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An explicit solution for an instantaneous two-phase Stefan problem with nonlinear thermal coefficients

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We consider a nonlinear heat conduction problem for a semi-infinite material $x > 0$, with phase-change temperature T_1, an initial temperature $T_2(>T_1)$ and a heat flux of the type $q(t) = q_0/\sqrt{t}$ imposed on the fixed face $x = 0$. We assume that the volumetric heat capacity and the thermal conductivity are particular nonlinear functions of the temperature in both solid and liquid phases.

We determine necessary and/or sufficient conditions on the parameters of the problem in order to obtain the existence of an explicit solution for an instantaneous nonlinear two-phase Stefan problem (solidification process).

Keywords: Stefan problem; free boundary problem; phase-change process; solidification; similarity solution; Kirchoff transformation.

1. Introduction

We consider the two-phase Stefan problem (solidification process) with nonlinear thermal coefficients for a semi-infinite region $x > 0$ with phase-change temperature T_1, an initial temperature $T_2 > T_1$ and an imposed heat flux of the type $q(t) = q_0/\sqrt{t}$ ($q_0 > 0$) on the fixed face $x = 0$. For $t > 0$ we are going to determine if there exist a temperature distribution $u(x, t)$ and a free boundary $x = y(t)$, where

$$u(x,t) = \begin{cases} u_1(x,t) < T_1, & 0 < x < y(t), \\ T_1, & x = y(t), \\ u_2(x,t) > T_1, & x > y(t). \end{cases} \tag{1.1}$$

The modelling of this type of system is a problem of great mathematical and industrial significance. Phase-change processes appear frequently in industrial processes and other problems of technological interest (Tarzia, 2000).

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The problem to be considered is given as follows:

$$C_1(u_1)\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial x}\left[K_1(u_1)\frac{\partial u_1}{\partial x}\right], \quad 0 < x < y(t), \quad t > 0, \tag{1.2}$$

$$C_2(u_2)\frac{\partial u_2}{\partial t} = \frac{\partial}{\partial x}\left[K_2(u_2)\frac{\partial u_2}{\partial x}\right], \quad x > y(t), \quad t > 0, \tag{1.3}$$

$$y(0) = 0, \tag{1.4}$$

$$u_2(x, 0) = T_2 > T_1, \quad x > 0, \tag{1.5}$$

$$u_1(y(t), t) = u_2(y(t), t) = T_1, \quad t > 0, \tag{1.6}$$

$$K_1(u_1)\frac{\partial u_1}{\partial x} - K_2(u_2)\frac{\partial u_2}{\partial x} = Ly'(t), \quad \text{on} \quad x = y(t), \quad t > 0, \tag{1.7}$$

$$K_1(u_1(0,t))\frac{\partial u_1}{\partial x}(0,t) = \frac{q_0}{\sqrt{t}}, \quad t > 0, \tag{1.8}$$

where x is spatial coordinate, t is time, $u_i(x,t)$ is temperature distribution for phase i, T_1 is phase-change or freezing temperature, T_2 is initial temperature, L is volumetric latent heat, $C_i(u_i)$ is volumetric heat capacity for phase i, $K_i(u_i)$ is thermal conductivity for phase i, $y(t)$ is the free boundary (solid–liquid interface) at time t, q_0 is a positive given constant which characterizes the heat flux on $x = 0$; here $i = 1$ is the solid phase, $i = 2$ is the liquid phase.

We assume that the volumetric heat capacity and the thermal conductivity for each phase $i (i = 1, 2)$ are related as follows:

$$C_i(u_i) = \frac{K_i(u_i)c_0}{k_0 a_i^2\left[b_i - \frac{1}{k_0}\int_0^{(u_i - T_1)/(T_2 - T_1)} K_i(T_1 + (T_2 - T_1)z)\,\mathrm{d}z\right]^2} \tag{1.9}$$

with the assumption that

$$\frac{1}{k_0(T_2 - T_1)}\int_{T_1}^{T_2} K_2(z)\,\mathrm{d}z < b_2, \tag{1.10}$$

where a_i, b_i $(i = 1, 2)$ are positive constants and k_0, c_0 are scales for the thermal conductivity and volumetric heat capacity respectively. The nonlinear relations (1.9) follow from the solidification of iron on a copper base (Tritscher & Broadbridge, 1994). Furthermore, these relations imply that the material is of Storm's type, that is to say (Briozzo *et al.*, 1999; Hill & Hart, 1986; Rogers, 1985; Storm, 1951)

$$\frac{1}{\sqrt{K_i(u_i)C_i(u_i)}}\frac{\mathrm{d}}{\mathrm{d}u_i}\left(\log\sqrt{\frac{C_i(u_i)}{K_i(u_i)}}\right) = \frac{a_i}{\sqrt{c_0 k_0}(T_2 - T_1)} = \text{const.}, \quad i = 1, 2.$$

The goal of this paper is to determine which conditions on the parameters of the problem (in particular q_0) must be satisfied in order to have an instantaneous phase-change process. The heat flux condition of the type (1.8) was first considered in Tarzia (1981) where an inequality for the coefficient q_0 was found in order to have an instantaneous two-phase Stefan problem with constant thermal coefficients, for both solid and liquid phases.

