A free boundary problem for a diffusion–convection equation

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ABSTRACT

One-dimensional free boundary problem for a nonlinear diffusion–convection equation with a Dirichlet condition at fixed face $x = 0$, variable in time, is considered. Through several transformations the problem is reduced to a free boundary problem for a diffusion equation and the integral formulation is obtained. By using fixed point theorems, the existence of at least a solution, for small time, to a system of coupled nonlinear integral equations is obtained.

1. Introduction

Free boundary problems (FBP) are of great importance, both physically and mathematically. FBP are boundary value problems for partial differential equations where an unknown moving boundary must be determined [1–4]. In this paper, we formulate a FBP for a nonlinear diffusion–convection equation namely Rosen–Fokas–Yorstos equation [5,6]. This equation describes fluid diffusion with convective effects in porous media and has multiple applications, for example, to ground water hydrology, oil reservoir engineering and other biological applications as the drug propagation in the arterial tissues.

In [7,8] a FBP on a finite interval is formulated and solved for a nonlinear diffusion–convection equation which describe drug diffusion in arterial tissues after the drug is released by an arterial stent and the problem is reduced to a system of nonlinear integral equations.

We will study a one-dimensional FBP for the diffusion–convection equation with a variable Dirichlet condition(which is the one novelty with respect to [7,8]) at the fixed face x = 0 and a Stefan like condition on the free boundary which has a convective term. The present paper is organized as follows: In Section 2, we introduce the FBP and through several transformations we map the FBP for the nonlinear diffusion–convection equation into an equivalent FBP for the linear heat-diffusion equation. In Section 3, we give an equivalent integral formulation to problem which requires to solve a system of three coupled nonlinear Volterra integral equations. Section 4 is subdivided into two subsections: in Section 4.1, fixed one unknown, we prove existence and uniqueness of the solution, local in time, by using Banach fixed point theorem, in Section 4.2 we use the Schauder fixed point theorem to prove that there exists at least one solution of this unknown.

We can remark that sequential transformations used on Section 2 have been previously used in different physical context as modelled, in particular, by moving boundary problems, for example [6,9–16].

2. Free boundary problem

We consider the free boundary $s = s(t) > 0$, defined for $t > 0$, and $u(x, t)$ which satisfy a diffusion–convection equation with the following conditions:

$$u_t = u^2(Du_{xx} - u_x) \ , \quad 0 < x < s(t) \ , \quad t > 0 \ , \tag{1}$$

$$u(0, t) = f(t) \ , \quad t > 0 \ , \tag{2}$$

$$u(s(t), t) = \beta > 0 \ , \quad t > 0 \ , \tag{3}$$

$$Du_x(s(t), t) - u(s(t), t) = -\dot{s}(t) \ , \quad t > 0 \ , \tag{4}$$

$$u(x, 0) = u_0(x) > \beta \ , \quad 0 \le x \le b \ , \tag{5}$$

$$s(0) = b \tag{6}$$

where D is the diffusivity, u_0 is the initial concentration and $f = f(t)$ is the concentration in the fixed face $x = 0$. We assume that:

$$f \in C^1[0, \sigma], \quad u_0 \in C^1[0, b], \quad u_0(0) = f(0), \quad u_0(b) = \beta, \quad f(t) > \frac{3\beta}{2} \tag{7}$$

Following [6–8] we will transform this problem in the one which is governed by the Burgers equation. We have:

Lemma 1. *(A) If $u = u(x, t)$, $s = s(t)$ is a solution to the problem* (1)–(6) *then $v = v(z, t)$, $z_0(t)$, $z_1(t)$ defined by:*

$$v(z, t) = u(x, t), \tag{8}$$

where

$$z(x, t) = C_1 + \int_0^t \left(u(0, \tau) - Du_x(0, \tau) \right) d\tau + \int_0^x \frac{1}{u(\eta, t)} d\eta \tag{9}$$

$$z_0(t) = z(0,t) = C_1 + \int_0^t \left(f(\tau) - D\frac{v_z(z_0(\tau), \tau)}{f(\tau)} \right) d\tau \tag{10}$$

$$z_1(t) = z(s(t), t) = C_2 + (\beta + 1)t - \frac{D(\beta + 1)}{\beta^2} \int_0^t v_z(z_1(\tau), \tau) d\tau \tag{11}$$

with C_1 an arbitrary constant, is a solution to the problem given by the Burgers equation

$$v_t = Dv_{zz} - 2vv_z \quad , \quad z_0(t) < z < z_1(t) \quad , \quad t > 0 \quad , \tag{12}$$

with the following initial and boundary conditions:

$$v(z_0(t), t) = f(t) \ , \quad t > 0 \ , \tag{13}$$

$$v(z_1(t), t) = \beta \ , \quad t > 0 \ , \tag{14}$$

$$D\frac{v_z(z_1(t), t)}{v(z_1(t), t)} - v(z_1(t), t) = -\frac{\beta}{\beta + 1}\dot{z}_1(t) \ , \quad t > 0 \ , \tag{15}$$

$$v(z, 0) = v_0(z) \ , \quad C_1 \le z \le C_2 \ , \tag{16}$$

$$z_0(0) = C_1 \ , \quad z_1(0) = C_2 \tag{17}$$

where

$$v_0(z) = u_0(g^{-1}(z)), \qquad g(x) = C_1 + \int_0^x \frac{1}{u_0(\eta)} d\eta \tag{18}$$

$$C_2 = C_1 + U_0 = C_1 + \int_0^b \frac{1}{u_0(\eta)} d\eta \tag{19}$$

and the constants b, C_1 and C_2 satisfy the following relation

$$b = \int_{C_1}^{C_2} v_0(z) dz \tag{20}$$

(B) Conversely if $v = v(z,t)$, $z_0(t)$, $z_1(t)$ is the solution to the problem (12)–(17) then $u = u(x,t)$, $s = s(t)$ given by

$$u(x,t) = v(z,t), \tag{21}$$

with

$$x(z,t) = \int_{z_0(t)}^z v(\eta, t) d\eta, \tag{22}$$

$$s(t) = x(z_1(t), t) = \int_{z_0(t)}^{z_1(t)} v(\eta, t) d\eta \tag{23}$$

is a solution to the problem (1)–(6).

Proof. (A) From (8), (9) and by (1) we have

$$z_x = \frac{1}{u(x,t)} = \frac{1}{v(z,t)}, \quad z_t = u(x,t) - Du_x(x,t) = v(z,t) - D\frac{v_z(z,t)}{v(z,t)},$$

and

$$u_x(x,t) = \frac{v_z(z,t)}{v(z,t)}, \quad u_{xx}(x,t) = \frac{v_{zz}(z,t)}{v^2(z,t)} - \frac{v_z^2(z,t)}{v^3(z,t)},$$

$$u_t(x,t) = v_t(z,t) + v_z \left(v(z,t) - D\frac{v_z(z,t)}{v(z,t)} \right).$$

Then, from (1) we get (12) which is the Burgers equation for the dependent variable $v(z,t)$.

Taking into account (9) the domain $D = \{(x,t)/0 < x < s(t), t > 0\}$ for $u(x,t)$ is transformed into the domain $D^* = \{(z,t)/z_0(t) < z < z_1(t), t > 0\}$ for $v(z,t)$, where $z_0(t)$ and $z_1(t)$ are given by

$$z_0(t) = z(0,t) = C_1 + \int_0^t \left(u(0,\tau) - Du_x(0,\tau) \right) d\tau$$

$$z_1(t) = z(s(t), t) = C_1 + \int_0^t \left(u(0,\tau) - Du_x(0,\tau) \right) d\tau + \int_0^{s(t)} \frac{1}{u(\eta, t)} d\eta$$

If we derivate z_1 respect to variable t and we use (1) and the conditions (2)–(6), we obtain the follow relation

$$\dot{z}_1(t) = \frac{\beta+1}{\beta}\dot{s}_1(t).$$

Then, from (4) we have (15) and the expression (11) for $z_1(t)$, where $z_1(0) = C_1 + \int_0^b \frac{1}{u_0(\eta)} d\eta = C_2$. Eqs. (13) and (14) follows inmediatly from (2) and (3) respectively.

For $t = 0$ we have that

$$z = C_1 + \int_0^x \frac{1}{u_0(\eta)} d\eta = g(x),$$

then (5) is equivalent to $v(z,0) = u_0\left(g^{-1}(z)\right)$ for $C_1 \le z \le C_2$ where $C_2 = C_1 + \int_0^b \frac{1}{u_0(\eta)} d\eta$. Therefore (16) holds.

To prove (B) we consider (21), (22) and the (12)–(17) which are satisfied by $v = v(z,t)$, $z_0(t)$, $z_1(t)$. We have

$$x_z = v(z,t), \quad x_t = Dv_z - v^2(z,t).$$

Moreover, for $z = z_0(t)$ is $x = 0$ and for $z = z_1(t)$ is $x = \int_{z_0(t)}^{z_1(t)} v(\eta, t) d\eta = s(t)$. Since

$$v_t = Du_x^2 u - u_x u^2 + u_t, \quad v_z = u_x u, \quad v_{zz} = u_{xx}u^2 + u_x^2 u$$

then (12) yields (1).

