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A Stefan problem for a non-classical heat equation with a convective condition

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ABSTRACT

We prove the existence and uniqueness, local in time, of the solution of a one-phase Stefan problem for a non-classical heat equation for a semi-infinite material with a convective boundary condition at the fixed face $x = 0$. Here the heat source depends on the temperature at the fixed face $x = 0$ that provides a heating or cooling effect depending on the properties of the source term. We use the Friedman–Rubinstein integral representation method and the Banach contraction theorem in order to solve an equivalent system of two Volterra integral equations. We also obtain a comparison result of the solution (the temperature and the free boundary) with respect to the one corresponding with null source term.

1. Introduction

The one-phase Stefan problem for a semi-infinite material for the classical heat equation requires the determination of the temperature distribution u of the liquid phase (melting problem) or of the solid phase (solidification problem), and the evolution of the free boundary $x = s(t)$. Phase-change problems appear frequently in industrial processes and in other problems of technological interest [1,2,6,8–12,18,21]. A large bibliography on the subject was given in [28]. Motivated by [30] the free boundary problem which we want to consider consists in determining the temperature $u = u(x,t)$ and the free boundary $x = s(t)$ which satisfy the following conditions

$$\begin{cases} (i) u_t - u_{xx} = -F(u(0,t)), & 0 < x < s(t), 0 < t < T, \\ (ii) u_x(0,t) = g(t)[u(0,t) - f(t)], f(t) \geqslant 0, & 0 < t < T, \\ (iii) u(s(t),t) = 0, & 0 < t < T, \\ (iv) u_x(s(t),t) = -\dot{s}(t), & 0 < t < T, \\ (v) u(x,0) = h(x) \geqslant 0, & 0 \leqslant x \leqslant b, \\ (vi) s(0) = b (b > 0). \end{cases} \tag{1}$$

Here, the control function F depends on the evolution of the temperature at the extremum $x = 0$ in which a convective boundary condition is imposed. In condition (1*ii*), the function $g(t)$ is the thermal transfer coefficient depending on time and $f(t)$ is the temperature of the external fluid which also depends on time.

The non-classical heat Eq. (1*i*) can be thought as motivated by the modelling of a system of temperature regulation in isotropic mediums, with a non-uniform source term which provides a cooling or heating effect depending upon the properties of F related to the course of the temperature $u(0,t)$. For example when

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$$F(0) = 0 \quad \text{and} \quad u(0,t)F(u(0,t) < 0 \text{ if } u(0,t) \neq 0, \tag{2}$$

the source term is a cooler if $u(0,t) < 0$ and a heater if $u(0,t) > 0$.

In the particular case of a bounded domain, a class of problems when the heat source is uniform and belongs to a given multivalued function from $\mathbb{R}$ into itself, was studied in [20] regarding existence, uniqueness and asymptotic behavior. Other references on the subject are in [15,16,19].

Non-classical heat conduction problem for a semi-infinite material was studied in [3,7,20,31,32], e.g. problems of the type

$$\begin{array}{ll} u_t - u_{xx} = -F(u_x(0,t)), & x > 0, t > 0, \\ u(0,t) = 0, & t > 0, \\ u(x,0) = h(x), & x > 0, \end{array} \tag{3}$$

where $h(x)$, $x > 0$, and $F(V),\ V \in \mathbb{R}$, are continuous functions. In this case, the heat source depends on the heat flux at the boundary $x = 0$. The function F, henceforth referred as control function, is assumed to fulfill the following condition

$$F(0) = 0.$$

As observed in [31,32] the heat flux $w(x,t) = u_x(x,t)$ for problem (3) satisfies a classical heat conduction problem with a non-linear convective condition at $x = 0$, which can be written in the form

$$\begin{cases} w_t - w_{xx} = 0, & x > 0, t > 0, \\ w_x(0,t) = F(w(0,t)), & t > 0, \\ w(x,0) = h'(x) \geqslant 0, & x > 0. \end{cases} \tag{4}$$

The literature concerning the classical problem (4) has increased rapidly since the publication of the papers [22,24,25]. In [29] a one-phase Stefan problems for a non-classical heat equation for a semi-infinite material with a source term that depends on the heat flux at $x = 0$ was shown. In [4,5] an existence and uniqueness result, local in time, is obtained for two different one-phase Stefan problems for the non-classical heat equation.