Other problems in this direction are given by Briozzo & Tarzia (1998), Hill & Hart (1986), Natale & Tarzia (2000), Rogers & Broadbridge (1986), Rogers (1985), Solomon *et al.* (1983), Tarzia & Turner (1992).

In Section 2 we consider the associated nonlinear heat conduction problem corresponding to the initial liquid temperature T_2 and the heat flux condition on $x = 0$ of the type $q_0/\sqrt{t}$ for $t > 0$. The nonlinear condition between the thermal conductivity heat capacity is supposed to be of the type (1.9). We give a necessary condition for the heat flux input coefficient q_0, that is,

$$q_0 > \frac{\sqrt{c_0 k_0}(T_2 - T_1)}{a_2} Q^{-1}\left((k_0 b_2 (T_2 - T_1))^{-1} \int_{T_1}^{T_2} K_2(z)\,\mathrm{d}z\right) \tag{1.11}$$

in order to obtain an instantaneous phase-change process, where Q is the real function defined by

$$Q(x) = \sqrt{\pi} x \exp(x^2)(1 - \operatorname{erf}(x)), \quad x > 0 \tag{1.12}$$

with the properties $Q(0) = 0$, $Q(+\infty) = 1$, $Q'(x) > 0$ for all $x > 0$.

In Section 3 we consider the nonlinear two-phase Stefan problem (1.2)–(1.8) and we prove that it admits a similarity solution if the condition (1.11) for the coefficient q_0 is satisfied.

In the text we will use the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-w^2)\,\mathrm{d}w.$$

2. A nonlinear heat conduction problem and its instantaneous phase-change process

We consider a semi-infinite slab $x \geqslant 0$ of a material that freezes at temperature T_1. We suppose that it is initially hot at the uniform temperature $T_2 > T_1$ and it has nonlinear heat transfer coefficients. However, what happens if a heat flux of the type $q_0/\sqrt{t}$ is imposed at $x = 0$? Our interest is in finding relations among data corresponding to obtaining an instantaneous phase-change process, that is, the temperature of the material at $x = 0$ must be less than T_1 for all positive time. Then, we consider the following nonlinear heat conduction problem corresponding to the initial phase (liquid phase) given by

$$C_2(u)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K_2(u)\frac{\partial u}{\partial x}\right], \quad x > 0, \quad t > 0, \tag{2.1}$$

$$u(x,0) = T_2, \quad x > 0, \tag{2.2}$$

$$K_2(u(0,t))\frac{\partial u}{\partial x}(0,t) = \frac{q_0}{\sqrt{t}}, \quad t > 0, \tag{2.3}$$

where K_2 and C_2 satisfy the relation (1.9) with the assumption (1.10).

Then the question that follows is: which conditions must be satisfied by the parameters q_0, T_1, T_2, K_2 and C_2 in order to have that the temperature $u(0, t) < T_1$ for all $t > 0$? If the answer is affirmative then we can be sure that the phase-change is instantaneous (Solomon *et al.*, 1983; Tarzia, 1981; Tarzia & Turner, 1992).

Below, we calculate the explicit solution to the problem (2.1)–(2.3) for the liquid phase and we demonstrate that this solution is constant at $x = 0$ for all t. Then we can answer affirmatively the previous question if (1.10) and (1.11) holds.

In order to obtain that explicit solution for the problem (2.1)–(2.3) we define the new variables and parameters

$$\left.\begin{array}{ll} x_* = x\sqrt{\dfrac{c_0}{k_0 t_s}}, & t_* = \dfrac{t}{t_s}, \\ u_*(x_*, t_*) = \dfrac{u(x,t) - T_1}{T_2 - T_1} > 0, & q_{0*} = \dfrac{q_0}{\sqrt{c_0 k_0}(T_2 - T_1)}, \\ K_{2*}(u_*) = \dfrac{K_2(u)}{k_0}, & C_{2*}(u_*) = \dfrac{C_2(u)}{c_0}, \end{array}\right\} \tag{2.4}$$

where t_s is a time scale. Following Broadbridge *et al.* (1993), Knight & Philip (1974), we consider the Kirchhoff transformation given by

$$\eta(x_*, t_*) = \mu(u_*(x_*, t_*)) = \int_0^{u_*(x_*,t_*)} K_{2*}(z)\,\mathrm{d}z, \quad \mu(\Psi) = \int_0^{\Psi} K_{2*}(z)\,\mathrm{d}z \tag{2.5}$$

and we define the new variables

$$\left.\begin{array}{llll} \chi(x_*, t_*) = \displaystyle\int_0^{x_*} \frac{1}{a_2(b_2 - \eta(z, t_*))}\,\mathrm{d}z, & & x_* > 0, & t_* > 0, \\ \tau = t_*, & \overline{\mu}(\chi, \tau) = \eta(x_*, t_*), & \chi > 0, & \tau > 0. \end{array}\right\} \tag{2.6}$$

