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Convergence of the solution of the one-phase Stefan problem when the heat transfer coefficient goes to zero

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ABSTRACT

We consider the one-phase unidimensional Stefan problem with a convective boundary condition at the fixed face, with a heat transfer coefficient (proportional to the Biot number) $h > 0$. We study the limit of the temperature θ_h and the free boundary s_h when h goes to zero, and we also obtain an order of convergence. The goal of this paper is to do the mathematical analysis of the physical behavior given in [C. Naaktgeboren, The zero-phase Stefan problem, Int. J. Heat Mass Transfer 50 (2007) 4614–4622].

1. Introduction

In this paper we consider the one-unidimensional free boundary problem (one-phase Stefan problem) with a convective boundary condition on the fixed boundary $\xi = 0$. It consists in determining the temperature $\theta = \theta(\xi, t)$ and the free boundary $\xi = s(t)$ which satisfy the following conditions

$$\begin{cases} \text{(i)}\ \ \rho c\theta_\tau - k\theta_{\xi\xi} = 0, & 0 < \xi < s(\tau),\ \tau > 0, \\ \text{(ii)}\ k\theta_\xi(0,\tau) = h\big[\theta(0,\tau) - f(\tau)\big], & \tau > 0, \\ \text{(iii)}\ \theta\big(s(\tau),\tau\big) = 0, & \tau > 0, \\ \text{(iv)}\ k\theta_\xi\big(s(\tau),\tau\big) = -\rho l \dfrac{ds}{d\tau}(\tau), & \tau > 0, \\ \text{(v)}\ \ \theta(\xi,0) = \varphi(\xi), & 0 \leqslant \xi \leqslant b, \\ \text{(vi)}\ s(0) = b \quad (b > 0) \end{cases} \tag{1}$$

where $h > 0$ is the thermal transfer coefficient, $\varphi(\xi) \geqslant 0$, $0 \leqslant \xi \leqslant b$, is the initial temperature, $f = f(\tau) \geqslant 0$, $\tau > 0$ is the temperature of the external fluid and the compatibility conditions $k\varphi'(0) = h(\varphi(0) - f(0))$ and $\varphi(b) = 0$ are assumed. The goal of this paper is to study the mathematical behavior of the solution $\theta = \theta_h(\xi, \tau)$, $s = s_h(\xi, \tau)$ of the problem (1) when $h \to 0$.

The Stefan problem was studied in the last decades, see for example [1,2,5,8–10] and a large bibliography on the subject was given in [15].

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Existence and uniqueness of solution to problem (1) is given in [6]. In [16] the behavior of the solution of the free boundary problem (1) with respect to the heat transfer coefficient h in the one-phase case was studied. A generalization of this result for the two-phase problem was considered in [17]. There it was proved that the asymptotic behavior when $t \to \infty$ of the one-phase free boundary problem with a convective boundary condition at the fixed face is the same that for the case where the temperature boundary condition, which is depending on time, is given on $x = 0$. Asymptotic behavior for the one-phase problem with temperature boundary condition on the fixed face was given by [3,4]. For the particular case $f(\tau) = \text{Const} > 0$, for the multidimensional case, the study of the asymptotic behavior when $h \to \infty$ is obtained by using the variational inequality [13,14] and for the one-dimensional case in [12]. In [17] the monotone dependence of the solution with respect to the data and with respect to the thermal transfer coefficient is proved to the two-phase Stefan problem. In [11], the classical one-phase Stefan problem is presented in dimensionless form with a time-varying-heat-power boundary condition. The asymptotic behavior of the solution for the generalized form of the Biot number $Bi \to 0$ is studied from a physical point of view. The goal of this paper is to obtain the mathematical analysis of this asymptotic behavior by obtaining an error convergence with respect to the Biot number.

We will make the following assumptions on the initial and boundary data:

(i) Let $\varphi = \varphi(\xi)$ be a positive and piecewise continuous function, with $\varphi'(\xi) \leqslant 0$.
(ii) Let $f = f(\tau)$ be a positive bounded piecewise continuous function, with $f'(\tau) \geqslant 0$.
(iii) Compatibility conditions: $f(0) > \varphi(\xi)$, $\forall \xi \in (0, b)$, $k\varphi'(0) = h(\varphi(0) - f(0))$ and $\varphi(b) = 0$.

