = \{0, 1\}$. For simplicity we will write $x \in \mathbb{R}$ instead of $\boldsymbol{x}$, $\Delta u(\boldsymbol{x}, t) = u_{xx}(x, t)$, etc. For fixed $0 < \alpha < 1$ and each $\beta > 0$, the initial-boundary value problems to be considered are the following

- The one dimensional Dirichlet problem, which we call D-IVBP-1d:

$$\begin{array}{lll} (i) & D^{C\alpha}_{0t} u(x,t) = u_{xx}(x,t), & x \in (0,1),\ 0 < t < T, \\ (ii) & u(x,0) = u_0(x), & x \in [0,1], \\ (iii) & u(0,t) = u(1,t) = 0, & 0 < t < T. \end{array} \tag{60}$$

- The one dimensional Robin problem, which we call β−IBVP-1d

$$\begin{array}{lll} (i) & {}_0^C D^{\alpha}_t u(x,t) = u_{xx}(x,t), & x \in (0,1),\ 0 < t < T, \\ (ii) & u(x,0) = u_0(x), & x \in [0,1], \\ (iii) & -u_x(0,t) + \beta u(0,t) = 0, & 0 < t < T, \\ & u_x(1,t) + \beta u(1,t) = 0, & 0 < t < T. \end{array} \tag{61}$$

Let us construct the solutions to both problems by proceding like in Section 3.

For the D-IVBP-1d problem we have the next Sturm-Liouville type problem with Dirichlet condition:

$$D - SL \begin{cases} \psi''(x) + \lambda\psi(x) = 0, & 0 < x < 1, \\ \psi(0) = \psi(1) = 0. \end{cases} \tag{62}$$

The solutions to this problems are, for $k \in \mathbb{N}$, the pairs of eigenvalues and eigenfunctions (λ_k, ψ_k) given by $\lambda_k = k^2\pi^2$ and $\psi_k(x) = \sqrt{2}\sin(k\pi x)$. The solution to the ordinary FDE linked to the time variable is given by the Mittag-Leffler function: $\eta(t) = E_\alpha(-\lambda t^\alpha)$. Finally, we couple this solutions to get that the formal solution to the D-IVBP-1d is given by

$$u(x,t) = \sum_{k=1}^{\infty} \left(u_0, \sqrt{2}\sin(k\pi\,\cdot)\right)\sqrt{2}\sin(k\pi x)E_\alpha(-k^2\pi^2 t^\alpha).$$

For problem β-IVBP-1d, the Sturm-Liouville type problem with Robin condition is

$$R - SL \begin{cases} \psi''(x) + \lambda\psi(x) = 0, & 0 < x < 1 \\ -\psi'(0) + \beta\psi(0) = 0, & \\ \psi'(1) + \beta\psi(1) = 0. & \end{cases} \tag{63}$$

Working with the general solutions to the second order differential equation above

$$\psi(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x),$$

and with the Robin conditions we get the following system:

$$\begin{cases} -A\sqrt{\lambda} + \beta B = 0, \\ (A\sqrt{\lambda} + \beta B)\cos(\sqrt{\lambda}) + (\beta A - B\sqrt{\lambda})\sin(\sqrt{\lambda}) = 0, \end{cases}$$

or equivalently

$$\begin{cases} B = A\frac{\sqrt{\lambda}}{\beta}, \\ 2\sqrt{\lambda}\cos(\sqrt{\lambda}) + \left(\beta - \frac{\lambda}{\beta}\right)\sin(\sqrt{\lambda}) = 0. \end{cases}$$

Note that from the last equation we get an implicit formula for λ, namely $\tan(\sqrt{\lambda}) = \frac{2\sqrt{\lambda}\beta}{\lambda - \beta^2}$, from where it can be observed that the equation becomes $\sin(\sqrt{\lambda}) = 0$, when β approaches infinity. This is precisely the equation that defines the eigenvalues for the Dirichlet case.

Now, let us consider the functions

$$h_\beta(\lambda) := 2\sqrt{\lambda}\cos(\sqrt{\lambda}) + \left(\beta - \frac{\lambda}{\beta}\right)\sin(\sqrt{\lambda}). \tag{64}$$

We know from Theorems 4 and 7 that for fixed β, the values of λ that satisfy $h_\beta(\lambda) = 0$ are countable, say $\{\lambda_k(\beta)\}$ and moreover, they converge to $\{k^2\pi^2\}$ when β goes to infinity. Hence we can isolate the roots of the functions h_β around $k^2\pi^2$ and approximate them numerically using some software. We used SageMath to compile the Table 1.

Table 1
Values for $\lambda_k(\beta)$ found as roots of function (64).