Now we assume a similarity solution of the type

$$g(\phi) = \frac{\overline{\mu}(\chi, \tau)}{\theta}, \quad \phi = \frac{\chi}{2\sqrt{\tau}}, \quad \theta = \int_0^1 K_{2*}(z)\,\mathrm{d}z, \tag{2.7}$$

then the problem (2.1)–(2.3) reduces to the problem (2.8), (2.9) for the unknown function g given by

$$2(\phi + \lambda)g'(\phi) + g''(\phi) = 0, \quad \phi > 0, \tag{2.8}$$

$$g(+\infty) = 1, \quad g'(0) = \frac{2\lambda}{\theta}(b_2 - \theta g(0)), \quad \lambda = a_2 q_{0*} \tag{2.9}$$

whose solution is given by

$$g(\phi) = A[\operatorname{erf}(\phi + \lambda) - \operatorname{erf}(\lambda)] + B, \quad \phi > 0, \tag{2.10}$$

$$A = \frac{\lambda(b_2 - \theta)\sqrt{\pi}}{\theta[\exp(-\lambda^2) - \sqrt{\pi}\lambda(1 - \operatorname{erf}(\lambda))]}, \quad B = 1 - \frac{\lambda(b_2 - \theta)\sqrt{\pi}(1 - \operatorname{erf}(\lambda))}{\theta[\exp(-\lambda^2) - \sqrt{\pi}\lambda(1 - \operatorname{erf}(\lambda))]}. \tag{2.11}$$

We obtain the following result.

THEOREM 1 The parametric solution to the problem (2.1)–(2.3) is given by

$$u(x,t) = T_1 + (T_2 - T_1)\mu^{-1}\left(\theta A\left(\operatorname{erf}\left(\frac{\chi}{2\sqrt{\tau}} + \lambda\right) - \operatorname{erf}(\lambda)\right) + \theta B\right), \tag{2.12}$$

where

$$x = a_2\sqrt{\frac{k_0 t_s \tau}{c_0}}\left\{(b_2 - B + A\,\mathrm{erf}(\lambda))\chi - 2A\sqrt{\tau}\left[\left(\frac{\chi}{2\sqrt{\tau}}+\lambda\right)\mathrm{erf}\left(\frac{\chi}{2\sqrt{\tau}}+\lambda\right)\right.\right.$$
$$\left.\left. + \frac{1}{\sqrt{\pi}}\exp\left(-\left(\frac{\chi}{2\sqrt{\tau}}+\lambda\right)^2\right) - \lambda\,\mathrm{erf}(\lambda) - \frac{1}{\sqrt{\pi}}\exp(-\lambda^2)\right]\right\}, \quad \chi > 0, \quad \tau > 0,$$
$$t = t_s\tau, \quad \tau > 0, \tag{2.13}$$

where A, B are defined in (2.11). Moreover, we have that

$$u(0,t) < T_1, \forall t > 0 \iff q_0 \text{ satisfies (1.11).} \tag{2.14}$$

Proof. If we invert the transformations (2.7), (2.6), (2.5) and (2.4) we obtain that

$$u(x,t) = T_1 + (T_2 - T_1)\mu^{-1}\left(\theta g\left(\frac{\chi}{2\sqrt{\tau}}\right)\right), \tag{2.15}$$

$$x = \sqrt{\frac{k_0 t_s \tau}{c_0}}x_* = \sqrt{\frac{k_0 t_s \tau}{c_0}}\int_0^{\chi} a_2(b_2 - \overline{\mu}(z,\tau))\,dz$$
$$= \sqrt{\frac{k_0 t_s \tau}{c_0}}\int_0^{\chi} a_2\left(b_2 - \theta g\left(\frac{z}{2\sqrt{\tau}}\right)\right)dz, \quad t = t_s\tau, \quad \tau > 0. \tag{2.16}$$

Then, if we replace the expression (2.10) for $g = g(\chi/2\sqrt{\tau})$ in (2.15) and (2.16) we obtain the parametric solution (2.12), (2.13) corresponding to the problem (2.1)–(2.3).