If we define the following transformation

$$u(x,t) = \frac{c}{l}\theta(\xi,\tau), \quad x = \frac{\xi}{b},\ t = \frac{\alpha}{b^2}\tau \tag{2}$$

where $\alpha = \frac{k}{\rho c}$ is the diffusion coefficient then the free boundary problem (1) becomes

$$\begin{cases} \text{(i)}\ \ u_t - u_{xx} = 0, & 0 < x < S(t),\ t > 0, \\ \text{(ii)}\ u_x(0,t) = H\big[u(0,t) - F(t)\big], & t > 0, \\ \text{(iii)}\ u\big(S(t),t\big) = 0, & t > 0, \\ \text{(iv)}\ u_x\big(S(t),t\big) = -\dot{S}(t), & t > 0, \\ \text{(v)}\ \ u(x,0) = \chi(x) \geqslant 0, & 0 \leqslant x \leqslant 1, \\ \text{(vi)}\ S(0) = 1 \end{cases} \tag{3}$$

where

$$F(t) = \frac{c}{l} f\left(\frac{b^2 t}{\alpha}\right) \geqslant 0, \qquad H = b\frac{h}{k} > 0 \quad \text{(the Biot number)}, \tag{4}$$

$$\chi(x) = \frac{c}{l}\varphi(bx), \qquad S(t) = \frac{1}{b} s\left(\frac{b^2 t}{\alpha}\right). \tag{5}$$

In Section 2 we enunciate some preliminary results for the solution to the problem (3). In Section 3 we study the convergence for the solution to the problem (3) when the Biot number H (proportional to the heat transfer coefficient h) goes to zero and we give an order of convergence for the corresponding temperature at the fixed face and free boundary.

2. Properties of the solution to problem (3)

Under the assumptions given in Introduction we have the following results:

Lemma 1. *(See [12,16,17].) The solution $u = u_H(x,t)$, $s = s_H(t)$ to the problem* (3) *satisfies the following inequalities*:

(a) $0 \leqslant u_H(x,t) \leqslant F(t)$;
(b) $H_1 < H_2 \Rightarrow u_{H_1}(x,t) \leqslant u_{H_2}(x,t)$;
(c) $H_1 < H_2 \Rightarrow S_{H_1}(t) \leqslant S_{H_2}(t)$;
(d) $u_H(x,t) \geqslant 0$, $u_x \leqslant 0$, $u_t = u_{xx} \geqslant 0$;
(e) $0 \leqslant \dot{S}_H(t) \leqslant H.F(t)$

and it has the integral representation given by

$$u_H(x,t) = \int_0^1 N(x,t;\xi,0)\chi(\xi)\,d\xi + \int_0^t N\big(x,t;S_H(\tau),\tau\big)V_H(\tau)\,d\tau$$

$$-H\int_0^t N(x,t;0,\tau)v_H(\tau)\,d\tau + H\int_0^t N(x,t;0,\tau)F(\tau)\,d\tau, \tag{6}$$

$$S_H(t) = 1 - \int_0^t V_H(\tau)\,d\tau \tag{7}$$

where the functions $V_H = V_H(t)$ and $v_H = v_H(t)$, defined as

$$\begin{cases} V_H(t) = u_x\big(S_H(t),t\big), & t>0, \\ v_H(t) = u_H(0,t), & t>0, \end{cases} \tag{8}$$

are the solutions of the following system of integral equations:

$$V_H(t) = 2\int_0^1 \chi'(\xi)G\big(S_H(t),t,\xi,0\big)\,d\xi - 2\int_0^t H\big[v_H(\tau)-F(\tau)\big]N_x\big(S_H(t),t,0,\tau\big)\,d\tau$$

$$+2\int_0^t V_H(\tau)N_x\big(S_H(t),t,S_H(\tau),\tau\big)\,d\tau, \tag{9}$$

$$v_H(t) = \int_0^1 \chi(\xi)N(0,t,\xi,0)\,d\xi - \int_0^t H\big[v_H(\tau)-F(\tau)\big]N(0,t,0,\tau)\,d\tau$$

$$+\int_0^t V_H(\tau)N\big(0,t,S_H(\tau),\tau\big)\,d\tau, \tag{10}$$

for $0<x<s_H(t)$, $0<t<T$, where G and N are the Green and Neumann functions defined by:

$$G(x,t,\xi,\tau) = K(x,t,\xi,\tau) - K(-x,t,\xi,\tau), \tag{11}$$

$$N(x,t,\xi,\tau) = K(x,t,\xi,\tau) + K(-x,t,\xi,\tau) \tag{12}$$

with

$$K(x,t,\xi,\tau) = \begin{cases} \frac{1}{2\sqrt{\pi(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right), & t>\tau, \\ 0, & t\leqslant\tau. \end{cases} \tag{13}$$