$\lambda_k(\beta)$	$\beta = 10^2$	$\beta = 10^3$	$\beta = 10^4$	$\beta = 10^5$	$\beta = 10^6$	$\lambda_k = k^2\pi^2$
k=1	9.486473204354914	9.830244232285152	9.865657743495532	9.869209628756735	9.869564922790271	9.869604401089359
k=2	37.947300586356484	39.320978472148354	39.462630975538794	39.47683851502818	39.47825969116076	39.47841760435743
k=3	85.38668247637567	88.47220734834096	88.79091970080098	88.82288665881923	88.8260843051117	88.82643960980423
k=4	151.81154366014752	157.28393857454202	157.85052392706672	157.90735406013744	157.91303876464283	157.9136704174297
k=5	237.23142256188495	245.75618294799932	246.64144366523558	246.73024071899425	246.73912306975478	246.7401100272340
k=6	341.6583215827787	353.888954347627	355.1636789293196	355.2915466354033	355.30433722044694	355.3057584392169
k=7	465.1065236769831	481.6822697315615	483.41722973644585	483.5912718093815	483.6086812167196	483.6106156533786
k=8	607.5923811691522	629.1361491341754	631.4020961068547	631.6294162409495	631.6521550585727	631.6546816697189
k=9	769.1340834087115	796.2506156625528	799.118278063901	799.4059799301307	799.4347587460061	799.4379564882380
k=10	949.7514101063597	983.0256954924237	986.5657756340526	986.920962876951	986.9564922790203	986.9604401089359

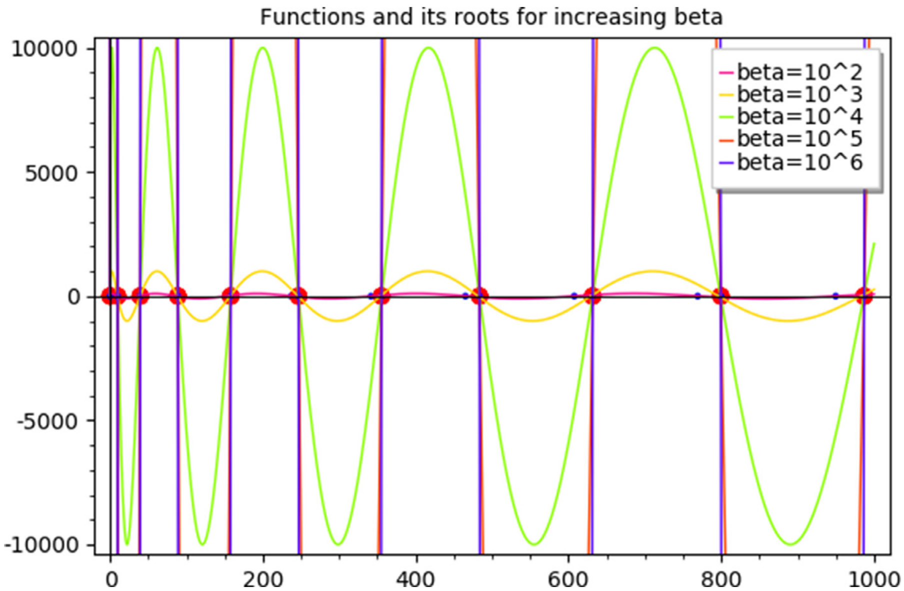

Fig. 1. Functions (64) and roots for different values of β.

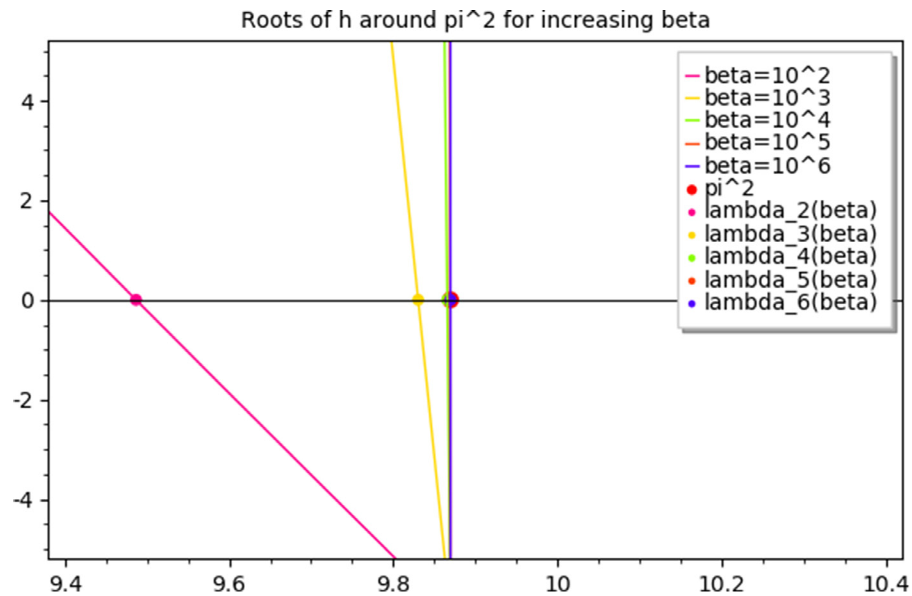

Fig. 2. Close up of functions for different values of β around π^2.