Next, we wish to control whether the temperature on $x = 0$ satisfies the inequality $u(0,t) < T_1$ for all $t > 0$. From (2.4) and (2.5), we have

$$u(0,t) = (T_2 - T_1)u_*(0,t_*) + T_1 \quad \text{and} \quad \eta(0,t_*) = \mu(u_*(0,t_*)) = \int_0^{u_*(0,t_*)} K_{2*}(z)\,dz.$$

Then, taking into account (2.6), (2.7) and (2.10), (2.11) we have

$$\eta(0,t_*) = \overline{\mu}(0,\tau) = \theta g(0) = \frac{\theta\exp(-\lambda^2) - \lambda b_2\sqrt{\pi}(1-\mathrm{erf}(\lambda))}{\exp(-\lambda^2) - \sqrt{\pi}\lambda(1-\mathrm{erf}(\lambda))}$$

which is a constant for all positive time. Then $u_*(0,t_*)$ is constant and therefore $u(0,t)$ is also a constant for all t. On the other hand, we have the following equivalences:

$$u(0,t) < T_1 \iff g(0) < 0 \iff Q(\lambda) > \frac{\theta}{b_2} \iff (1.11).$$

□

In the next section we will assume that the heat flux input coefficient q_0 satisfies the inequality (1.11) and we will obtain the explicit solution to the free boundary problem (1.2)–(1.8).

3. Explicit solution for the instantaneous two-phase Stefan process with nonlinear thermal coefficients

From now on we will consider the problem (1.2)–(1.8) and we will prove that it is well posed for $t > 0$ when data satisfy condition (1.11) and assumption (1.10).

In order to obtain the explicit solution corresponding to the problem (1.2)–(1.8) we will consider the same kind of transformations used for problem (2.1)–(2.3) and we define the new variables and parameters

$$\left.\begin{array}{lll} x_* = x\sqrt{\dfrac{c_0}{k_0 t_s}}, & t_* = \dfrac{t}{t_s}, & y_*(t_*) = y(t)\sqrt{\dfrac{c_0}{k_0 t_s}}, \\ u_{i*}(x_*, t_*) = \dfrac{u_i(x,t) - T_1}{T_2 - T_1}, & K_{i*}(u_{i*}) = \dfrac{K_i(u_i)}{k_0}, & \\ C_{i*}(u_{i*}) = \dfrac{C_i(u_i)}{c_0}, & L_* = \dfrac{L}{c_0(T_2 - T_1)}. & \end{array}\right\} \tag{3.1}$$

By considering the Kirchhoff transformation given by

$$\eta_i(x_*, t_*) = \mu_i(u_{i*}(x_*, t_*)) = \int_0^{u_{i*}(x_*,t_*)} K_{i*}(z)\,\mathrm{d}z, \quad \mu_i(\Psi) = \int_0^{\Psi} K_{i*}(z)\,\mathrm{d}z, \quad i = 1, 2, \tag{3.2}$$

we have that the thermal diffusivity for each phase i $(i = 1, 2)$ is given by $D_{i*}(\mu_i) = K_{i*}(\mu_i)/C_{i*}(\mu_i) = a_i^2(b_i - \mu_i)^2$. The one-dimensional diffusion equation with this class of diffusivity was solved previously by Knight & Philip (1974), Reeves (1975). Now, in order to linearize the nonlinear differential equations we define the new variables through the Storm transformation given by Knight & Philip (1974), Storm (1951)

$$\left.\begin{array}{ll} \chi_1(x_*, t_*) = \displaystyle\int_0^{x_*} \frac{1}{a_1(b_1 - \eta_1(z, t_*))}\,\mathrm{d}z, & 0 < x_* < y_*(t_*), \\ \chi_2(x_*, t_*) = \displaystyle\int_{y_*(t_*)}^{x_*} \frac{1}{a_2(b_2 - \eta_2(z, t_*))}\,\mathrm{d}z, & x_* > y_*(t_*), \\ \tau = t_*, \quad \overline{\mu_i}(\chi_i, \tau) = \eta_i(x_*, t_*), & i = 1, 2, \end{array}\right\} \tag{3.3}$$

and the free boundary is now given by

$$S(\tau) = \chi_1(y_*(\tau), \tau) = \int_0^{y_*(\tau)} \frac{1}{a_1(b_1 - \eta_1(z, \tau))}\,\mathrm{d}z. \tag{3.4}$$