3. Asymptotic behavior of the solution u_H, s_H when $H \to 0$

Motivated by the physical study given in [11], we will study the behavior of the solution $u = u_H(x,t)$, $S = S_H(t)$ of the problem (3) when $H \to 0$. We will prove that the solution to problem (3) converge to the solution of the following parabolic free boundary problem (14):

$$\begin{cases} \text{(i)}\ \ u_{0t} - u_{0_{xx}} = 0, & 0<x<S_0(t),\ t>0, \\ \text{(ii)}\ \ u_{0_x}(0,t) = 0, & t>0, \\ \text{(iii)}\ u_0\big(S_0(t),t\big) = 0, & t>0, \\ \text{(iv)}\ u_{0_x}\big(S_0(t),t\big) = -\dot{S}_0(t), & t>0, \\ \text{(v)}\ \ u_0(x,0) = \chi(x) \geqslant 0, & 0\leqslant x\leqslant 1, \\ \text{(vi)}\ S_0(0) = 1 \end{cases} \tag{14}$$

when $H \to 0$. The problem (14) has the following integral representation

$$u_0(x,t) = \int_0^1 N(x,t;\xi,0)\chi(\xi)\,d\xi + \int_0^t N\big(x,t;S_0(\tau),\tau\big)u_{0_x}\big(S_0(\tau),\tau\big)\,d\tau. \tag{15}$$

We will use some integral relations satisfied by the solutions $u = u_H(x,t)$, $S = S_H(t)$, and $u = u_0(x,t)$, $S = S_0(t)$ to problems (3) and (14) respectively.

Lemma 2. *For problem* (3), *we have the following integral relations*:

$$S_H(t) = 1 - H\int_0^t \big[u_H(0,\tau) - F(\tau)\big]\,d\tau - \int_0^{S_H(t)} u_H(x,t)\,dx + \int_0^1 \chi(x)\,dx, \quad 0 < x < S_H(t),\ t > 0, \tag{16}$$

$$S_H^2(t) = 1 - 2\int_0^{S_H(t)} x u_H(x,t)\,dx + 2\int_0^1 x\chi(x)\,dx + 2\int_0^t u_H(0,\tau)\,d\tau, \quad 0 < x < S_H(t),\ t > 0, \tag{17}$$

$$\int_0^{S_H(t)} u_H^2(x,t)\,dx - \int_0^1 \chi^2(x)\,dx + 2\int_0^t \int_0^{S_H(\tau)} u_{H_x}^2(x,\tau)\,dx\,d\tau \leqslant H\int_0^t F^2(\tau)\,d\tau, \quad 0 < x < S_H(t),\ t > 0. \tag{18}$$

For problem (14), *we have the following integral relations*:

$$S_0(t) = 1 - \int_0^{S_0(t)} u_0(x,t)\,dx + \int_0^1 \chi(x)\,dx, \quad 0 < x < S_0(t),\ t > 0, \tag{19}$$

$$S_0^2(t) = 1 - 2\int_0^{S_0(t)} x u_0(x,t)\,dx + 2\int_0^1 x\chi(x)\,dx + 2\int_0^t u_0(0,\tau)\,d\tau, \quad 0 < x < S_0(t),\ t > 0, \tag{20}$$

$$\int_0^{S_0(t)} u_0^2(x,t)\,dx - \int_0^1 \chi^2(x)\,dx + 2\int_0^t \int_0^{S_0(\tau)} u_{0_x}^2(x,\tau)\,dx\,d\tau = 0, \quad 0 < x < S_0(t),\ t > 0. \tag{21}$$