The conditions (2), (3) and (5) follows inmediatly from (13), (14) and (16) respectively.

To prove (4), from (23) we calculate $\dot{s}(t)$ and use (12) and (14). We have

$$\dot{s}(t) = v(z_1(t), t)\dot{z}_1(t) - v(z_0(t), t)\dot{z}_0(t) + \int_{z_0(t)}^{z_1(t)} v_t(\eta, t) d\eta$$

$$= \beta - D\frac{v_z(z_1(t), t)}{\beta} = \beta - Du_x(s(t), t)$$

and (4) holds. □

Remark 1. Eq. (9) is equivalent to the relations

$$z_x = \frac{1}{u(x,t)}, \qquad z_t = u(x,t) - Du_x(x,t). \tag{24}$$

Eq. (22) is equivalent to

$$x_z = v(z,t), \quad x_t = Dv_z - v^2(z,t). \tag{25}$$

Now we introduce the Galilean Transformation given by

$$V(y,t) = v(z,t) - \beta, \qquad y = z - 2\beta t \qquad t > 0 \tag{26}$$

to obtain the following result:

Lemma 2. Under the transformation (26) the problem (12)–(19) is equivalent to the following FBP:

$$V_t = DV_{yy} - 2VV_y \quad , \quad y_0(t) < y < y_1(t) \quad , \quad t > 0 \quad , \tag{27}$$

$$V(y_0(t), t) = f(t) - \beta \ , \quad t > 0 \ , \tag{28}$$

$$V(y_1(t), t) = 0 \ , \quad t > 0 \ , \tag{29}$$

$$D\frac{V_y(y_1(t), t)}{\beta} = \frac{\beta(1 - \beta) - \beta\dot{y}_1(t)}{\beta + 1} \ , \quad t > 0 \ , \tag{30}$$

$$V(y, 0) = V_0(y) \ , \quad C_1 \le y \le C_2 \ , \tag{31}$$

$$y_0(0) = C_1 \ , \quad y_1(0) = C_2 \tag{32}$$

where

$$V_0(y) = v_0(y) - \beta \tag{33}$$

$$y_0(t) = C_1 - 2\beta t + \int_0^t \left(f(\tau) - D\frac{V_y(y_0(\tau), \tau)}{f(\tau)} \right) d\tau \tag{34}$$

$$y_1(t) = C_2 + (1 - \beta)t - \frac{D(\beta + 1)}{\beta^2} \int_0^t V_y(y_1(\tau), \tau) d\tau. \tag{35}$$

Proof. The Galilean transformation (26) leaves invariant the Burgers equation (12). The free boundaries $y_0(t)$ and $y_1(t)$ given by (34)–(35) are obtained from (10)–(11). The conditions (28)–(32) follows from (13)–(17).

Conversely, if we define

$$v(z,t) = V(y,t) + \beta, \qquad z = y + 2\beta t \qquad t > 0,$$

from (27)–(35) we obtain (12)–(19) with $z_0(t)$ and $z_1(t)$ given by (10) and (11) respectively. □

Let us now transform problem (27)–(35) in the one which is governed by a heat-diffusion equation using the Hopf Cole transformation given by

$$w(y,t) = C(t)V(y,t)\eta(y,t), \quad y_0(t) \le y \le y_1(t) \ , \ t > 0, \tag{36}$$

with

$$C(t) = 1 - \int_0^t w_y(y_1(\tau),\tau)d\tau, \tag{37}$$

and

$$\eta(y,t) = exp\left(\frac{1}{D}\int_y^{y_1(t)} V(\xi,t)d\xi\right). \tag{38}$$

We have the following result:

Theorem 1. *Under transformation* (36)–(38) *problem* (27)–(35) *is equivalent to the free boundary problem* (39)–(47) *given by:*

$$w_t = Dw_{yy} \ , \ y_0(t) < y < y_1(t) \ , \ t > 0 \ , \tag{39}$$

$$w(y_0(t),t) = (f(t)-\beta)\left(C(t) + \frac{1}{D}\int_{y_0(t)}^{y_1(t)} w(\xi,t)d\xi\right) \ , \ t > 0 \ , \tag{40}$$

$$w(y_1(t),t) = 0 \ , \ t > 0 \ , \tag{41}$$

$$\frac{Dw_y(y_1(t),t)}{\beta C(t)} = \frac{\beta(1-\beta) - \beta \dot{y}_1(t)}{\beta+1} \ , \ t > 0 \ , \tag{42}$$

$$w(y,0) = F(y) \ , \ C_1 \le y \le C_2 \ , \tag{43}$$

$$y_0(0) = C_1 \ , \ y_1(0) = C_2 \tag{44}$$

where

$$F(y) = V_0(y)exp\left(\frac{1}{D}\int_y^{C_2} V_0(\xi)d\xi\right) = V_0(y)\left(1 - \frac{1}{D}\int_y^{C_2} w(\xi,0)d\xi\right) \tag{45}$$

and the free boundaries $y_0 = y_0(t)$ *and* $y_1 = y_1(t)$ *are given by:*

$$y_0(t) = C_1 - \beta^2 \int_0^t \frac{1}{f(\tau)}d\tau - D\int_0^t \frac{w_y(y_0(\tau),\tau)}{w(y_0(\tau),\tau)}\left(1 - \frac{\beta}{f(\tau)}\right)d\tau, \tag{46}$$

$$y_1(t) = C_2 + (1-\beta)t + \frac{D(\beta+1)}{\beta^2}\log\left(1 - \int_0^t w_y(y_1(\tau),\tau)d\tau\right). \tag{47}$$

Proof. To prove the equivalence of the two problems we will deduce the inverse transformation to the relation (36) by considering the definition (38), we have

$$log(\eta(y,t)) = \frac{1}{D}\int_y^{y_1(t)} V(\xi,t)d\xi$$

then

$$\eta_y(y,t) = -\frac{1}{D}V(y,t)\eta(y,t) = -\frac{1}{D}\frac{w(y,t)}{C(t)}.$$

Integrating on variable y, it follows that

$$\eta(y,t) = \frac{C(t) + \frac{1}{D}\int_y^{y_1(t)} w(\xi,t)d\xi}{C(t)}.$$

Therefore, we have that the inverse relation to the generalized Hopf–Cole transformation (36) is expressed by:

$$V(y,t) = \frac{w(y,t)}{C(t) + \frac{1}{D}\int_y^{y_1(t)} w(\xi,t)d\xi}. \tag{48}$$

Under transformation (48) the Burgers equation (27) is mapped into the linear heat-diffusion equation (39). The initial and boundary conditions (40)–(44) are easily obtained from (28)–(32). The expressions (46) and (47) for the free boundaries are obtained from (34) and (35) respectively.

The converse is proved analogously. $\square$

3. Integral formulation

In this section, we give an integral formulation of the free boundary problem (39)–(47). We have the following equivalence theorem.

Theorem 2. *Let* (7) *and* $0 < D < 2$ *be. The solution to the free boundary problem* (39)–(47) *has the following integral representation*

$$w(y,t) = \int_{C_1}^{C_2} G(y,t;\xi,0)F(\xi)d\xi + D\int_0^t \phi_1(\tau)G(y,t;y_1(\tau),\tau)d\tau \tag{49}$$

$$+ \beta^2 \int_0^t \frac{h(\tau)}{f(\tau)}G(y,t;y_0(\tau),\tau)d\tau - D\beta\int_0^t \frac{\phi_2(\tau)}{f(\tau)}G(y,t;y_0(\tau),\tau)d\tau$$

$$- D\int_0^t h(\tau)N_y(y,t;y_0(\tau),\tau)d\tau,$$

with

$$h(t) = (f(t)-\beta)\left(C(t) + \frac{1}{D}\int_{y_0(t)}^{y_1(t)} w(\xi,t)d\xi\right), \tag{50}$$

$$y_0(t) = C_1 - \beta^2 \int_0^t \frac{1}{f(\tau)}d\tau - D\int_0^t \frac{\phi_2(\tau)}{h(\tau)}\left(1 - \frac{\beta}{f(\tau)}\right)d\tau, \tag{51}$$

$$y_1(t) = C_2 + (1-\beta)t + \frac{D(\beta+1)}{\beta^2}ln\left(1 - \int_0^t \phi_1(\tau)d\tau\right) \tag{52}$$

and ϕ_1, ϕ_2 *are defined by*

$$\phi_1(t) = \frac{\partial w}{\partial y}(y_1(t),t) \ , \ \phi_2(t) = \frac{\partial w}{\partial y}(y_0(t),t) \tag{53}$$

if and only if it satisfies the following system of two Volterra integral equations:

$$\phi_1(t) = \frac{2}{2-D}\left\{\int_{C_1}^{C_2} N(y_1(t),t;\xi,0)F'(\xi)d\xi\right.$$

$$+ D\int_0^t \phi_1(\tau)G_y(y_1(t),t;y_1(\tau),\tau)d\tau$$

$$+ \beta^2 \int_0^t \frac{h(\tau)}{f(\tau)}G_y(y_1(\tau),t;y_0(\tau),\tau)d\tau$$

$$- D\beta\int_0^t \frac{\phi_2(\tau)}{f(\tau)}G_y(y_1(\tau),t;y_0(\tau),\tau)d\tau,$$