The goal of this paper is to prove in Section 2 the existence and uniqueness, local in time, of the solution to the one-phase Stefan problem (1) for a non-classical heat equation for a semi-infinite material with a convective boundary condition at the fixed face $x = 0$. First, we prove that the non-classical Stefan problem (1) is equivalent to a system of two Volterra integral equations (8) and (9) [17,23] following the Friedman–Rubinstein's method given in [13], [14, pp. 220–221] , [26] for the classical Stefan problem through a Volterra integral equation. Then, we prove, by using the Banach contraction theorem, that the system (8), (9) (i.e. the non-classical free boundary (1)) has a unique local solution. The mean difference of our result with respect to the analysis by Friedman [13], [14, p. 221] is the dimension of the Volterra integral equation. Another difference is the boundary condition at the fixed face $x = 0$: convective in our case and Dirichlet in [13], [14, p.216]. We remark that the convective boundary condition has an important physical meaning and it is, in general, not usually considered in mathematics.

In Section 3 we consider a source term which verifies the condition (2). We obtain a comparison result of the temperature and the free boundary of the non-classical Stefan problem (1) with respect to the temperature and the free boundary corresponding to the classical problem (43), which is our previous problem (1) with null source term. This result can be interpreted as the source term accelerates the fusion process in our case; another similar result for the only heat transfer process can be found in [3].

2. Existence and uniqueness of solutions

Let be $f,\ g \in C^0(\mathbb{R}_0^+),\ h \in C^1[0,b],\ h(b) = 0,\ h'(0) = g(0)[h(0) - f(0)],\ F$ is a Lipschitz function over $C^0(\mathbb{R}_0^+)$ with a Lipschitz constant $L > 0$.

We have the following equivalence for the existence of solutions to the non-classical free boundary problem (1).

Theorem 1. *The solution to the free boundary problem (1) is given by the following expression*

$$\begin{aligned} u(x,t) = &\int_0^b N(x,t;\xi,0)h(\xi)d\xi + \int_0^t N(x,t;0,\tau)g(\tau)[W(\tau) - f(\tau)]d\tau + \int_0^t N(x,t;s(\tau),\tau)w(\tau)d\tau \\ &- \int\int_{D(t)} N(x,t;\xi,\tau)F(W(\tau))d\xi d\tau \end{aligned} \tag{5}$$

and

$$s(t) = b - \int_0^t w(\tau)d\tau, \tag{6}$$

where $D(t) = \{(x,\tau)/0 < x < s(\tau), 0 < \tau < t\}$, and the functions w, W defined by

$$w(t) = u_x(s(t),t), W(t) = u(0,t), \tag{7}$$

must satisfy the following system of two Volterra integral equations:

$$w(t) = 2\int_0^b h'(\xi)G(s(t),t,\xi,0)d\xi + 2\int_0^t g(\tau)[W(\tau)-f(\tau)]N_x(s(t),t,0,\tau)d\tau + 2\int_0^t w(\tau)N_x(s(t),t,s(\tau),\tau)d\tau$$
$$+ 2\int_0^t G(s(t),t,s(\tau),\tau)F(W(\tau))d\tau, \tag{8}$$

$$W(t) = \int_0^b h(\xi)N(0,t,\xi,0)d\xi + \int_0^t g(\tau)[W(\tau)-f(\tau)]N(0,t,0,\tau)d\tau + \int_0^t w(\tau)N(0,t,s(\tau),\tau)d\tau$$
$$- \int\int_{D(t)} N(0,t,\xi,\tau)F(W(\tau))d\tau d\xi, \tag{9}$$

where G, N are the Green and Neumann functions and K is the fundamental solution to the heat equation, defined respectively by

$$G(x,t,\xi,\tau) = K(x,t,\xi,\tau) - K(-x,t,\xi,\tau), \tag{10}$$

$$N(x,t,\xi,\tau) = K(x,t,\xi,\tau) + K(-x,t,\xi,\tau), \tag{11}$$

$$K(x,t,\xi,\tau) = \begin{cases} \frac{1}{2\sqrt{\pi(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) & t > \tau \\ 0 & t \leqslant \tau \end{cases} \tag{12}$$

and s(t) is given by(6).

