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Free Boundary Problems

Theory and Applications

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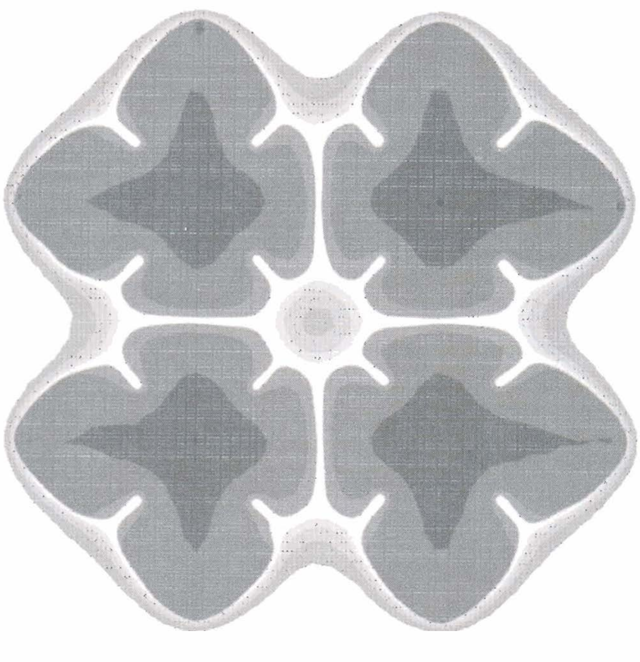

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Existence, Uniqueness and an Explicit Solution for a One-Phase Stefan Problem for a Non-classical Heat Equation

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Abstract. Existence and uniqueness, local in time, of the solution of a one-phase Stefan problem for a non-classical heat equation for a semi-infinite material is obtained by using the Friedman-Rubinstein integral representation method through an equivalent system of two Volterra integral equations. Moreover, an explicit solution of a similarity type is presented for a non-classical heat source depending on time and heat flux on the fixed face $x = 0$.

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Keywords. Stefan problem, Non-classical heat equation, Free boundary problems, Similarity solution, Nonlinear heat sources, System of Volterra integral equations.

1. Introduction

The one-phase Stefan problem for a semi-infinite material is a free boundary problem for the classical heat equation which requires the determination of the temperature distribution u of the liquid phase (melting problem) or of the solid phase (solidification problem), and the evolution of the free boundary $x = s(t)$. Phase-change problems appear frequently in industrial processes and other problems of technological interest [2, 4, 6, 9, 12]. A large bibliography on the subject was given in [20].

Non-classical heat conduction problem for a semi-infinite material was studied in [3, 5, 10, 22, 23], e.g., problems of the type

$$\begin{array}{ll} u_t - u_{xx} = -F(u_x(0,t)), & x > 0,\ t > 0, \\ u(0,t) = 0, & t > 0 \\ u(x,0) = h(x), & x > 0 \end{array} \tag{1.1}$$

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where $h(x), x > 0$, and $F(V), V \in \mathbb{R}$, are continuous functions. The function F, henceforth referred as control function, is assumed to fulfill the following condition

$$(H1) \qquad F(0) = 0.$$

As it was observed in [22, 23] the heat flux $w(x,t) = u_x(x,t)$ for problem (1.1) satisfies a classical heat conduction problem with a nonlinear convective condition at $x = 0$, which can be written in the form

$$\begin{cases} w_t - w_{xx} = 0, & x > 0,\ t > 0, \\ w_x(0,t) = F(w(0,t)), & t > 0, \\ w(x,0) = h'(x) \geq 0, & x > 0. \end{cases} \tag{1.2}$$

The literature concerning problem (1.2) has constantly increased from the appearance of the papers [13, 15, 17]. In [21] a one-phase Stefan problem for a non-classical heat equation for a semi-infinite material was presented. The free boundary problem consists in determining the temperature $u = u(x,t)$ and the free boundary $x = s(t)$ with a control function F which depends on the evolution of the heat flux at the boundary $x = 0$, satisfying the following conditions

$$\begin{cases} u_t - u_{xx} = -F(u_x(0,t)), & 0 < x < s(t), 0 < t < T, \\ u(0,t) = f(t) \geq 0, & 0 < t < T, \\ u(s(t),t) = 0, u_x(s(t),t) = -\dot{s}(t), & 0 < t < T, \\ u(x,0) = h(x) \geq 0, & 0 \leq x \leq b = s(0) \ \ (b > 0). \end{cases} \tag{1.3}$$

In Section 2 we present a result on the local existence and uniqueness in time of the solution of the one-phase Stefan problem (1.3) for a non-classical heat equation with temperature boundary condition at the fixed face $x = 0$. First, we prove that the free boundary problem (1.3) is equivalent to a system of two Volterra integral equations (2.4)–(2.5) [8, 14] following the Friedman-Rubinstein's method given in [7, 18](see also [19]). Then, we prove that the problem (2.4)–(2.5) has a unique local solution in time by using the Banach contraction theorem.