$$\left. - \int_0^t h'(\tau)N(y_1(t),t;y_0(\tau),\tau)d\tau\right\}, \tag{54}$$

$$\phi_2(t) = \frac{2f(t)}{2f(t)-D\beta}\left\{-\beta^2 \frac{h(t)}{f(t)} + \int_{C_1}^{C_2} N(y_0(t),t;\xi,0)F'(\xi)d\xi\right.$$

$$+ D\int_0^t G_y(y_0(t),t;y_1(\tau),\tau)\phi_1(\tau)d\tau$$

$$+ \beta^2 \int_0^t \frac{h(\tau)}{f(\tau)}G_y(y_0(t),t;y_0(\tau),\tau)d\tau$$

$$- D\beta\int_0^t \frac{\phi_2(\tau)}{f(\tau)}G_y(y_0(t),t;y_0(\tau),\tau)d\tau$$

$$\left. - \int_0^t h'(\tau)N(y_0(t),t;y_0(\tau),\tau)d\tau\right\}, \tag{55}$$

where G, N *are the Green and Neumann functions respectively, and* K *is the fundamental solution to the heat equation, defined by*

$$G(x,t,\xi,\tau) = K(x,t,\xi,\tau) - K(-x,t,\xi,\tau), \tag{56}$$

$$N(x,t,\xi,\tau) = K(x,t,\xi,\tau) + K(-x,t,\xi,\tau), \tag{57}$$

$$K(x,t,\xi,\tau) = \begin{cases} \frac{1}{2\sqrt{\pi D(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4D(t-\tau)}\right) & t > \tau \\ 0 & t \le \tau \end{cases} \tag{58}$$

and y_0 , y_1 *are given by* (51) *and* (52) *respectively. Moreover, function* $h(t) = w(y_0(t),t)$ *must satisfy the integral relation*

$$h(t) = (f(t)-\beta)\left(1 - \int_0^t \phi_1(\tau)d\tau + \frac{1}{D}\int_{y_0(t)}^{y_1(t)} w(y,t)dy\right). \tag{59}$$

Proof. Let $w(y,t)$, $y_0(t)$, $y_1(t)$ be the solution to the problem (39)–(47). We integrate on the domain

$$D_{t,\epsilon} = \left\{ (\xi,\tau)\,/\,y_0(\tau) < \xi < y_1(\tau)\,, \epsilon < \tau < t-\epsilon \right\} \ (\epsilon > 0),$$

the Green identity

$$D\left(Gw_\xi - wG_\xi\right)_\xi - (Gw)_\tau = 0 \tag{60}$$

and we let $\epsilon \to 0$, to obtain the integral representation for $w(y,t)$ [3,17]

$$w(y,t) = \int_{C_1}^{C_2} G(y,t;\xi,0)w(\xi,0)d\xi + D\int_0^t w_\xi(y_1(\tau),\tau)G(y,t;y_1(\tau),\tau)d\tau \tag{61}$$

$$+ \beta^2 \int_0^t \frac{w(y_0(\tau),\tau)}{f(\tau)}G(y,t;y_0(\tau),\tau)d\tau$$

$$- D\beta \int_0^t \frac{w_\xi(y_0(\tau),\tau)}{f(\tau)}G(y,t;y_0(\tau),\tau)d\tau$$

$$+ D\int_0^t w(y_0(\tau),\tau)G_\xi(y,t;y_0(\tau),\tau)d\tau.$$

By using the definitions of ϕ_1 and ϕ_2 given by (53), the definition of h and boundary conditions we have (49). If we differentiate (61) in variable y and we let $y \to y_0^+(t)$ and $y \to y_1^-(t)$, by using the jump relations [17] we obtain the system of integral equations (54) and (55) for ϕ_1 and ϕ_2. Moreover, from (37) and (50) we have Eq. (59).

Conversely, the function $w(y,t)$ defined by (49), where ϕ_1 and ϕ_2 are the solutions of (54) and (55), satisfies the conditions (39), (42)–(44). In order to prove the conditions (40) and (41) we define

$$\mu_1(t) = w(y_1(t),t) \ \text{ and } \ \mu_2(t) = h(t) - w(y_0(t),t).$$

If we integrate the Green identity (60) over the domain $D_{t,\epsilon}$ $(\epsilon > 0)$ and we let $\epsilon \to 0$, we obtain that

$$w(y,t) = \int_{C_1}^{C_2} G(y,t;\xi,0)w(\xi,0)d\xi + D\int_0^t G(y,t;y_1(\tau),\tau)\phi_1(\tau)d\tau$$

$$- D\int_0^t G_y(y,t;y_1(\tau),\tau)w(y_1(\tau),\tau)d\tau + \int_0^t G(y,t;y_1(\tau),\tau)w(y_1(\tau),\tau)y_1'(\tau)d\tau$$

$$- \int_0^t G(y,t;y_0(\tau),\tau)\left[w(y_0(\tau),\tau)y_0'(\tau) - D\phi_2(\tau)\right]d\tau$$

$$+ D\int_0^t G_\xi(y,t;y_0(\tau),\tau)w(y_0(\tau),\tau)d\tau. \tag{62}$$

Then, if we compare this last expression (62) with (49) we deduce that

$$\int_0^t G(y,t;y_0(\tau),\tau)\left[\frac{\beta^2}{f(\tau)}\mu_2(\tau) - \frac{D\phi_2(\tau)}{f(\tau)}\right.$$

$$\left. \times \left(\beta + \frac{w(y_0(\tau),\tau)(f(\tau)-\beta)}{h(\tau)}\right) + D\phi_2(\tau)\right]d\tau$$

$$+ D\int_0^t G_y(y,t;y_0(\tau),\tau)\mu_2(\tau)d\tau + D\int_0^t G_y(y,t;y_1(\tau),\tau)\mu_1(\tau)d\tau$$

$$- \int_0^t G(y,t;y_1(\tau),\tau)\mu_1(\tau)\left[(1-\beta) - \frac{D\phi_1(\tau)(\beta+1)}{\beta^2 C(\tau)}\right]d\tau = 0. \tag{63}$$

By taking $y \to y_1^-(t)$ and $y \to y_0^+(t)$ in (63), and the jump relations we obtain that μ_1 and μ_2 must satisfy the following system of Volterra integral equations:

$$\mu_1(t) = \frac{-2}{D}\int_0^t \left\{ DG_y(y_1(t),t;y_1(\tau),\tau) - G(y,t;y_1(\tau),\tau) \right.$$

$$\left. \times \left[(1-\beta) - \frac{D\phi_1(\tau)(\beta+1)}{\beta^2 C(\tau)}\right]\right\}\mu_1(\tau),$$

$$+ \left\{ DG_y(y_1(t),t;y_0(\tau),\tau) + G(y_1(t),t;y_0(\tau),\tau) \right.$$

$$\left. \times \left[\frac{\beta}{f(\tau)} - D\phi_2(\tau)\left(\frac{\beta}{f(\tau)} - \frac{1}{h(\tau)}\right)\right]\right\}\mu_2(\tau)d\tau \tag{64}$$

$$\mu_2(t) = \frac{2}{D} + \int_0^t \left\{ DG_y(y_0(t),t;y_0(\tau),\tau) - G(y_0(t),t;y_0(\tau),\tau) \right. \tag{65}$$

$$\left[\frac{\beta}{f(\tau)} - D\phi_2(\tau)\left(\frac{\beta}{f(\tau)} - \frac{1}{h(\tau)}\right)\right]\right\}\mu_2(\tau)$$

$$+ \left\{ DG_y(y_0(t),t;y_1(\tau),\tau) - G(y_0(t),t;y_1(\tau),\tau) \right.$$

$$\left. \times \left[(1-\beta) - \frac{D\phi_1(\tau)(\beta+1)}{\beta^2 C(\tau)}\right]\right\}\mu_1(\tau)d\tau.$$

Following [18], it is easy to see that there exist a unique solution $\mu_1 \equiv \mu_2 \equiv 0$ to the system of Volterra integral equations (64)–(65). Then (40) and (41) are verified and the result holds. $\square$

4. Existence of the solution

In order to prove existence of solution $w = w(y,t)$, $y = y_0(t)$ and $y = y_1(t)$ of (39)–(47) and taking into account the result of Theorem 1 we will demonstrate that there exists at least a local solution ϕ_1, ϕ_2 and h to the coupled nonlinear integral equations (54), (55) and (59).

We will proceed in the following way: Fixed positive constants H, R, S and σ we define the set $\Pi = \Pi(H,R,S,\sigma)$ given by

$$\Pi := \left\{ h \in C^1[0,\sigma]\,/\,h(t) \geq H, \|h\| \leq R, \|h'\| \leq S \right\} \tag{66}$$

where $\|h\| = \max_{t\in[0,\sigma]}|h(t)|$. Clearly Π is a compact and convex set in $C^1[0,\sigma]$.

For each fixed function $h \in \Pi_1 = \{h \in C^1[0,1]\,/\,h(t) \geq H, \|h\| \leq R, \|h'\| \leq S\}$ we will use the Banach fixed point Theorem in order to prove that there exist unique solutions ϕ_1, $\phi_2 \in C^0[0,\sigma]$ to the system of two Volterra integral equations (54) and (55). Then for suitable H,R,S and σ, by using Schauder's fixed point Theorem we will demonstrate that there exists at least a solution $h \in \Pi_1$ of (59).