Proof. See [2,7,17]. □

Lemma 3. *We have $S_0(t) < S_H(t)$, for all $t > 0$, $H > 0$.*

Proof. We suppose that the assertion of the Lemma 3 is false, that is there exists $t_1 > 0$ such that

$$S_0(t) < S_H(t), \quad \forall 0 < t < t_1 \tag{22}$$

and

$$S_0(t_1) = S_H(t_1).$$

If we define

$$w_H(x,t) = u_H(x,t) - u_0(x,t), \quad 0 < x < S_0(t),\ 0 < t < t_1$$

we have the following properties:

$$\begin{cases} \text{(i)}\ \ w_{H_t} - w_{H_{xx}} = 0, & 0 < x < S_0(t),\ 0 < t < t_1, \\ \text{(ii)}\ \ w_{H_x}(0,t) = u_{H_x}(0,t) - u_{0_x}(0,t) = H\big[u_H(0,t) - F(t)\big] < 0, & 0 < t < t_1, \\ \text{(iii)}\ w_H\big(S_0(t),t\big) = u_H\big(S_0(t),t\big) \geqslant 0, & 0 < t < t_1, \\ \text{(iv)}\ w_{H_x}\big(S_0(t),t\big) = u_{H_x}\big(S_0(t),t\big) + \dot{S}_0(t), & 0 < t < t_1, \\ \text{(v)}\ \ w_H(x,0) = 0, & 0 \leqslant x \leqslant 1, \\ \text{(vi)}\ S_0(0) = 1 \end{cases} \tag{23}$$

and

$$w_H\big(S_0(t_1),t_1\big) = u_H\big(S_0(t_1),t_1\big) = u_H\big(S_H(t_1),t_1\big) = 0.$$

By the maximum principle we deduce that $w_H(0,t) \geqslant 0$, $0 < t < t_1$ by using the condition (23)(ii). Therefore we have a minimum value $w_H(S_0(t_1),t_1) = 0$ and then we get $w_{H_x}(S_0(t_1),t_1) < 0$. But on the other hand we have

$$w_{H_x}\big(S_0(t_1),t_1\big) = u_{H_x}\big(S_0(t_1),t_1\big) - u_{0x}\big(S_0(t_1),t_1\big) = -\dot{S}_H(t_1) + \dot{S}_0(t_1) \geqslant 0$$

which is a contradiction. □

Lemma 4. *We have*

$$u_H(x,t) \geqslant u_0(x,t), \quad \text{for all } 0 < x < S_0(t),\ t > 0. \tag{24}$$

Proof. If we consider

$$w_H(x,t) = u_H(x,t) - u_0(x,t), \quad 0 < x \leqslant S_0(t),\ t > 0$$

then w_H has the following properties:

$$\begin{cases} \text{(i)}\ \ w_{H_t} - w_{H_{xx}} = 0, & 0 < x < S_0(t),\ t > 0, \\ \text{(ii)}\ w_{H_x}(0,t) = u_{H_x}(0,t) - u_{0_x}(0,t) = H\big[u_H(0,t) - F(t)\big] < 0, & t > 0, \\ \text{(iii)}\ w_H\big(S_0(t),t\big) = u_H\big(S_0(t),t\big) \geqslant 0, & t > 0, \\ \text{(iv)}\ w_{H_x}\big(S_0(t),t\big) = -u_{H_x}\big(S_0(t),t\big) + \dot{S}_0(t), & t > 0, \\ \text{(v)}\ \ w_H(x,0) = 0, & \\ \text{(vi)}\ S_H(0) = 1. & \end{cases} \tag{25}$$

By the maximum principle, as in Lemma 3, we analyze the sign of $w_H(0,t)$ and we obtain that the minimum of $w_H(x,t)$ is a positive value and it is on the parabolic boundary. Then we get (24). □

Lemma 5. *If* $\int_0^{+\infty} F(\tau)\,d\tau < \infty$ *then we have the following limit*:

$$\lim_{H\to 0} S_H(t) = S_0(t) \tag{26}$$

for each t in a compact set in $\mathbb{R}^+$, *with the following order of convergence given by*:

$$0 \leqslant S_H(t) - S_0(t) \leqslant H \int_0^t F(\tau)\,d\tau. \tag{27}$$

Proof. According to Lemma 2 it follows

$$\begin{aligned} S_H(t) - S_0(t) &= \int_0^{S_0(t)} u_0(x,t)\,dx - \int_0^{S_H(t)} u_H(x,t)\,dx - H\int_0^t \big[u_H(0,\tau) - F(\tau)\big]\,d\tau \\ &= \int_0^{S_H(t)} u_{0H}(x,t)\,dx - \int_0^{S_H(t)} u_H(x,t)\,dx + H\int_0^t \big[F(\tau) - u_H(0,\tau)\big]\,d\tau \end{aligned}$$

where u_{0H} is defined as an extension of u_0 by 0 as follows:

$$u_{0H}(x,t) = \begin{cases} u_0(x,t), & 0 < x \leqslant S_0(t),\ t > 0, \\ 0, & S_0(t) < x \leqslant S_H(t),\ t > 0. \end{cases}$$