In Section 3 we show an explicit solution of a similarity type for a one-phase Stefan problem for a non classical control function F which depends on time and heat flux on the fixed face $x = 0$.

2. Existence and uniqueness of the non-classical free boundary problem

We have the following equivalence:

Theorem 2.1. *The solution of the free boundary problem* (1.3) *is given by*

$$\begin{aligned} u(x,t) \ = \ & \int_0^b G(x,t;\xi,0)h(\xi)d\xi + \int_0^t G_\xi(x,t;0,\tau)f(\tau)d\tau \\ & + \int_0^t G(x,t;s(\tau),\tau)v(\tau)d\tau - \iint_{D(t)} G(x,t;\xi,\tau)F(V(\tau))d\xi d\tau\ , \end{aligned} \tag{2.1}$$

$$s(t) = b - \int_0^t v(\tau)d\tau \tag{2.2}$$

where $D(t) = \{(x,\tau)/\ 0 < x < s(\tau), 0 < \tau < t\}$, *with* $f \in C^1[0,T)$, $h \in C^1[0,b]$, $h(b) = 0$, $h(0) = f(0)$, F *is a Lipschitz function over* $C^0[0,T]$, *and the functions* $v \in C^0[0,T]$, $V \in C^0[0,T]$ *defined by*

$$v(t) = u_x(s(t),t) \quad , \quad V(t) = u_x(0,t) \tag{2.3}$$

must satisfy the following system of two Volterra integral equations

$$\begin{aligned} v(t) &= 2\int_0^b N(s(t),t;\xi,0)h'(\xi)d\xi \\ &-2\int_0^t N(s(t),t;0,\tau)\,\dot{f}(\tau)d\tau + 2\int_0^t G_x(s(t),t;s(\tau),\tau)v(\tau)d\tau \\ &+2\int_0^t \left[N(s(t),t;s(\tau),\tau) - N(s(t),t;0,\tau)\right]F(V(\tau))d\tau\ , \end{aligned} \tag{2.4}$$

$$\begin{aligned} V(t) &= \int_0^b N(0,t;\xi,0)h'(\xi)d\xi \\ &-\int_0^t N(0,t;0,\tau)\,\dot{f}(\tau)d\tau + \int_0^t G_x(0,t;s(\tau),\tau)v(\tau)d\tau \\ &+\int_0^t \left[N(0,t;s(\tau),\tau) - N(0,t;0,\tau)\right]F(V(\tau))d\tau\ , \end{aligned} \tag{2.5}$$

where G, N *are the Green and Neumann functions and* K *is the fundamental solution of the heat equation, defined respectively by*

$$G(x,t,\xi,\tau) = K(x,t,\xi,\tau) - K(-x,t,\xi,\tau) \tag{2.6}$$

$$N(x,t,\xi,\tau) = K(x,t,\xi,\tau) + K(-x,t,\xi,\tau) \tag{2.7}$$

$$K(x,t,\xi,\tau) = \begin{cases} \frac{1}{2\sqrt{\pi(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) & t > \tau \\ 0 & t \le \tau \end{cases} \tag{2.8}$$

where $s(t)$ *is given by* (2.2).

In order to prove the local existence and uniqueness of solution v, $V \in C^0[0,\sigma]$ (σ is a positive small number) to the system of two Volterra integral equations (2.4)–(2.5) we will use the Banach fixed point theorem. Let us define the Banach space:

$$C_{M,\sigma} = \left\{ \overrightarrow{w} = \begin{pmatrix} v \\ V \end{pmatrix} / \ v, V : [0,\sigma] \to \mathbb{R}, \text{continuous, with } \left\|\overrightarrow{w}\right\|_\sigma \le M \right\}$$

with the norm

$$\left\|\overrightarrow{w}\right\|_\sigma := \|v\|_\sigma + \|V\|_\sigma := \max_{t\in[0,\sigma]}|v(t)| + \max_{t\in[0,\sigma]}|V(t)|. \tag{2.9}$$