4.1. Existence and uniqueness of ϕ_1, ϕ_2

We consider the Banach space

$$\mathbf{C}[0,\sigma] = \left\{ \overrightarrow{\phi^*} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\,/\,\phi_i : [0,\sigma] \to \mathbb{R}, \quad i=1,2, \text{ continuous} \right\}$$

with the norm

$$\left\|\overrightarrow{\phi^*}\right\|_\sigma := \max_{t\in[0,\sigma]}|\phi_1(t)| + \max_{t\in[0,\sigma]}|\phi_2(t)|$$

and the subset:

$$C_{M,\sigma} = \left\{ \overrightarrow{\phi^*} \in \mathbf{C}[0,\sigma]\,/\,\left\|\overrightarrow{\phi^*}\right\|_\sigma \leq M \right\}$$

with σ and M positive numbers to be determinate.

We define the map $\chi : C_{M,\sigma} \longrightarrow C_{M,\sigma}$, such that

$$\chi\left(\overrightarrow{\phi^*}\right)(t) = \begin{pmatrix} \chi_1(\phi_1(t),\phi_2(t)) \\ \chi_2(\phi_1(t),\phi_2(t)) \end{pmatrix}$$

where

$$\chi_1(\phi_1(t),\phi_2(t)) = \frac{2}{2-D}\left\{ \int_{C_1}^{C_2} N(y_1(t),t;\xi,0)F'(\xi)d\xi \right.$$

$$+ D\int_0^t \phi_1(\tau)G_y(y_1(t),t;y_1(\tau),\tau)d\tau$$

$$+ \beta^2 \int_0^t \frac{h(\tau)}{f(\tau)}G_y(y_1(\tau),t;y_0(\tau),\tau)d\tau$$

$$- D\beta \int_0^t \frac{\phi_2(\tau)}{f(\tau)}G_y(y_1(\tau),t;y_0(\tau),\tau)d\tau,$$

$$\left. - \int_0^t h'(\tau)N(y_1(t),t;y_0(\tau),\tau)d\tau \right\}, \tag{67}$$

$$\chi_2(\phi_1(t),\phi_2(t)) = \frac{2f(t)}{2f(t)-D\beta}\left\{ -\beta^2 \frac{h(t)}{f(t)} + \int_{C_1}^{C_2} N(y_0(t),t;\xi,0)F'(\xi)d\xi \right.$$

$$+ D\int_0^t G_y(y_0(t),t;y_1(\tau),\tau)\phi_1(\tau)d\tau$$

$$+ \beta^2 \int_0^t \frac{h(\tau)}{f(\tau)}G_y(y_0(t),t;y_0(\tau),\tau)d\tau$$

$$- D\beta \int_0^t \frac{\phi_2(\tau)}{f(\tau)}G_y(y_0(t),t;y_0(\tau),\tau)d\tau$$

$$- \int_0^t h'(\tau) N(y_0(t), t; y_0(\tau), \tau) d\tau \Bigg\}. \tag{68}$$

We will prove that for suitable M and σ, the map χ is well defined and it is also a contraction, therefore by the Banach fixed point Theorem it has a unique fixed point.

Firstly, we give some preliminary results

Lemma 3. *Let $f(t) > \frac{3\beta}{2}$, $0 < D < 2$ and $\phi_i \in C^0[0,\sigma]$, $\max_{t\in[0,\sigma]} |\phi_i(t)| \le M, (i = 1,2)$. If*

$$2(1+\beta)\left(1 + \frac{M}{\beta^2}\right)\sigma \le C_2, \qquad 2\left(\beta + 2\frac{MD}{H}\right)\sigma \le C_1$$

then y_0 and y_1 defined by (51) and (52) satisfies

$$|y_0(t) - y_0(\tau)| \le \left(\beta + 2\frac{DM}{H}\right)|t - \tau| \;,\; \forall \tau, t \in [0,\sigma], \tag{69}$$

$$\frac{C_1}{2} \le y_0(t) \le 3\frac{C_1}{2}, \forall t \in [0,\sigma], \tag{70}$$

$$|y_1(t) - y_1(\tau)| \le (1+\beta)\left(1 + \frac{M}{\beta^2}\right)|t - \tau| \;,\; \forall \tau, t \in [0,\sigma], \tag{71}$$

$$\frac{C_2}{2} \le y_1(t) \le 3\frac{C_2}{2}, \forall t \in [0,\sigma]. \tag{72}$$

Proof. It follows inmediatly from definitions (51)–(52) and assumptions on data. $\square$

To prove the following Lemmas we need to use the classical inequality

$$\frac{\exp\left(\frac{-x^2}{\alpha(t-\tau)}\right)}{(t-\tau)^{\frac{n}{2}}} \le \left(\frac{n\alpha}{2ex^2}\right)^{\frac{n}{2}} \;,\; \alpha, x > 0 \;,\; t > \tau \;,\; n \in \mathbb{N}. \tag{73}$$

Lemma 4. *Let $\sigma \le 1$. For each function $h \in \Pi_1$ under the hypothesis of Lemma 3 and $C_1 < \frac{U_0}{2}$ we have that following properties are satisfied*

$$\int_{C_1}^{C_2} |F'(\xi)| |N(y_1(t), t; \xi, 0)| d\xi \le \|F'\| \le A_1(u_0, U_0, \beta, D), \tag{74}$$

$$D\int_0^t |G_y(y_1(t), t; y_1(\tau), \tau)\phi_1(\tau)| d\tau \le A_2(M, D, U_0, C_1)\sqrt{\sigma}, \tag{75}$$

$$\beta^2 \int_0^t \left|G_y(y_1(t), t; y_0(\tau), \tau)\frac{h(\tau)}{f(\tau)}\right| d\tau \le A_3(R, D, \beta, C_2, C_1)\sqrt{\sigma}, \tag{76}$$

$$\beta D \int_0^t \left|G_y(y_1(t), t; y_0(\tau), \tau)\frac{\phi_2(\tau)}{f(\tau)}\right| d\tau \le A_4(M, D, C_2, C_1)\sqrt{\sigma}, \tag{77}$$

$$\int_0^t |h'(\tau)| |N(y_1(t), t; y_0(\tau), \tau)| d\tau \le A_5(S, D)\sqrt{\sigma}, \tag{78}$$

$$\int_{C_1}^{C_2} |F'(\xi)| |N(y_0(t), t; \xi, 0)| d\xi \le \|F'\| \le A_1(u_0, U_0, \beta, D), \tag{79}$$

$$D\int_0^t |G_y(y_0(t), t; y_1(\tau), \tau)\phi_1(\tau)| d\tau \le A_4(D, M, C_1, C_2)\sqrt{\sigma}, \tag{80}$$

$$\beta^2 \int_0^t \left|G_y(y_0(t), t; y_0(\tau), \tau)\frac{h(\tau)}{f(\tau)}\right| d\tau \le A_6(R, M, D, \beta, C_2, C_1)\sqrt{\sigma}, \tag{81}$$

$$\beta D \int_0^t \left|G_y(y_0(t), t; y_0(\tau), \tau)\frac{\phi_2(\tau)}{f(\tau)}\right| d\tau \le A_7(M, D, C_1)\sqrt{\sigma}, \tag{82}$$

$$\int_0^t |h'(\tau)| |N(y_0(t), t; y_0(\tau), \tau)| d\tau \le A_5(S, D)\sqrt{\sigma}, \tag{83}$$

$$\beta^2 \frac{h(t)}{f(t)} \le \beta R, \tag{84}$$

where

$$A_1(u_0, U_0, \beta, D) = \exp\left(\frac{\|u_0\| + \beta U_0}{D}\right)\left[\left\|\frac{u_0'}{u_0}\right\| + \frac{\|u_0\| + \beta}{D}\right],$$

$$A_2(M, D, U_0, C_1) = \frac{M\sqrt{D}}{2\sqrt{\pi}}\left[2M + \frac{3}{C_2^2}\left(\frac{2D}{3e}\right)^{3/2}\right],$$

$$A_3(R, D, \beta, C_2, C_1) = \frac{R\beta}{2\sqrt{D\pi}}(A_{31} + A_{32}),$$

$$A_{31} = \frac{3C_2 - C_1}{2}\left(\frac{24D}{e(C_2 - 3C_1)^2}\right)^{\frac{3}{2}}, \qquad A_{32} = \frac{18\sqrt{6}}{e^{3/2}(C_1 + C_2)^2},$$

$$A_4(M, D, C_2, C_1) = \frac{M\sqrt{D}}{2\sqrt{\pi}}(A_{31} + A_{32}),$$

$$A_5(S, D) = 2\frac{S}{\sqrt{D\pi}},$$

$$A_6(R, M, D, \beta, C_1) = \frac{\beta R M \sqrt{D}}{2\sqrt{\pi}}\left[2M + \frac{3}{C_1^2}\left(\frac{2D}{3e}\right)^{3/2}\right],$$

$$A_7(M, D, \beta, C_1) = DM\left[2M + \frac{3}{C_1^2}\left(\frac{2D}{3e}\right)^{3/2}\right],$$

Proof.