Then we have the estimation (27) because $u_{0H}(x,t) - u_H(x,t) \leqslant 0$ and $u_H(0,\tau) \geqslant 0$. Then, the thesis holds. □

Lemma 6. *If* $\int_0^{+\infty} F(\tau)\,d\tau < \infty$ *then we have the following limit*:

$$\lim_{H\to 0} u_H(0,t) = u_0(0,t) \tag{28}$$

for each t in a compact set in $\mathbb{R}^+$ *with the following order of convergence given by*

$$\big|u_H(0,t) - u_0(0,t)\big| \leqslant L(t)H$$

where

$$L(t) = I(t) + J(t)\int_0^t F(\tau)\,d\tau \tag{29}$$

with

$$I(t) = \left(\frac{2}{\sqrt{\pi}}\sqrt{t} + \left(\frac{6}{e}\right)^{3/2} \frac{t^2 \|F\|_t}{2\sqrt{\pi}}\right)\|F\|_t, \tag{30}$$

$$J(t) = \frac{1 + \int_0^1 \chi(x)\,dx}{2\sqrt{\pi}} \left(\frac{6}{e}\right)^{\frac{3}{2}} t\|\dot{S}_0\|_t + \left(\frac{10}{e}\right)^{\frac{3}{2}} \frac{1}{2\sqrt{\pi}}\left[\frac{\|S_0\|_t^2}{2} t + t^2\right]. \tag{31}$$

Proof. By using (6) and (15) we have

$$\begin{aligned}
u_H(0,t) - u_0(0,t) &= -\int_0^t H\big[u_H(0,\tau) - F(\tau)\big] N(0,t,0,\tau)\,d\tau \\
&\quad + \int_0^t u_{H_x}\big(S_H(\tau),\tau\big) N\big(0,t,S_H(\tau),\tau\big)\,d\tau - \int_0^t u_{0\xi}\big(S_0(\tau),\tau\big) N\big(x,t;S_0(\tau),\tau\big)\,d\tau \\
&= -\int_0^t H\big[u_H(0,\tau) - F(\tau)\big] N(0,t,0,\tau)\,d\tau \\
&\quad + \int_0^t \big[N\big(0,t;S_0(\tau),\tau\big)\dot{S}_0(\tau) - N\big(0,t;S_H(\tau),\tau\big)\dot{S}_H(\tau)\big]\,d\tau := A_1 + A_2,
\end{aligned}$$

where

$$A_1 = \int_0^t H\big[F(\tau) - u_H(0,\tau)\big] N(0,t;0,\tau)\,d\tau \leqslant H \int_0^t \frac{F(\tau)}{\sqrt{\pi(t-\tau)}}\,d\tau \leqslant \frac{2H\|F\|_t}{\sqrt{\pi}}\sqrt{t} \tag{32}$$

and

$$\begin{aligned}
A_2 &= \int_0^t \big[N\big(0,t;S_0(\tau),\tau\big)\dot{S}_0(\tau) - N\big(0,t,S_H(\tau),\tau\big)\dot{S}_H(\tau)\big]\,d\tau \\
&= \int_0^t \dot{S}_H(\tau)\big[N\big(0,t;S_0(\tau),\tau\big) - N\big(0,t;S_H(\tau),\tau\big)\big]\,d\tau \\
&\quad + \int_0^t N\big(0,t;S_0(\tau),\tau\big)\big[\dot{S}_0(\tau) - \dot{S}_H(\tau)\big]\,d\tau := A_3 + A_4,
\end{aligned} \tag{33}$$

and

$$\|F\|_t = \max\big\{F(\tau),\ 0 < \tau < t\big\}. \tag{34}$$

If we take into account that

$$\begin{aligned}
N\big(0,t;S_0(\tau),\tau\big) - N\big(0,t,S_H(\tau),\tau\big) &= N_\xi(0,t;c,\tau)\big[S_0(\tau) - S_H(\tau)\big] \\
&= \frac{-c\exp(-\frac{c^2}{4(t-\tau)})}{2\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\big[S_0(\tau) - S_H(\tau)\big],
\end{aligned}$$