We define the map $A : C_{M,\sigma} \longrightarrow C_{M,\sigma}$, such that

$$\overrightarrow{\widetilde{w}}\ (t) = A\left(\overrightarrow{w}\ (t)\right) = \begin{pmatrix} A_1(v(t), V(t)) \\ A_2(v(t), V(t)) \end{pmatrix} \tag{2.10}$$

where

$$A_1(v(t), V(t)) = F_1(v(t)) + 2\int_0^t \left[N(s(t), t, s(\tau), \tau) - N(s(t), t, 0, \tau)\right] F(V(\tau))d\tau \tag{2.11}$$

with

$$F_1(v(t)) = 2\int_0^b N(s(t), t, \xi, 0)h'(\xi)d\xi - 2\int_0^t N(s(t), t, 0, \tau)\ \dot{f}\ (\tau)d\tau$$
$$+2\int_0^t G_x(s(t), t, s(\tau), \tau)v(\tau)d\tau$$

and

$$A_2(v(t), V(t)) = F_2(v(t)) + \int_0^t \left[N(0, t, s(\tau), \tau) - N(0, t, 0, \tau)\right] F(V(\tau))d\tau. \tag{2.12}$$

with

$$F_2(v(t)) = \int_0^b N(0, t, \xi, 0)h'(\xi)d\xi - \int_0^t N(0, t, 0, \tau)\ \dot{f}\ (\tau)d\tau \tag{2.13}$$
$$+\int_0^t G_x(0, t, s(\tau), \tau)v(\tau)d\tau$$

Then we have the following property:

Theorem 2.2. *If $f \in C^1[0,T]$, $h \in C^1[0,b]$, $f(0) = h(0)$, $h(b) = 0$ and F is a Lipschitz function over $C^0[0,T]$, then the map $A : C_{M,\sigma} \longrightarrow C_{M,\sigma}$ is well defined and is a contraction map if $\sigma > 0$ is small enough. Then there exists an unique solution on $C_{M,\sigma}$ to the system of integral equations* (2.4), (2.5).

3. Explicit solution of a one-phase Stefan problem for a non-classical heat equation

Now, we consider a free boundary problem which consists in determining the temperature $u = u(x,t)$ and the free boundary $x = s(t)$ with a control function F which depends on time and the evolution of the heat flux at the boundary $x = 0$, satisfying the following conditions

$$\rho c u_t - k u_{xx} = -\gamma F(u_x(0,t), t)\ ,\ 0 < x < s(t)\ ,\ t > 0, \tag{3.1}$$

$$u(0,t) = f = \text{Const.} > 0,\ t > 0, \tag{3.2}$$

$$u(s(t),t) = 0,\ \ k u_x(s(t),t) = -\rho l\ \dot{s}\ (t)\ ,\ \ t > 0, \tag{3.3}$$

$$s(0) = 0, \tag{3.4}$$

where the thermal coefficients $k,\ \rho,\ c,\ l,\ \gamma > 0$ and the control function F is given by the expression

$$F(V,t) = \frac{\lambda_0}{\sqrt{t}}\, V \ \ (\lambda_0 > 0)\,. \tag{3.5}$$

In order to obtain an explicit solution of a similarity type, we define

$$\Phi(\eta) = u(x,t),\ \eta = \frac{x}{2a\sqrt{t}} \tag{3.6}$$

where $a^2 = k/\rho c$ is the diffusion coefficient.

After some elementary computations we obtain

$$\Phi(\eta) = f\left[1 - \frac{E(\eta)}{E(\eta_0)}\right]\ ,\ 0 < \eta < \eta_0, \tag{3.7}$$

where

$$E(x) = erf\,(x) + \frac{4\lambda}{\sqrt{\pi}}\int_0^x f_1(r)dr,\ \ \lambda = \frac{\gamma\lambda_0}{\rho c a} > 0\ ,\ [\lambda] = 1 \tag{3.8}$$

and

$$f_1(x) = \exp(-x^2)\int_0^x \exp(r^2)dr \tag{3.9}$$

is Dawson's integral [1] and η_0 is an unknown positive parameter to be determined which characterizes the free boundary given by

$$s(t) = 2a\eta_0\sqrt{t}. \tag{3.10}$$

We remark that Dawson's integral also appears in the explicit solution for the supercooled one-phase Stefan problem with a constant temperature boundary condition on the fixed face [16].

Taking into account the Stefan condition we have that $\eta_0 = \eta_0(\lambda, Ste)$ must be the solution of the following equation

$$\frac{Ste}{\sqrt{\pi}}[\exp(-x^2) + 2\lambda f_1(x)] = x[erf(x) + \frac{4\lambda}{\sqrt{\pi}}\int_0^x f_1(z)dz]\ \ ,\ \ x > 0 \tag{3.11}$$

where $Ste = \dfrac{fc}{l} > 0$ is the Stefan number and

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-z^2)dz. \tag{3.12}$$

The equation (3.11) is equivalent to the equation

$$W_1(x) = 2\lambda W_2(x)\ ,\ x > 0 \tag{3.13}$$

where functions W_1 and W_2 are defined by

$$W_1(x) = Ste\ \exp\left(-x^2\right) - \sqrt{\pi} erf(x) x \tag{3.14}$$

$$W_2(x) = 2x\int_0^x f_1(r)dr - Ste\ f_1(x). \tag{3.15}$$

Remark 3.1. If $\lambda = 0$ (that is $\gamma = 0$) in the free boundary problem (3.1)–(3.4) we obtain the classical Lamé-Clapeyron [11] solution and there exists a unique solution η_{00} of the equation (3.11) which is given now by