Owing to the condition on the free boundary and following (Tritscher & Broadbridge, 1994) we have that the interface between the two phases must move as $y_*(t_*) = \delta\sqrt{t_*}$, and the flux of η_2 on the free boundary takes the explicit form $\partial\eta_2/\partial x_*(y_*(t_*), t_*) = \gamma/\sqrt{t_*}$, where the positive constants δ and γ must be determined. Now the free boundary $S(\tau)$ may be expressed in terms of the transformed coordinates as follows:

$$S(\tau) = 2(\Lambda_1 - \lambda_1)\sqrt{\tau}, \quad \tau > 0, \quad \Lambda_1 > \lambda_1 > 0, \tag{3.5}$$

where

$$\lambda_1 = a_1 q_{0*}, \quad \Lambda_1 = a_1\gamma + \frac{\delta}{2}\left[\frac{1}{a_1 b_1} + a_1 L_*\right], \tag{3.6}$$

and a two-phase Stefan problem with convective terms in both heat equations and a convective boundary condition on the fixed face are obtained. If we assume a similarity solution of the following type:

$$g_1(\phi_1) = \overline{\mu_1}(\chi_1, \tau), \quad \phi_1 = \frac{\chi_1}{2\sqrt{\tau}}, \tag{3.7}$$

$$g_2(\phi_2) = \frac{\overline{\mu_2}(\chi_2, \tau)}{\theta_2}, \quad \phi_2 = \frac{\chi_2}{2\sqrt{\tau}}, \quad \theta_2 = \int_0^1 K_{2*}(z)\,dz, \tag{3.8}$$

then it reduces to the following problem:

$$2(\phi_1 + \lambda_1)g_1'(\phi_1) + g_1''(\phi_1) = 0, \quad 0 < \phi_1 < \Lambda_1 - \lambda_1, \tag{3.9}$$

$$2(\phi_2 + \lambda_2)g_2'(\phi_2) + g_2''(\phi_2) = 0, \quad 0 < \phi_2, \tag{3.10}$$

$$g_2(+\infty) = 1, \quad g_1'(0) = 2q_{0*}a_1(b_1 - g_1(0)) \tag{3.11}$$

$$g_1(\Lambda_1 - \lambda_1) = g_2(0) = 0, \quad \frac{g_1'(\Lambda_1 - \lambda_1)}{a_1 b_1} - \frac{g_2'(0)\theta_2}{a_2 b_2} = L_*\delta, \tag{3.12}$$

for the unknown functions g_1 and g_2, and the unknown coefficients Λ_1 and λ_2, where

$$\gamma = \frac{\Lambda_1 a_1 b_1 - a_2 b_2 \lambda_2 [1 + a_1^2 b_1 L_*]}{a_1^2 b_1 (1 - a_2^2 b_2 L_*) - a_2^2 b_2}, \quad \delta = (\lambda_2 - a_2\gamma) 2 a_2 b_2. \tag{3.13}$$

The solution of (3.9)–(3.12) is given by

$$\left.\begin{aligned} g_1(\phi_1) &= b_1 \frac{\mathrm{erf}(\phi_1 + \lambda_1) - \mathrm{erf}(\Lambda_1)}{\widetilde{g}(\lambda_1) - \mathrm{erf}(\Lambda_1)}, \quad 0 < \phi_1 < \Lambda_1 - \lambda_1, \\ g_2(\phi_2) &= \frac{\mathrm{erf}(\phi_2 + \lambda_2) - \mathrm{erf}(\lambda_2)}{1 - \mathrm{erf}(\lambda_2)}, \quad 0 < \phi_2, \end{aligned}\right\} \tag{3.14}$$

where

$$\widetilde{g}(z) = \mathrm{erf}(z) + \frac{1}{\sqrt{\pi}} \frac{\exp(-z^2)}{z} = g\left(z, \frac{1}{\sqrt{\pi}}\right)$$

and $g(z, p)$ was defined in Briozzo *et al.* (1999) and we have the following useful properties:

$$\widetilde{g}(+\infty) = 1, \quad \widetilde{g}(0) = +\infty, \quad \widetilde{g}'(z) < 0, \quad \forall z > 0. \tag{3.15}$$

Then, the new unknown coefficients Λ_1 and λ_2 must satisfy the system of equations

$$\left.\begin{aligned} &\text{(i)} \quad \lambda_2 = \frac{a_1 b_1}{a_2 b_2 (1 + a_1^2 b_1 L_*)}[\Lambda_1 - \overline{\Lambda}_1 G(\Lambda_1)], \\ &\text{(ii)} \quad G(\Lambda_1) = \frac{\theta_2}{\sqrt{\pi} a_1 b_1 a_2 b_2} F(\lambda_2), \end{aligned}\right\} \tag{3.16}$$

where

$$\overline{A}_1 = a_1^2 b_1 (1 - a_2^2 b_2 L_*) - a_2^2 b_2, \tag{3.17}$$

$$\left.\begin{aligned} G(z) &= \frac{(1 + a_1^2 b_1 L_*)\exp(-z^2)}{\sqrt{\pi} a_1^2 b_1 [\widetilde{g}(\lambda_1) - \operatorname{erf}(z)]} - L_* z, \quad z > 0, \\ F(z) &= \frac{\exp(-z^2)}{1 - \operatorname{erf}(z)}, \quad z > 0. \end{aligned}\right\} \tag{3.18}$$

In order to obtain the solution to our problem (3.16) we will first prove some preliminary results.