where $c = c(H,\tau) \in (S_0(\tau), S_H(\tau))$ and by using the inequality

$$\frac{\exp(\frac{-x^2}{\alpha(t-\tau)})}{(t-\tau)^{\frac{n}{2}}} \leqslant \left(\frac{n\alpha}{2ex^2}\right)^{\frac{n}{2}}, \quad \alpha, x > 0,\ t > \tau,\ n \in \mathbb{N}, \tag{35}$$

then we have

$$\begin{aligned}
N\big(0,t;S_0(\tau),\tau\big)-N\big(0,t,S_H(\tau),\tau\big) &\leqslant \left(\frac{6}{e}\right)^{3/2}\frac{1}{2\sqrt{\pi}c^2}\big[S_H(\tau)-S_0(\tau)\big]\\
&\leqslant \left(\frac{6}{e}\right)^{3/2}\frac{1}{2\sqrt{\pi}S_0^2(\tau)}\big[S_H(\tau)-S_0(\tau)\big]\\
&\leqslant \left(\frac{6}{e}\right)^{3/2}\frac{1}{2\sqrt{\pi}}\big[S_H(\tau)-S_0(\tau)\big].
\end{aligned}$$

Therefore by using Lemma 1 and (27) we have

$$\begin{aligned}
|A_3| &\leqslant \left(\frac{6}{e}\right)^{3/2}\frac{t}{2\sqrt{\pi}}\|\dot{S}_H\|_t\|S_H-S_0\|_t\\
&\leqslant \left(\frac{6}{e}\right)^{3/2}\frac{t}{2\sqrt{\pi}}H^2\|F\|_t\int_0^t F(\tau)\,d\tau.
\end{aligned}\tag{36}$$

Moreover, we have

$$\begin{aligned}
A_4 &= \int_0^t N\big(0,t;S_0(\tau),\tau\big)\big[\dot{S}_0(\tau)-\dot{S}_H(\tau)\big]\,d\tau\\
&= -\int_0^t \frac{\partial}{\partial\tau}\big(N\big(0,t,S_0(\tau),\tau\big)\big)\big[S_0(\tau)-S_H(\tau)\big]\,d\tau
\end{aligned}\tag{37}$$

where

$$\frac{\partial}{\partial\tau}\big(N\big(0,t,S_0(\tau),\tau\big)\big)=N_\xi\big(0,t,S_0(\tau),\tau\big)\dot{S}_0(\tau)+N_\tau\big(0,t,S_0(\tau),\tau\big).\tag{38}$$

Taking into account (35) and

$$N_\xi\big(0,t,S_0(\tau),\tau\big)=\frac{-S_0(\tau)\exp(-\frac{S_0^2(\tau)}{4(t-\tau)})}{2\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}$$

and

$$N_\tau\big(0,t,S_0(\tau),\tau\big)=2K_\tau\big(0,t,S_0(\tau),\tau\big)=-\frac{\exp(-\frac{S_0^2(\tau)}{4(t-\tau)})S_0^2(\tau)}{4\sqrt{\pi}(t-\tau)^{\frac{5}{2}}}-\frac{\exp(-\frac{S_0^2(\tau)}{4(t-\tau)})S_0^2(\tau)}{2\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}$$

we have

$$\big|N_\xi\big(0,t,S_0(\tau),\tau\big)\big|\leqslant\frac{1+\int_0^1\chi(x)\,dx}{2\sqrt{\pi}}\left(\frac{6}{e}\right)^{\frac{3}{2}}$$

and

$$\big|N_\tau\big(0,t,S_0(\tau),\tau\big)\big|\leqslant\frac{\exp(-\frac{1}{4(t-\tau)})}{2\sqrt{\pi}(t-\tau)^{\frac{5}{2}}}\left[\frac{S_0^2(\tau)}{2}+(t-\tau)\right]\leqslant\left(\frac{10}{e}\right)^{\frac{3}{2}}\frac{1}{2\sqrt{\pi}}\left[\frac{S_0^2(\tau)}{2}+t\right].$$

Therefore we obtain

$$\begin{aligned}
|A_4| &\leqslant \int_0^t\big[\big|N_\xi\big(0,t,S_0(\tau),\tau\big)\big|\big|\dot{S}_0(\tau)\big|+\big|N_\tau\big(0,t,S_0(\tau),\tau\big)\big|\big]\big|S_0(\tau)-S_H(\tau)\big|\,d\tau\\
&\leqslant\left\{\frac{1+\int_0^1\chi(x)\,dx}{2\sqrt{\pi}}\left(\frac{6}{e}\right)^{\frac{3}{2}}t\|\dot{S}_0\|_t+\left(\frac{10}{e}\right)^{\frac{3}{2}}\frac{1}{2\sqrt{\pi}}\left[\frac{\|S_0\|_t^2}{2}t+t^2\right]\right\}\|S_0-S_H\|_t.
\end{aligned}\tag{39}$$