To prove (74) we consider

$$\int_{C_1}^{C_2} |F'(\xi)| |N(y_1(t), t; \xi, 0)| d\xi \le \|F'\|\int_0^\infty |N(y_1(t), t; \xi, 0)| d\xi \le \|F'\|.$$

From (45) we have

$$F'(y) = \exp\left(\frac{1}{D}\int_y^{C_2} V_0(\xi)d\xi\right)\left[V_0'(y) - \frac{1}{D}V_0^2(y)\right]$$

then

$$\|F'\| \le \exp\left(\frac{\|V_0\|(C_2 - C_1)}{D}\right)\left[\|V_0'\| + \frac{1}{D}\|V_0\|\right]$$

$$\le \exp\left(\frac{(\|u_0\| + \beta)U_0}{D}\right)\left[\left\|\frac{u_0'}{u_0}\right\| + \frac{\|u_0\| + \beta}{D}\right] = A_1(u_0, U_0, \beta, D).$$

Following the proof given in [19,20] and taking $C_1 < \frac{U_0}{2}$ we obtain (75), (76), (77), (80), (81) and (82).

To prove (78) we take into account that

$$|N(y_1(t), t; y_0(\tau), \tau)| \le \frac{1}{\sqrt{\pi(t-\tau)}}$$

so, we obtain

$$\int_0^t |h'(\tau)| |N(y_1(t), t; y_0(\tau), \tau))| d\tau \le 2\sqrt{\frac{t}{D\pi}}S.$$

The inequalities (79) and (83) are proved in the same way as (74) and (78) respectively. $\square$

Lemma 5. *Let y_{01} and y_{02} be the functions corresponding to ϕ_{21} and ϕ_{22} in $C^0[0,\sigma]$ respectively, and y_{11} and y_{12} be the functions corresponding to ϕ_{11} and ϕ_{12} in $C^0[0,\sigma]$ respectively with $\max_{t\in[0,\sigma]} |\phi_{ij}(t)| \le M$, $i, j = 1, 2$. Under hypothesis Lemma 3 we have*

$$\begin{cases} |y_{01}(t) - y_{02}(t)| \le \frac{2D}{H}\sigma\|\phi_{11} - \phi_{12}\|_\sigma, \\ |y_{01}(t) - y_{02}(\tau)| \le \left(\beta + 2\frac{DM}{H}\right)|t - \tau|, i = 1, 2, \\ \frac{C_1}{2} \le y_{0i}(t) \le \frac{3C_1}{2}, \forall t \in [0,\sigma], i = 1, 2, \end{cases} \tag{85}$$

and

$$\begin{cases} |y_{11}(t) - y_{12}(t)| \le \frac{\beta+1}{\beta^2}\sigma\|\phi_{21} - \phi_{22}\|_\sigma, \\ |y_{1i}(t) - y_{1i}(\tau)| \le (1+\beta)\left(1 + \frac{M}{\beta^2}\right)|t - \tau|, i = 1, 2, \\ \frac{C_2}{2} \le y_{1i}(t) \le \frac{3C_2}{2}, \forall t \in [0,\sigma], i = 1, 2. \end{cases} \tag{86}$$

Proof. It follows inmediatly from definitions (51)–(52) and assumptions on data. $\square$

Lemma 6. *If we take $\sigma \le 1$, $\frac{4M}{H}\left(\beta + \frac{2DM}{H}\right)\sigma \le 1$ and we assume the hypothesis of Lemma 5 then we have*

$$\int_{C_1}^{C_2} |F'(\xi)| |N(y_{11}(t), t; \xi, 0) - N(y_{12}(t), t; \xi, 0)| d\xi \tag{87}$$

$$\leq \frac{2\,\|F'\|_{[C_1,C_2]}}{D\sqrt{\pi}}\,\|\phi_{11}-\phi_{12}\|_\sigma\,\sqrt{\sigma} \leq P_1(u_0,D,\beta,U_0)\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|\sqrt{\sigma},$$

$$D\int_0^t\left|\phi_{11}(\tau)G_y(y_{11}(t),t;y_{11}(\tau),\tau)-\phi_{12}(\tau)G_y(y_{12}(t),t;y_{12}(\tau),\tau)\right|d\tau \quad (88)$$

$$\leq P_2(M,D,C_2)\sigma\,\|\phi_{11}-\phi_{12}\| \leq P_2(M,D,C_2)\,\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|\sqrt{\sigma},$$

$$\beta^2\int_0^t\frac{h(\tau)}{f(\tau)}\left|G_y(y_{11}(t),t;y_{01}(\tau),\tau)-G_y(y_{12}(t),t;y_{02}(\tau),\tau)\right|d\tau \quad (89)$$

$$\leq P_3(R,\beta,C_1,C_2)\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|\sqrt{\sigma},$$

$$D\beta\int_0^t\frac{1}{f(\tau)}\left|\phi_{21}(\tau)G_y(y_{11}(t),t;y_{01}(\tau),\tau)-\phi_{22}(\tau)G_y(y_{12}(t),t;y_{02}(\tau),\tau)\right|d\tau \quad (90)$$

$$\leq P_4(D,M,C_1,C_2)\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|\sqrt{\sigma},$$

$$\int_0^t\left|h'(\tau)\right|\left|N(y_{11}(t),t;y_{01}(\tau),\tau)-N(y_{12}(t),t,y_{02}(\tau),\tau)\right|d\tau \quad (91)$$

$$\leq P_5(S,D,C_1,C_2)\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|\sqrt{\sigma},$$

$$\int_{C_1}^{C_2}\left|F'(\xi)\right|\left|N(y_{01}(t),t;\xi,0)-N(y_{02}(t),t;\xi,0)\right|d\xi \quad (92)$$

$$\leq \frac{2\,\|F'\|_{[C_1,C_2]}}{D\sqrt{\pi}}\,\|\phi_{2,1}-\phi_{2,2}\|_\sigma\,\sqrt{\sigma} \leq P_1(u_0,D,\beta,U_0)\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|\sqrt{\sigma},$$

$$D\int_0^t\left|\phi_{11}(\tau)G_y(y_{01}(t),t;y_{11}(\tau),\tau)-\phi_{12}(\tau)G_y(y_{02}(t),t;y_{12}(\tau),\tau)\right|d\tau \quad (93)$$

$$\leq P_4(D,M,C_1,C_2)\,\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|\sqrt{\sigma},$$

$$\beta^2\int_0^t\frac{h(\tau)}{f(\tau)}\left|G_y(y_{01}(t),t;y_{01}(\tau),\tau)-G_y(y_{02}(t),t;y_{02}(\tau),\tau)\right|d\tau \quad (94)$$

$$\leq P_6(D,H,R,M,C_1)\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|\,\sqrt{\sigma},$$

$$D\int_0^t\left|\phi_{21}(\tau)G_y(y_{01}(t),t;y_{01}(\tau),\tau)-\phi_{22}(\tau)G_y(y_{02}(t),t;y_{02}(\tau),\tau)\right|d\tau \quad (95)$$

$$\leq P_7(M,D,C_1)\,\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|\,\sqrt{\sigma},$$

$$\int_0^t\left|h'(\tau)\right|\left|N(y_{11}(t),t;y_{01}(\tau),\tau)-N(y_{12}(t),t,y_{02}(\tau),\tau)\right|d\tau \quad (96)$$

$$\leq P_5(S,D,C_1,C_2)\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|\sqrt{\sigma},$$

where

$$P_1(u_0,D,\beta,U_0) = \frac{2}{D\sqrt{\pi}}\left[\left(\|u_0\|+\beta\right)\exp\left(\frac{U_0}{\beta}\left(\|u_0\|+\beta\right)\right)+\frac{(\|u_0\|+\beta)^2}{D}\right], \quad (97)$$

$$P_2(M,D,C_2) = \frac{\sqrt{D}}{4\sqrt{\pi}}\left[6M+\frac{3}{C_2^2}\left(\frac{2}{3e}\right)^{3/2}+\frac{6M}{C_2^2}\left(\frac{6}{e}\right)^{3/2}\right], \quad (98)$$

$$P_3(R,\beta,C_1,C_2) = R\beta(P_{31}+P_{32}), \quad (99)$$

with

$$P_{31}(C_1,C_2) = \frac{1}{\sqrt{\pi}e^{3/2}}\left[\frac{\sqrt{6}(3C_2-C_1)^2}{16(C_2-3C_1)^3}+\frac{27\sqrt{3}}{4}+\frac{12\sqrt{6}}{(C_2-3C_1)^3}+\frac{6\sqrt{3}}{(C_2+C_1)^3}\right], \quad (100)$$

$$P_{32}(C_1,C_2) = \frac{12\sqrt{6}}{\sqrt{\pi}e^{3/2}}\left[\frac{1}{(C_2-3C_1)^3}+\frac{9}{8}+\frac{(3C_2-C_1)^2}{8(C_2-3C_1)^3}+\frac{1}{(C_2+C_1)^2}\right], \quad (101)$$

$$P_4(D,M,C_1,C_2) = D\left[M(P_{31}+P_{32})+P_{41}\right], \quad (102)$$

where

$$P_{41}(C_1,C_2) = \frac{\sqrt{6}}{\sqrt{\pi}e}\left[\frac{1}{(C_2-3C_1)^2}+\frac{1}{(C_2+C_1)^2}\right], \quad (103)$$

$$P_5(S,D,C_1,C_2) = \frac{6^{3/2}SD}{\sqrt{\pi}e^{3/2}}\left[\frac{3C_2-C_1}{(C_2-3C_1)^3}+\frac{3}{(C_2+C_1)^2}\right], \quad (104)$$

$$P_6(D,H,R,M,C_1) = \beta R\left\{(2D)^{-1}(D\pi)^{-1/2}\left[\frac{2D}{H}+\frac{2}{H}\left(\beta+\frac{2DM}{H}\right)^2\right]\right.$$

$$\left.+\left(\frac{6}{eC_1^2}\right)^{3/2}\frac{18C_1^2+1}{4\sqrt{\pi}}\frac{4D}{H}\right\}, \quad (105)$$

$$P_7(M,D,C_1) = \frac{\sqrt{D}}{4\sqrt{\pi}}\left[6M+\frac{3}{C_1^2}\left(\frac{2}{3e}\right)^{3/2}+\frac{6M}{C_1^2}\left(\frac{6}{e}\right)^{3/2}\right]. \quad (106)$$