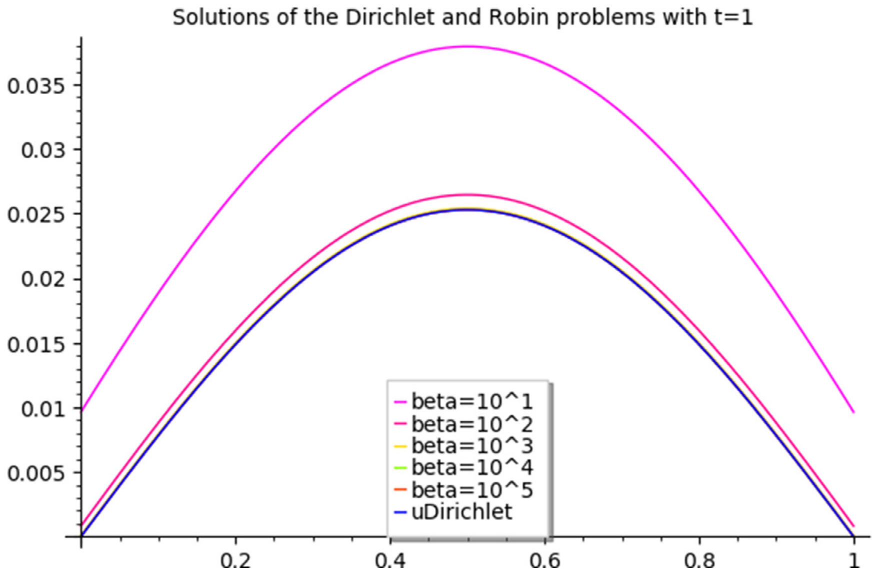

Fig. 3. Solutions for the β-IVBP-1d problems vs. the solution for the D-IVBP-1d problem at time $t = 1$.

Also by the aid of SageMath we were able to visualize the functions h_β and its roots in Fig. 1. A zoomed version around π^2 can also be seen in Fig. 2. For the entries in the above table, we have that

$$\psi_k(\beta; x) = A_k(\beta)\left(\sin(\sqrt{\lambda_k(\beta)}x) + \frac{\sqrt{\lambda_k(\beta)}}{\beta}\cos(\sqrt{\lambda_k(\beta)}x)\right),$$

where the coefficients $A_k(\beta)$ are such that the $\psi_k(\beta; \cdot)$'s are orthonormal in $L^2([0, 1])$. By performing some straightforward calculations keeping in mind that (64) holds, we get that

$$A_k(\beta) = \frac{\sqrt{2}\beta}{\sqrt{\beta^2 + \beta + \lambda_k(\beta)}},$$

which, as expected, converges to $\sqrt{2}$ when β goes to infinity.

Last but not least, for finding the appropiate coefficients $(u_0, \psi_k(\beta; \cdot))$ we resort to the initial value. Again, coupling these solutions provides us with the formal solutions

$$u_\beta(x, t) = \quad \sum_{k=1}^{\infty} (u_0, \psi_k(\beta; \cdot))A_k(\beta)\left(\sin\sqrt{\lambda_k(\beta)}x + \frac{\sqrt{\lambda_k(\beta)}}{\beta}\cos\sqrt{\lambda_k(\beta)}x\right)E_\alpha(-\lambda_k(\beta)t^\alpha),$$

and, if we observe the explicit formulation for $u_\beta(x, t)$ when β goes to infinity, it is clear that we obtain $u(x, t)$.

If we fix some more values and functions we will be able to have an idea of what these solutions look like. For the software implementation we will be considering $\alpha = 0.8$, $u_0 = \sin(\pi x)$ and $\beta = 10^l$ for $l = 1, 2, 3, 4, 5$. Of course, all of these values could be changed at will. We chose this initial data in order to have the solution to the D-IVBP-1d consisting of only one term, and the values of β bigger than 10^4 provided us with no visible changes on the outcome. The solution to the D-IVBP-1d problem is then

$$u(x, t) = \sin(\pi x)E_\alpha(-\pi^2 t^\alpha).$$

Let us remark that in order to plot this function for fixed time $t = 1$, we had to use the integral expression for the Mittag-Leffler functions given in Theorem 2.1 from [27], since from the definition only it seemed that the software brought precision loss due to numeric issues. Also, as it is expected, only the first term of the Fourier series defining u_β has a visual impact, from the second one on it really doesn't affect the visualization. Fig. 3 shows how the solutions of the β-IVBP-1d approaches the solution of the D-IVBP-1d.

6. Conclusion

We have proved existence and uniqueness of solutions to a family of initial boundary value problems with Robin condition for the FDE where the parameter of the family was the Newton heat transfer coefficient linked to the Robin condition on the boundary.The proofs where done following the Fourier approach and the convergence result lead us to a limit problem which is the initial boundary value problem for the FDE with an homogeneous Dirichlet condition. Finally we have visualized the previous results by considering a one-dimensional case with the aid of SageMath software.

Data availability

No data was used for the research described in the article.

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