Owing to (27) and the fact we can take $H\leqslant 1$, then we get

$$
\begin{aligned}
\left|u_H(0,t)-u_0(0,t)\right| &\leqslant \frac{2H\|F\|_t}{\sqrt{\pi}}\sqrt{t}+\left(\frac{6}{e}\right)^{3/2}\frac{t}{2\sqrt{\pi}}H^2\|F\|_t\int_0^t F(\tau)\,d\tau \\
&\quad+\left\{\frac{1+\int_0^1\chi(x)\,dx}{2\sqrt{\pi}}\left(\frac{6}{e}\right)^{\frac{3}{2}}t\|\dot{S}_0\|_t+\left(\frac{10}{e}\right)^{\frac{3}{2}}\frac{1}{2\sqrt{\pi}}\left[\frac{\|S_0\|_t^2}{2}t+t^2\right]\right\}\|S_0-S_H\|_t \\
&\leqslant \frac{2H\|F\|_t}{\sqrt{\pi}}\sqrt{t}+\left(\frac{6}{e}\right)^{3/2}\frac{t^2}{2\sqrt{\pi}}H\|F\|_t^2 \\
&\quad+\left\{\frac{1+\int_0^1\chi(x)\,dx}{2\sqrt{\pi}}\left(\frac{6}{e}\right)^{\frac{3}{2}}t\|\dot{S}_0\|_t+\left(\frac{10}{e}\right)^{\frac{3}{2}}\frac{1}{2\sqrt{\pi}}\left[\frac{\|S_0\|_t^2}{2}t+t^2\right]\right\}\|S_0-S_H\|_t \\
&\leqslant I(t)H+J(t)\|S_0-S_H\|_t\leqslant L(t)H
\end{aligned}
\tag{40}
$$

and the thesis holds. □

Theorem 7. *If $\int_0^{+\infty}F(\tau)\,d\tau<\infty$ then we have*

$$u_H(x,t)\underset{H\to 0}{\longrightarrow}u_0(x,t),\quad \textit{for all compact set in the domain } 0<x<S_0(t),\ t>0.$$

Proof. Taking into account Lemma 2 we have

$$
\begin{aligned}
0 &\leqslant \int_0^{S_0(t)} x\left[u_H(x,t)-u_0(x,t)\right]dx+\frac{S_H^2(t)}{2}-\frac{S_0^2(t)}{2} \\
&= \int_{S_0(t)}^{S_H(t)} xu_H(x,t)\,dx+\int_0^t\left[u_H(0,\tau)-u_0(0,\tau)\right]d\tau.
\end{aligned}
$$

By using (40), Lemmas 1 and 6, and taking $H\leqslant 1$ we have

$$
\begin{aligned}
&\int_{S_0(t)}^{S_H(t)} xu_H(x,t)\,dx+\int_0^t\left[u_H(0,\tau)-u_0(0,\tau)\right]d\tau \\
&\quad\leqslant \left\|u_H(.,t)\right\|_{[S_0(t),S_H(t)]}\left(\frac{S_H^2(t)}{2}-\frac{S_0^2(t)}{2}\right)+H\int_0^t L(\tau)\,d\tau \\
&\quad\leqslant \|F\|_t\frac{1}{2}\big(S_H(t)+S_0(t)\big)\big(S_H(t)-S_0(t)\big)+H\int_0^t L(\tau)\,d\tau\leqslant M(t)H
\end{aligned}
\tag{41}
$$

where $M(t)$ is given by

$$M(t)=\int_0^t L(\tau)\,d\tau+\|F\|_t\left(S_0(t)+\frac{1}{2}\int_0^t F(\tau)\,d\tau\right)\int_0^t F(\tau)\,d\tau \tag{42}$$

and the thesis holds. □

4. Conclusions

The asymptotic behavior of the solution to the Stefan problem with a convective boundary condition at the fixed face when the heat transfer coefficient (proportional to the Biot number) goes to zero has been obtained given an order of convergence.

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