Proof. The inequalities (87)–(93) and (95)–(96) are obtained following [20].

We will show the proof of (94), following [21]. We write

$$\left|G_y(y_{01}(t),t;y_{01}(\tau),\tau)-G_y(y_{02}(t),t;y_{02}(\tau),\tau)\right|$$

$$\leq\left|K_y(y_{01}(t),t;y_{01}(\tau),\tau)-K_y(y_{02}(t),t;y_{02}(\tau),\tau)\right|$$

$$+\left|K_y(-y_{01}(t),t;y_{01}(\tau),\tau)-K_y(-y_{02}(t),t;y_{02}(\tau),\tau)\right|.$$

Taking into account that

$$\left|K_y(y_{01}(t),t;y_{01}(\tau),\tau)-K_y(y_{02}(t),t;y_{02}(\tau),\tau)\right|$$

$$\leq(2D(t-\tau))^{-1}\left|K(y_{01}(t),t;y_{01}(\tau),\tau)\left[\left(y_{01}(t)-y_{01}(\tau)\right)-\left(y_{02}(t)-y_{02}(\tau)\right)\right]\right.$$

$$\left.+\left[K(y_{01}(t),t;y_{01}(\tau),\tau)-K(y_{02}(t),t;y_{02}(\tau),\tau)\right]\left(y_{02}(t)-y_{02}(\tau)\right)\right|$$

$$\leq(2D(t-\tau))^{-1}K(y_{01}(t),t;y_{01}(\tau),\tau)\left|\left[\left(y_{01}(t)-y_{01}(\tau)\right)-\left(y_{02}(t)-y_{02}(\tau)\right)\right]\right.$$

$$\left.+[1-exp(m(t,\tau))]\left(y_{02}(t)-y_{02}(\tau)\right)\right|,$$

where

$$m(t,\tau) = \frac{\left(y_{01}(t)-y_{01}(\tau)\right)^2-\left(y_{02}(t)-y_{02}(\tau)\right)^2}{4D(t-\tau)}$$

$$= \frac{\left[\left(y_{01}(t)-y_{01}(\tau)\right)-\left(y_{02}(t)-y_{02}(\tau)\right)\right]\left[\left(y_{01}(t)-y_{01}(\tau)\right)+\left(y_{02}(t)-y_{02}(\tau)\right)\right]}{4D(t-\tau)}.$$

We have

$$\left|\left(y_{01}(t)-y_{01}(\tau)\right)-\left(y_{02}(t)-y_{02}(\tau)\right)\right|$$

$$\leq D\int_\tau^t\frac{1}{h(\eta)}\left|1-\frac{\beta}{f(\eta)}\right|\left|\phi_{21}(\eta)-\phi_{22}(\eta)\right|d\eta$$

$$\leq \frac{2D}{H}\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|(t-\tau),$$

and

$$\left|\left(y_{01}(t)-y_{01}(\tau)\right)+\left(y_{02}(t)-y_{02}(\tau)\right)\right| \leq 2\left(\beta+\frac{2DM}{H}\right)(t-\tau)$$

$$\leq 4\left(\beta+\frac{2DM}{H}\right)\sigma,$$

then

$$|m(t,\tau)| \leq \frac{2}{H}\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|\left(\beta+\frac{2DM}{H}\right)\sigma,$$

and taking into account that $\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|\leq 2M$ we have

$$|m(t,\tau)| \leq \frac{4M}{H}\left(\beta+\frac{2DM}{H}\right)\sigma.$$

If we assume that σ satisfies

$$\frac{4M}{H}\left(\beta+\frac{2DM}{H}\right)\sigma\leq 1,$$

we obtain that

$$|1-exp(m(t,\tau))| \leq 2\,|m(t,\tau)| \leq \frac{2}{H}\left(\beta+\frac{2DM}{H}\right)\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|\sigma.$$

Therefore

$$\left|K_y(y_{01}(t),t;y_{01}(\tau),\tau)-K_y(y_{02}(t),t;y_{02}(\tau),\tau)\right|\leq$$

$$\leq(2D)^{-1}K(y_{01}(t),t;y_{01}(\tau),\tau)\left[\frac{2D}{H}+\frac{2}{H}\left(\beta+\frac{2DM}{H}\right)^2\sigma\right]\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|$$

$$\leq(4D)^{-1}(D\pi(t-\tau))^{-1/2}\left[\frac{2D}{H}+\frac{2}{H}\left(\beta+\frac{2DM}{H}\right)^2\sigma\right]\left\|\vec{\phi}_1^*-\vec{\phi}_2^*\right\|.$$

Using the mean value theorem we may write

$$\left|K_y(-y_{01}(t),t;y_{01}(\tau),\tau)-K_y(-y_{02}(t),t;y_{02}(\tau),\tau)\right|$$

$$\leq\left|K(n(t,\tau),t;0,\tau)\left(\frac{n^2(t,\tau)}{4D^2(t-\tau)^2}-\frac{1}{2D(t-\tau)}\right)\right|$$

$$\times\left|y_{01}(t)+y_{01}(\tau)-y_{02}(t)-y_{02}(\tau)\right|$$

where $n=n(t,\tau)$ is between $y_{01}(t)+y_{01}(\tau)$ and $y_{02}(t)+y_{02}(\tau)$.

Since

$$\left| y_{01}(t) + y_{01}(\tau) - y_{02}(t) - y_{02}(\tau) \right| \leq \frac{4D}{H} \sigma \left\| \vec{\phi}_1^* - \vec{\phi}_2^* \right\|,$$

and

$$C_1 \leq n(t,\tau) \leq 6C_1,$$

by using (73) we have

$$\left| K(n(t,\tau),t;0,\tau) \left(\frac{n^2(t,\tau)}{4D^2(t-\tau)^2} - \frac{1}{2D(t-\tau)} \right) \right| \leq \left(\frac{6}{eC_1^2} \right)^{3/2} \frac{18C_1^2 + 1}{4\sqrt{\pi}},$$

then

$$\left| K_y(-y_{01}(t),t;y_{01}(\tau),\tau) - K_y(-y_{02}(t),t;y_{02}(\tau),\tau) \right|$$

$$\leq \left(\frac{6}{eC_1^2} \right)^{3/2} \frac{18C_1^2 + 1}{4\sqrt{\pi}} \frac{4D}{H} \sigma \left\| \vec{\phi}_1^* - \vec{\phi}_2^* \right\|.$$

Collecting the results we have

$$\left| G_y(y_{01}(t),t;y_{01}(\tau),\tau) - G_y(y_{02}(t),t;y_{02}(\tau),\tau) \right|$$

$$\leq \left\{ (4D)^{-1}(D\pi(t-\tau))^{-1/2} \left[\frac{2D}{H} + \frac{2}{H}\left(\beta + \frac{2DM}{H}\right)^2 \right] \right.$$

$$\left. + \left(\frac{6}{eC_1^2} \right)^{3/2} \frac{18C_1^2 + 1}{4\sqrt{\pi}} \frac{4D}{H} \right\} \left\| \vec{\phi}_1^* - \vec{\phi}_2^* \right\|,$$

then

$$\beta^2 \int_0^t \frac{h(\tau)}{f(\tau)} \left| G_y(y_{01}(t),t;y_{01}(\tau),\tau) - G_y(y_{02}(t),t;y_{02}(\tau),\tau) \right| d\tau$$

$$\leq P_6(D,H,R,M,C_1) \left\| \vec{\phi}_1^* - \vec{\phi}_2^* \right\| \sqrt{\sigma},$$

where

$$P_6(D,H,R,M,C_1) = \beta R \left\{ (2D)^{-1}(D\pi)^{-1/2} \left[\frac{2D}{H} + \frac{2}{H}\left(\beta + \frac{2DM}{H}\right)^2 \right] \right.$$

$$\left. + \left(\frac{6}{eC_1^2} \right)^{3/2} \frac{18C_1^2 + 1}{4\sqrt{\pi}} \frac{4D}{H} \right\}.$$