LEMMA 2 The real function $G = G(z)$, restricted to the domain $(\lambda_1, +\infty)$, has the following properties:

$$\begin{aligned} &\text{(i)}\ \ G(\lambda_1) = \frac{\lambda_1}{a_1^2 b_1}, \\ &\text{(ii)}\ \ G(+\infty) = -\infty, \\ &\text{(iii)}\ \ G'(z) < 0, \quad z > \lambda_1, \\ &\text{(iv)}\ \ G'(\lambda_1) = G'(+\infty) = -L_*, \\ &\text{(v)}\ \ \exists z_0 > \lambda_1 / G''(z) \begin{cases} < 0, & \lambda_1 < z < z_0, \\ = 0, & z = z_0, \\ > 0, & z > z_0. \end{cases} \end{aligned} \tag{3.19}$$

In order to solve the system (3.16) we define $\lambda_2 = \lambda_2(A_1)$ from (3.16 (i)). Then, we obtain the following results.

LEMMA 3 The function $\lambda_2 = \lambda_2(x)$, defined for $x > \lambda_1$, has the following properties:

(i) $\lambda_2(\lambda_1) = \dfrac{a_2 \lambda_1}{a_1}$,

(ii)

$$\lambda_2(+\infty) = \begin{cases} -\infty & \text{if } L_* > \dfrac{1}{a_2^2 b_2}, \\ +\infty & \text{if } L_* < \dfrac{1}{a_2^2 b_2}, \\ 0^+ & \text{if } L_* = \dfrac{1}{a_2^2 b_2}, \end{cases} \tag{3.20}$$

(iii)

$$\lambda_2'(x) = \begin{cases} < 0 & \text{if } L_* \geqslant \dfrac{1}{a_2^2 b_2}, \\ > 0 & \text{if } 0 < L_* \leqslant \max\left\{0, \dfrac{1}{a_2^2 b_2} - \dfrac{1}{a_1^2 b_1}\right\}, \\ \left.\begin{array}{ll} > 0 & \text{if } G'(x) > \dfrac{1}{\overline{A}_1} \\ = 0 & \text{if } G'(x) = \dfrac{1}{\overline{A}_1} \\ < 0 & \text{if } G'(x) < \dfrac{1}{\overline{A}_1} \end{array}\right\} & \text{and if } \max\left\{0, \dfrac{1}{a_2^2 b_2} - \dfrac{1}{a_1^2 b_1}\right\} < L_* < \dfrac{1}{a_2^2 b_2}, \end{cases} \tag{3.21}$$

where G is the real function defined in (3.18). Furthermore, in the case when $L_* > 1/a_2^2 b_2$ we have that

$$\lambda_2(x) > 0 \Longleftrightarrow \lambda_1 < x < \Lambda_1^*, \tag{3.22}$$

where Λ_1^* is the unique solution to the equation

$$\widetilde{g}(\lambda_1) = g\left(x, \frac{\overline{A}_1}{[\overline{A}_1 + a_2^2 b_2]\sqrt{\pi}}\right), \quad x > \lambda_1. \tag{3.23}$$

Proof. (i) This follows from elementary computations.

(ii) We have

$$\lambda_2(+\infty) = \lim_{x \longrightarrow +\infty} \frac{a_1 b_1}{a_2 b_2 (1 + a_1^2 b_1 L_*)} x \left(1 - \overline{A}_1 \frac{G(x)}{x}\right).$$

Then if we take into account that $\lim_{x \longrightarrow +\infty} G(x)/x = -L_*$, we obtain (3.20) from conditions on $\overline{A}_1$ and $1 + \overline{A}_1 L_*$.

(iii) We have

$$\lambda_2'(x) = \frac{a_1 b_1}{a_2 b_2 (1 + a_1^2 b_1 L_*)} [1 - \overline{A}_1 G'(x)].$$

In order to prove the results (iii) we must analyse all cases by considering the sign of $1 - \overline{A}_1 G'(x)$. Taking into account (3.17) we can complete this proof. Furthermore, in the case when $L_* > 1/a_2^2 b_2$, we have obtained that $\lambda_2(+\infty) = -\infty$ and $\lambda_2'(x) < 0$. As we are only interested in $\lambda_2(x) > 0$, we can solve this inequality and we obtain the equivalence (3.22), where Λ_1^* is the unique solution to the equation (3.23). □

COROLLARY 4 The composition real function $\Psi = F \circ \lambda_2$, defined in $(\lambda_1, +\infty)$, has the following properties:

(i) $$\Psi(+\infty) = \begin{cases} +\infty & \text{if } L_* < \dfrac{1}{a_2^2 b_2}, \\ 1 & \text{if } L_* = \dfrac{1}{a_2^2 b_2}, \end{cases}$$

(ii) $$\Psi(\lambda_1) = \frac{\exp\left(-\left(\frac{a_2\lambda_1}{a_1}\right)^2\right)}{1-\operatorname{erf}\left(\frac{a_2\lambda_1}{a_1}\right)}.$$

For the case when $L_* > 1/a_2^2 b_2$ we have that $\Psi(\Lambda_1^*) = 1$.