$$F_0(x) = \frac{Ste}{\sqrt{\pi}} \, , \; x > 0 \tag{3.16}$$

where

$$F_0(x) = erf(x) \exp(x^2) x \, . \tag{3.17}$$

Theorem 3.2. *For each $\lambda > 0$ there exists a unique solution η_0 of Eq.* (3.13). *This solution $\eta_0 = \eta_0(\lambda)$ has the following properties*

$$\begin{aligned} &\text{(i)} \;\; \eta_0(0^+) = \eta_{00} > 0 \\ &\text{(ii)} \;\; \eta_0(+\infty) = x_4 < +\infty \\ &\text{(iii)} \;\; \eta_0 = \eta_0(\lambda) \; \textit{is an increasing function on } \lambda \end{aligned} \tag{3.18}$$

where η_{00} is the unique solution of Equation (3.16) *and $x_4 > 0$ is the unique positive zero of W_2.*

Theorem 3.3. *For each $\lambda > 0$ the free boundary problem* (3.1)–(3.4) *has a unique similarity solution of the type*

$$u(x,t,\lambda) = f\left[1 - \frac{E(n,\lambda)}{E\left(\eta_0(\lambda),\lambda\right)}\right] \, , \;\; 0 < \eta = \frac{x}{2a\sqrt{t}} < \eta_0(\lambda) \tag{3.19}$$

$$s(t,\lambda) = 2a\, \eta_0(\lambda)\sqrt{t} \tag{3.20}$$

where

$$E(\eta,\lambda) = erf(\eta) + \frac{4\lambda}{\sqrt{\pi}} \int_0^\eta f_1(r)dr \tag{3.21}$$

and $\eta_0 = \eta_0(\lambda)$ is the unique solution of Eq. (3.13), *with $\eta_{00} < \eta_0(\lambda) < x_4$.*

Theorem 3.4. *The explicit solution* (3.19), (3.20) *of the problem* (3.1)–(3.4) *has the following properties:*

(i) $u_x(0,t,\lambda) = \dfrac{-f}{aE\left(\eta_0(\lambda),\lambda\right)} \dfrac{1}{\sqrt{\pi t}} < 0 \, , \; \forall t > 0$

(ii) $u(x,t,\lambda) \geq u_0(x,t) \, , \; \forall \, 0 \leq x \leq s_0(t) \, , \; t > 0$

(iii) $s(t,\lambda) \geq s_0(t) \, , \; \forall \, t > 0$

where $u_0(x,t) = f\left[1 - \dfrac{erf(\eta)}{erf(\eta_{00})}\right] \, , \; 0 < \eta = \dfrac{x}{2a\sqrt{t}} < \eta_{00} \, , \; t > 0$

$s_0(t) = s(t,0) = 2a\eta_{00}\sqrt{t}$

(iv) $1 \leq \dfrac{u(x,t,\lambda)}{u_0(x,t)} \leq \dfrac{1}{1 - \dfrac{\eta(x,t)}{\eta_{00}}} \left[1 - \dfrac{2}{Ste} \dfrac{\eta_0(\lambda)\left(1 + 2\lambda \left\|f_1\right\|_\infty\right)}{\exp\left(-\eta_0^2(\lambda)\right) + 2\lambda f_1\left(\eta_0(\lambda)\right)} \, \eta(x,t)\right]$

(v) $\displaystyle\lim_{t \to +\infty} \frac{u(x,t,\lambda)}{u_0(x,t)} = 1$ *uniformly* $\forall \, x \in$ *compact sets* $\subset [0, s_0(t))$.

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Free Boundary Problems
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This book gathers a collection of refereed articles containing original results reporting the recent contributions of the lectures and communications presented at the Free Boundary Problems Conference that took place at the University of Coimbra, Portugal, from June 7 to 12, 2005 (FBP2005).

They deal with the mathematics of a broad class of models and problems involving nonlinear partial differential equations arising in physics, engineering, biology and finance. Among the main topics, the talks considered free boundary problems in biomedicine, in porous media, in thermodynamic modeling, in fluid mechanics, in image processing, in financial mathematics or in computations for inter-scale problems.

The mathematical analysis and fine properties of solutions and interfaces in free boundary problems have been an active subject in the last three decades and their mathematical understanding continues to be an important interdisciplinary tool for the scientific applications, on one hand, and an intrinsic aspect of the current development of several important mathematical disciplines.

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