Then, (94) has been proved $\quad\square$

Theorem 3. *Let hypothesis (7) be. Fixed $0 < C_1 < \frac{U_0}{2}$ and $h \in \Pi$. If σ satisfies the following inequalities*

$$\sigma \leq 1, \quad 2(1+\beta)\left(1 + \frac{M}{\beta^2}\right)\sigma \leq C_2, \tag{107}$$

$$\left(\beta + 2\frac{MD}{H}\right)\sigma \leq C_1, \quad \frac{4M}{H}\left(\beta + \frac{2DM}{H}\right)\sigma \leq 1, \tag{108}$$

$$H_1(C_1,C_2,U_0,M,D,\beta,R,S,\sigma) \leq 1, \tag{109}$$

$$H_2(C_1,C_2,U_0,M,D,\beta,R,S,\sigma) \leq 1, \tag{110}$$

where M is given by

$$M\left(u_0,U_0,f,D,\beta,R\right) = 1 + \left(\frac{1}{2-D} + \frac{\|f\|}{\beta(3-D)}\right)2A_1 + 2\frac{\|f\|}{3-D}R, \tag{111}$$

and

$$H_1\left(C_1,U_0,f,M,D,\beta,R,S,\sigma\right) = \left\{ \left(\frac{2}{2-D}\right)\left[A_2 + A_3 + A_4 + \frac{2S}{\sqrt{\pi D}}\right] \right.$$

$$\left. + \frac{2\|f\|}{\beta(3-D)}\left[A_4 + A_5 + A_6 + \frac{2S}{\sqrt{\pi D}}\right] \right\}\sqrt{\sigma}, \tag{112}$$

$$H_2\left(C_1,U_0,f,M,D,\beta,R,S,\sigma\right) = \left\{ \frac{2}{2-D}\left[P_1 + P_2 + P_3 + P_4 + P_5\right] \right.$$

$$\left. + \frac{2\|f\|}{\beta(3-D)}\left[P_1 + P_4 + P_5 + P_6 + P_7\right] \right\}\sqrt{\sigma}, \tag{113}$$

then the map $\chi : C_{M,\sigma} \longrightarrow C_{M,\sigma}$ is well defined and it is a contraction map. Therefore there exists a unique solution ϕ_1^*, ϕ_2^* on $C_{M,\sigma}$ to the system of integral equations (54) and (55).

Proof. Firstly, we demonstrate that χ maps $C_{M,\sigma}$ into itself, that is

$$\left\| X\left(\vec{\phi}^*\right) \right\|_\sigma = \max_{t\in[0,\sigma]} \left| \chi_1(\phi_1(t),\phi_2(t)) \right| + \max_{t\in[0,\sigma]} \left| \chi_2(\phi_1(t),\phi_2(t)) \right| \leq M$$

Taking into account Lemma 4 we have

$$\left| \chi_1(\phi_1(t),\phi_2(t)) \right| \leq \frac{2}{2-D}\left\{ A_1 + \left[A_2 + A_3 + A_4 + \frac{2S}{\sqrt{\pi D}}\right]\sqrt{\sigma} \right\},$$

$$\left| \chi_2(\phi_1(t),\phi_2(t)) \right| \leq \frac{2\|f\|}{\beta(3-D)}\left\{ \beta R + A_1 + \left[A_4 + A_5 + A_6 + \frac{2S}{\sqrt{\pi D}}\right]\sqrt{\sigma} \right\},$$

and then

$$\left\| \chi\left(\vec{\phi}^*\right) \right\|_\sigma \leq 2A_1\left[\frac{1}{2-D} + \frac{\|f\|}{\beta(3-D)} \right] + \frac{2\|f\|R}{(3-D)}$$

$$+ H_1(C_1,C_2,U_0,M,D,\beta,R,S,\sigma)$$

where H_1 is given by (112).

Selecting M by (111) and σ such that (116) holds, we obtain

$$\left\| \chi\left(\vec{\phi}^*\right) \right\|_\sigma \leq M.$$

Now, we will prove that

$$\left\| \chi\left(\vec{\phi}_1^*\right) - \chi\left(\vec{\phi}_2^*\right) \right\|_\sigma \leq H_2\left(C_1,C_2,U_0,M,D,\beta,R,S,\sigma\right)\left\| \vec{\phi}_1^* - \vec{\phi}_2^* \right\|_\sigma$$

where $\vec{\phi}_1^* = \begin{pmatrix}\phi_{11}\\\phi_{12}\end{pmatrix}$, $\vec{\phi}_2^* = \begin{pmatrix}\phi_{21}\\\phi_{22}\end{pmatrix} \in C_{M,\sigma}$.

Taking into account Lemma 6 we have

$$\left\| \chi\left(\vec{\phi}_1^*\right) - \chi\left(\vec{\phi}_2^*\right) \right\|_\sigma = \max_{t\in[0,\sigma]}\left| \chi_1\left(\phi_{11}(t),\phi_{12}(t)\right) - \chi_1\left(\phi_{21}(t),\phi_{22}(t)\right) \right|$$

$$+ \max_{t\in[0,\sigma]}\left| \chi_2\left(\phi_{11}(t),\phi_{12}(t)\right) - \chi_2\left(\phi_{21}(t),\phi_{22}(t)\right) \right|$$

$$\leq \left\{ \frac{2}{2-D}\left[P_1 + P_2 + P_3 + P_4 + P_5\right] \right.$$

$$\left. + \frac{2\|f\|}{\beta(3-D)}\left[P_1 + P_4 + P_5 + P_6 + P_7\right] \right\}\sqrt{\sigma}\left\| \vec{\phi}_2^* - \vec{\phi}_1^* \right\|_\sigma \tag{114}$$

$$= H_2\left(C_1,C_2,U_0,M,D,\beta,R,S,\sigma\right)\left\| \vec{\phi}_2^* - \vec{\phi}_1^* \right\|_\sigma.$$

By hypothesis (107)–(111) we have that χ is a contraction and therefore, there exists a unique fixed point $\phi^* = \begin{pmatrix}\phi_1\\\phi_2\end{pmatrix}$ such that $\chi(\phi^*) = \phi^*$ this is

$$\chi_1(\phi_1(t),\phi_2(t)) = \phi_1(t), \qquad \chi_2(\phi_1(t),\phi_2(t)) = \phi_2(t). \quad\square$$

Theorem 4. *For each $h \in \Pi_1$, under hypothesis of Theorem 3 there exists a unique integral representation for w, y_0 and y_1 given by (49), (51) and (52) respectively, where ϕ_1 and ϕ_2 are the unique solutions of (54) and (55)*

4.2. Existence of at least a solution of $w_h(y_{0h}(t),t) = h(t)$

In this subsection we assume that all the hypothesis of Theorem 3 are valid, which guarantee the existence and uniqueness of $w = w_h$, $y_0 = y_{0h}$ and $y_1 = y_{1h}$ for each $h \in \Pi_1$.

Now we will prove that for suitable values of H, R, S and σ there exists $h \in \Pi(H,R,S,\sigma)$ such that

$$w_h(y_{0h}(t),t) = h(t) \tag{115}$$

for $t \in [0,\sigma]$, where w_h and y_{0h} are established for above theorem.

We define the map Z on Π such that for each $h \in \Pi \subset \Pi_1$ ($\sigma \leq 1$)

$$Z(h)(t) = w_h(y_{0h}(t),t),$$

this is

$$Z(h)(t) = (f(t)-\beta)\left(1 - \int_0^t \phi_{1h}(\tau)d\tau + \frac{1}{D}\int_{y_{0h}(t)}^{y_{1h}(t)} w_h(\xi,t)d\xi \right), \tag{116}$$

where ϕ_{1h}, w_h, y_{0h} and y_{1h} are the solutions obtained in above section. We will use the Schauder's fixed point theorem which states: *For any continuous function L mapping a compact convex set to itself there is x_0 such that $L(x_0) = x_0$.*

Lemma 7. *If*

$$(\|f\| + \beta)(2C_1 + 3U_0) < 2D \tag{117}$$

then, for $h \in \Pi$ function $Z(h) \in C^1[0,\sigma]$ and satisfies

$$Z(h)(t) > \frac{\beta}{2}, \tag{118}$$

$$\|Z(h)\| \leq \frac{2D(\|f\| + \beta)(1 + M)}{2D - (\|f\| + \beta)(2C_1 + 3U_0)}, \tag{119}$$

$$\|Z'(h)\| \leq \frac{2D(\|f\|+\beta)(1+M)}{2D-(\|f\|+\beta)(2C_1+3U_0)} \left\{ \frac{2\|f'\|}{\beta} + (\|f\|+\beta)\left[\frac{\beta}{D} + M\left(\frac{2}{\beta} + \frac{1}{D}\right)\right]\right\} \tag{120}$$

$$+ M(\|f\| + \beta).$$

Proof. From definition of $Z(h)$ and (7) we have $Z(h)(t) > \frac{\beta}{2}$. Taking into account (116) and the fact that

$$w_h(\xi, t) \leq w_h(y_0(t), t), \quad y_0(t) \leq \xi \leq y_1(t)$$

we have

$$Z(h)(t) \leq (f(t) + \beta)\left(1 - \int_0^t \phi_{1h}(\tau)d\tau + w_h(y_0(t), t)\left(y_1(t) - y_0(t)\right)\right).$$