(iii) We have

$$\Psi'(x) = \begin{cases} <0 & \text{if } L_* \geqslant \dfrac{1}{a_2^2 b_2}, \\ >0 & \text{if } 0 < L_* \leqslant \max\left\{0, \dfrac{1}{a_2^2 b_2} - \dfrac{1}{a_1^2 b_1}\right\}, \\ \left.\begin{array}{ll} >0 & \text{if } G'(x) > \frac{1}{\overline{A}_1} \\ =0 & \text{if } G'(x) = \frac{1}{\overline{A}_1} \\ <0 & \text{if } G'(x) < \frac{1}{\overline{A}_1} \end{array}\right\} & \text{and if } \max\left\{0, \dfrac{1}{a_2^2 b_2} - \dfrac{1}{a_1^2 b_1}\right\} < L_* < \dfrac{1}{a_2^2 b_2}. \end{cases}$$

Taking into account that L_* is from a physical point of view the inverse of the Stefan number we can obtain now the existence theorem for the solution in order to have an instantaneous phase-change process for problem (1.2)–(1.8) as a function of this important physical number.

Then, taking into account the above lemmas and corollary we have the following.

LEMMA 5. If q_0 satisfies the inequality (1.11), then (3.16) admits a unique solution $\widetilde{\Lambda}_1$, $\widetilde{\lambda}_2 = \lambda_2(\widetilde{\Lambda}_1)$ when

$$\frac{L}{c_0(T_2-T_1)} = L_* \geqslant \frac{1}{a_2^2 b_2} \quad \text{or} \quad 0 < \frac{L}{c_0(T_2-T_1)} = L_* \leqslant \max\left\{0, \frac{1}{a_2^2 b_2} - \frac{1}{a_1^2 b_1}\right\}$$

and at least one solution $\widetilde{\Lambda}_1$, $\widetilde{\lambda}_2 = \lambda_2(\widetilde{\Lambda}_1)$ when

$$\max\left\{0, \frac{1}{a_2^2 b_2} - \frac{1}{a_1^2 b_1}\right\} < \frac{L}{c_0(T_2-T_1)} < \frac{1}{a_2^2 b_2}.$$

Proof. We consider the results of Lemmas 2, 3 and Corollary 4. If

$$L_* \geqslant \frac{1}{a_2^2 b_2} \quad \text{or} \quad 0 < L_* \leqslant \max\left\{0, \frac{1}{a_2^2 b_2} - \frac{1}{a_1^2 b_1}\right\}$$

the equation (3.16) (ii) admits a unique solution $\widetilde{\Lambda}_1$ when

$$\frac{\theta_2}{\sqrt{\pi}\, a_1 b_1 a_2 b_2} \Psi(\lambda_1) < G(\lambda_1)$$

or equivalently, when q_0 satisfies the inequality (1.11). When we have the complementary condition $\max\{0, 1/a_2^2 b_2 - 1/a_1^2 b_1\} < L_* < 1/a_2^2 b_2$ we obtain at least one solution $\widetilde{\Lambda}_1$, if q_0 satisfies the inequality (1.11).

Next, we replace this solution $\widetilde{\Lambda}_1$ in (3.16) (i) and obtain $\widetilde{\lambda}_2 = \lambda_2(\widetilde{\Lambda}_1)$. □

Then, we have obtained the following theorem in terms of the original data of the problem (1.2)–(1.8).

THEOREM 6 If q_0 satisfies the inequality (1.11) then, an explicit solution to the problem (1.2)–(1.8) is given by

$$\left.\begin{aligned} u_1(x,t) &= T_1 + (T_2 - T_1)\mu_1^{-1}\left(b_1 \frac{\operatorname{erf}\left(\frac{\chi_1}{2\sqrt{\tau}} + \lambda_1\right) - \operatorname{erf}(\widetilde{\Lambda_1})}{\widetilde{g}(\lambda_1) - \operatorname{erf}(\widetilde{\Lambda_1})}\right), \\ & 0 < \chi_1 < S(\tau),\ \tau > 0, \\ u_2(x,t) &= T_1 + (T_2 - T_1)\mu_2^{-1}\left(\frac{\operatorname{erf}\left(\frac{\chi_2}{2\sqrt{\tau}} + \widetilde{\lambda_2}\right) - \operatorname{erf}(\widetilde{\lambda_2})}{1 - \operatorname{erf}(\widetilde{\lambda_2})}\right), \\ & \chi_2 > 0,\ \tau > 0, \end{aligned}\right\} \tag{3.24}$$