Since $\|\phi_{1h}\| \leq M$, $\sigma \leq 1$ and taking into account Lemma 5 we obtain

$$\|Z(h)\| \leq (\|f\| + \beta)\left(1 + M + \|Z(h)\| \frac{2C_1 + 3U_0}{2D}\right),$$

or equivalently

$$\|Z(h)\| \leq \frac{2D(\|f\| + \beta)(1 + M)}{2D - (\|f\| + \beta)(2C_1 + 3U_0)}.$$

If we derivate $Z(h)$ respect to variable t, we get

$$(Z(h))'(t) = f'(t)\left(1 - \int_0^t \phi_{1h}(\tau)d\tau + \frac{1}{D}\int_{y_{0h}(t)}^{y_{1h}(t)} w_h(\xi, t)d\xi\right)$$

$$+ (f(t) - \beta)\left[-\phi_{1h}(t) + \frac{1}{D}w_h(y_{0h}(t), t)y_{0h}'(t) + \frac{1}{D}\int_{y_{0h}(t)}^{y_{1h}(t)} w_{ht}(\xi, t)d\xi\right]$$

and using Eqs. (39), (51) and (53) we obtain

$$(Z(h))'(t) = f'(t)\left(1 - \int_0^t \phi_{1h}(\tau)d\tau + \frac{1}{D}\int_{y_{0h}(t)}^{y_{1h}(t)} w_h(\xi, t)d\xi\right)$$

$$+ (f(t) - \beta)\left[-\phi_{1h}(t) - \frac{1}{D}w_h(y_{0h}(t), t)y_{0h}'(t) + \phi_{h1}(t) - \phi_{h2}(t)\right]$$

$$= f'(t)\frac{Z(h)(t)}{f(t) - \beta} + (f(t) - \beta)\left[\frac{Z(h)\beta^2}{Df(t)} + \phi_{h2}(t)\left(\frac{Z(h)(t)}{h(t)} - 1 - \frac{Z(h)(t)\beta}{Df(t)}\right)\right].$$

Then we have

$$|(Z(h))'(t)| \leq \|f'\| \frac{2\|Z(h)\|}{\beta} + (\|f\| + \beta)\left[\frac{\|Z(h)\|\beta}{D}\right.$$

$$+ M\left(\frac{2\|Z(h)\|}{\beta} + 1 + \frac{\|Z(h)\|}{D}\right)\right]$$

$$\leq \|Z(h)\| \left\{ \frac{2\|f'\|}{\beta} + (\|f\| + \beta)\left[\frac{\beta}{D} + M\left(\frac{2}{\beta} + \frac{1}{D}\right)\right]\right\} + M(\|f\| + \beta),$$

$$\leq \frac{2D(\|f\|+\beta)(1+M)}{2D-(\|f\|+\beta)(2C_1+3U_0)} \left\{ \frac{2\|f'\|}{\beta} + (\|f\|+\beta)\left[\frac{\beta}{D} + M\left(\frac{2}{\beta} + \frac{1}{D}\right)\right]\right\}$$

$$+ M(\|f\| + \beta)$$

and the lemma holds. □

Next, we define

$$E_1 = 1 + 2\left(\frac{1}{2-D} + \frac{\|f\|}{\beta(3-D)}\right)A_1, \qquad E_2 = 2\frac{\|f\|}{(3-D)}, \tag{121}$$

and

$$E_3 = \frac{2D(\|f\| + \beta)}{2D - (\|f\| + \beta)(2C_1 + 3U_0)}.$$

Lemma 8. *We assume* (117) *and*

$$E_3 E_2 < 1. \tag{122}$$

If we take

$$H = \frac{\beta}{2}, \qquad R = \frac{E_3(1 + E_1)}{1 - E_3 E_2}, \tag{123}$$

$$S = E_3\left\{ \frac{2\|f'\|}{\beta} + (\|f\| + \beta)\left[\frac{\beta}{D} + M\left(\frac{2}{\beta} + \frac{1}{D}\right)\right]\right\} \tag{124}$$

where M is given by

$$M = \frac{E_1 + E_2 E_3}{1 - E_3 E_2} \tag{125}$$

then $Z_h \in \Pi$.

Proof. From (111) we have

$$M = E_1 + E_2 R,$$

then by (119) we have

$$E_3(1 + M) = R \quad \Leftrightarrow \quad E_3(1 + E_1) + E_3 E_2 R = R.$$

Therefore, if we define $R = \frac{E_3(1+E_1)}{1-E_3 E_2}$ we have $\|Z(h)\| \leq R$. Moreover we have $Z(h)(t) > \frac{\beta}{2} = H$ and by Lemma 7 we have $\|(Z(h))'\| \leq S$. This yields $Z(h) \in \Pi$ and the proof is complete. □

Remark 2. Assumption (122) is equivalent to

$$\frac{4D\|f\|(\|f\| + \beta)}{\left[2D - (\|f\| + \beta)(2C_1 + 3U_0)\right](3 - D)} < 1 \tag{126}$$

Theorem 5. *We assume hypothesis of Lemma 8. There exists at least a solution $h^* \in \Pi$ such that $Z(h^*) = h^*$.*

Proof. Taking into account above lemmas and using Schauder's fixed-point theorem we obtain that there exists at least a solution $h^* \in \Pi$ such that $Z(h^*) = h^*$. □

We can now formulate our main result.

Theorem 6. *Fixed $C_1 < \frac{U_0}{2}$. Let H, R, S and M given by (123), 8 and 125 respectively. If (117) and (126) hold,*

$$\sigma \leq 1, \quad 2(1 + \beta)\left(1 + \frac{M}{\beta^2}\right)\sigma \leq C_2, \tag{127}$$

$$\left(\beta + 2\frac{MD}{H}\right)\sigma \leq C_1 \qquad \frac{4M}{H}\left(\beta + \frac{2DM}{H}\right)\sigma \leq 1 \tag{128}$$

$$H_1 \leq 1, \qquad H_2 \leq 1 \tag{129}$$

where H_1 and H_2 are given by (112)–(113) *then there exists solution to the free boundary problem* (39)–(47) *given by*

$$w^*(y, t) = \int_{C_1}^{C_2} G(y, t; \xi, 0)F(\xi)d\xi + D\int_0^t \phi_1^*(\tau)G(y, t; y_1^*(\tau), \tau)d\tau \tag{130}$$

$$+ \beta^2 \int_0^t \frac{h^*(\tau)}{f(\tau)}G(y, t; y_0^*(\tau), \tau)d\tau - D\beta \int_0^t \frac{\phi_2^*(\tau)}{f(\tau)}G(y, t; y_0^*(\tau), \tau)d\tau$$

$$- D\int_0^t h^*(\tau)N_y(y, t; y_0^*(\tau), \tau)d\tau$$

and

$$y_0^*(t) = C_1 - \beta^2 \int_0^t \frac{1}{f(\tau)}d\tau - D\int_0^t \frac{1}{h^*(\tau)}\left(1 - \frac{\beta}{f(\tau)}\right)\phi_2^*(\tau)d\tau \tag{131}$$

$$y_1^*(t) = C_2 + (1 - \beta)t + \frac{D(\beta+1)}{\beta^2}\ln\left(1 - \int_0^t \phi_1^*(\tau)d\tau\right) \tag{132}$$

where ϕ_1^*, ϕ_2^* are the unique solutions to the system of two Volterra integral equations (54) *and* (55) *corresponding to h^* a solution of $Z(h) = h$.*

5. Parametric solution to the problem (1) –(6)

Assuming hypothesis of Theorem 6, if we invert the transformations given by (8), (26) and (36) we obtain the explicit parametric representation of the solution to free boundary problem (1)–(6) given by

$$u^*(x,t) = \frac{w^*(y,t)}{1 - \int_0^t \phi_1^*(\tau)d\tau + \frac{1}{D}\int_y^{y_1^*(t)} w^*(\xi,t)d\xi} + \beta \tag{133}$$

$$x = \int_{y_0^*(t)+2\beta t}^{y+2\beta t} \left[\frac{w^*(\mu,t)}{1 - \int_0^t \phi_1^*(\tau)d\tau + \frac{1}{D}\int_\mu^{y_1^*(t)} w^*(\xi,t)d\xi} + \beta \right] d\mu \tag{134}$$

with

$$y_0^*(t) < y < y_1^*(t), \qquad 0 < t < \sigma$$

and

$$s(t) = \int_{y_0^*(t)+2\beta t}^{y_1^*(t)+2\beta t} \left[\frac{w^*(\mu,t)}{1 - \int_0^t \phi_1^*(\tau)d\tau + \frac{1}{D}\int_\mu^{y_1^*(t)} w^*(\xi,t)d\xi} + \beta \right] d\mu \tag{135}$$

where $w^* = w^*(y,t)$ is given by (130), $y_0^*(t)$, $y_1^*(t)$ are given by (131) and (132) respectively, with ϕ_1^*, ϕ_2^* the unique solutions to (54) and (55) corresponding to the solution h^* to Eq. (115).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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