with

$$x = 2\frac{\sqrt{\frac{k_0 t_s \tau}{c_0}}\, a_1 b_1\left(\frac{\chi_1}{2\sqrt{\tau}} + \lambda_1\right)}{\widetilde{g}(\lambda_1) - \operatorname{erf}(\widetilde{\Lambda_1})}\left[\widetilde{g}(\lambda_1) - \widetilde{g}\left(\frac{\chi_1}{2\sqrt{\tau}} + \lambda_1\right)\right],\ 0 < \chi_1 < S(\tau),\ \tau > 0,$$

$$\begin{aligned} x &= \left[a_2 b_2 + \theta_2 a_2 \frac{\operatorname{erf}(\widetilde{\lambda_2})}{\operatorname{erf}c(\widetilde{\lambda_2})}\right]\chi_2 \\ &\quad - \frac{2\sqrt{\tau} a_2\theta_2}{1 - \operatorname{erf}(\widetilde{\lambda_2})}\left[\left(\frac{\chi_2}{2\sqrt{\tau}} + \widetilde{\lambda_2}\right)\widetilde{g}\left(\frac{\chi_2}{2\sqrt{\tau}} + \widetilde{\lambda_2}\right) - \widetilde{\lambda_2}\widetilde{g}(\widetilde{\lambda_2})\right] + \widetilde{\delta}\sqrt{\tau},\ \chi_2 > 0,\ \tau > 0, \end{aligned}$$

$$t = t_s\tau, \qquad \tau > 0, \tag{3.25}$$

where $y(t) = \sqrt{k_0/c_0}\widetilde{\delta}\sqrt{t}$ is the free boundary, and the coefficients $\widetilde{\gamma}$ and $\widetilde{\delta}$ are given by

$$\widetilde{\gamma} = \frac{\widetilde{\Lambda_1} a_1 b_1 - a_2 b_2 \widetilde{\lambda_2}[1 + a_1^2 b_1 L_*]}{a_1^2 b_1 (1 - a_2^2 b_2 L_*) - a_2^2 b_2}, \quad \widetilde{\delta} = (\widetilde{\lambda_2} - a_2\widetilde{\gamma}) 2 a_2 b_2, \tag{3.26}$$

where $\widetilde{\Lambda_1}$ and $\widetilde{\lambda_2} = \lambda_2(\widetilde{\Lambda_1})$ are the solution of the system (3.23).Moreover this solution is unique when

$$\frac{L}{c_0(T_2 - T_1)} = L_* \geqslant \frac{1}{a_2^2 b_2} \quad \text{or} \quad 0 < \frac{L}{c_0(T_2 - T_1)} = L_* \leqslant \max\left\{0, \frac{1}{a_2^2 b_2} - \frac{1}{a_1^2 b_1}\right\}.$$

Proof. From (3.14) and Lemma 5 the solution to the problem (3.9)–(3.12) is given by

$$\left.\begin{aligned} g_1(\phi_1) &= b_1 \frac{\operatorname{erf}(\phi_1 + \lambda_1) - \operatorname{erf}(\widetilde{\Lambda_1})}{\widetilde{g}(\lambda_1) - \operatorname{erf}(\widetilde{\Lambda_1})}, \quad 0 < \phi_1 < \widetilde{\Lambda_1} - \lambda_1, \\ g_2(\phi_2) &= \frac{\operatorname{erf}(\phi_2 + \widetilde{\lambda_2}) - \operatorname{erf}(\widetilde{\lambda_2})}{1 - \operatorname{erf}(\widetilde{\lambda_2})}, \quad \phi_2 > 0, \end{aligned}\right\} \tag{3.27}$$

where $\widetilde{\Lambda_1}, \widetilde{\lambda_2} = \lambda_2(\widetilde{\Lambda_1})$ are the solutions to the system (3.16).

Now, we invert the Storm transformations (3.3) and (3.1) in order to obtain the explicit solution (3.24) in the original variables. And, from (3.7), (3.8), (3.3) and (3.1) we complete the parametric solution. □

4. Conclusion

A nonlinear heat conduction problem for semi-infinite material $x > 0$, with phase-change temperature T_1, an initial temperature $T_2(>T_1)$ and a heat flux of the type $q(t) = q_0/\sqrt{t}$ imposed on the fixed face $x = 0$ is considered. Volumetric heat capacity and thermal conductivity are taken to be nonlinear functions of the temperature in both solid and liquid phases.

Necessary and/or sufficient conditions on the parameters of the problem are established in order to obtain an instantaneous nonlinear two-phase Stefan problem (solidification process) and the explicit solution is given.

Before that, the explicit solution to the corresponding nonlinear heat conduction problem for the initial (liquid) phase is obtained which has a constant value for all time on the fixed face $x = 0$. An inequality on the parameter q_0 is obtained in order to get that this constant is less than T_1, that is, a free boundary separating both phases is starting from $x = 0$.

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