Proof. Let $u(x,t)$ be the solution to the problem (1) and we integrate on the domain $D_{t,\varepsilon} = \{(\xi,\tau)/0 < \xi < s(\tau), \varepsilon < \tau < t-\varepsilon\}(\varepsilon > 0)$, the Green identity

$$(Nu_\xi - uN_\xi)_\xi - (Nu)_\tau = NF(u(0,\tau)). \tag{13}$$

Now we let $\varepsilon \to 0$, to obtain for $u(x,t)$ the integral representation [4,13,14,26]

$$u(x,t) = \int_0^b N(x,t;\xi,0)h(\xi)d\xi + \int_0^t N(x,t;0,\tau)g(\tau)[W(\tau)-f(\tau)]d\tau + \int_0^t N(x,t;s(\tau),\tau)u_\xi(s(\tau),\tau)d\tau$$
$$- \int\int_{D(t)} N(x,t;\xi,\tau)F(u(0,\tau))d\xi d\tau,$$

which is the integral representation (5) for $u(x,t)$ by using the definitions of $w(t)$ and $W(t)$ given by (7). Moreover, if we differentiate (5) in variable x and we let $x \to 0^+$ and $x \to s(t)^-$, by using the jump relations [13], we obtain the system of integral equations (8) and (9) for w and W.

Conversely the function $u(x,t)$ defined by (5), where w and W are the solutions of (8) and (9), satisfy the conditions (1)(i), (ii), (iv) and (v). In order to prove condition (1) (iii), we define $\psi(t) = u(s(t),t)$. Taking into account that u satisfy the conditions (1)(i), (ii), (iv) and (v), if we integrate the Green identity (13) over the domain $D_{t,\varepsilon}$ ($\varepsilon > 0$) and we let $\varepsilon \to 0$, we obtain that

$$u(x,t) = \int_0^b N(x,t;\xi,0)h(\xi)d\xi + \int_0^t N(x,t;s(\tau),\tau)w(\tau)d\tau + \int_0^t N(x,t;0,\tau)g(\tau)[W(\tau)-f(\tau)]d\tau$$
$$- \int_0^t \psi(\tau)[N_\xi(x,t;s(\tau),\tau) + N(x,t;s(\tau),\tau)w(\tau)]d\tau - \int\int_{D(t)} N(x,t;\xi,\tau)F(W(\tau))d\xi d\tau. \tag{14}$$

Then, if we compare this last expression (14) with (5) we deduce that

$$\int_0^t \psi(\tau)[N_\xi(x,t;s(\tau),\tau) + N(x,t;s(\tau),\tau)w(\tau)]d\tau \equiv 0, \tag{15}$$

for $0 < x < s(t)$, $0 < t < \sigma$. We let in (15) $x \to s(t)$ and by using the jump relations we have that ψ must satisfy the integral equation

$$\frac{1}{2}\psi(t) + \int_0^t \psi(\tau)[N_\xi(s(t),t;s(\tau),\tau) + N(s(t),t;s(\tau),\tau)w(\tau)]d\tau = 0.$$

Then we deduce that

$$|\psi(t)| \leqslant C\int_0^t \frac{|\psi(\tau)|}{\sqrt{t-\tau}}d\tau \leqslant C^2\int_0^t \frac{d\tau}{\sqrt{t-\tau}}\int_0^\tau \frac{|\psi(\eta)|}{\sqrt{\tau-\eta}}d\eta = C^2\int_0^t |\psi(\eta)|d\eta\int_\eta^t \frac{d\tau}{[(t-\tau)(\tau-\eta)]^{\frac{1}{2}}} = \pi C^2\int_0^t |\psi(\eta)|d\eta,$$

where $C = C(t)$, therefore by using the Gronwall inequality we have that $\psi(t) = 0$ over $[0,\sigma]$, that is the ($1iii$) holds. □

Then, we use the Banach fixed point Theorem in order to prove the local existence and uniqueness of solution w, $W \in C^0[0,\sigma]$ to the system of two Volterra integral Eqs. (8) and (9), where σ, is a positive small number $(0 < \sigma \leqslant T)$ to be determined. Consider the Banach Space:

$$C_{R,\sigma} = \left\{ \vec{v} = \begin{pmatrix} w \\ W \end{pmatrix} \Big/ w, W : [0,\sigma] \to \mathbb{R}, \text{continuous}, \text{with} \|\vec{v}\|_\sigma \leqslant R \right\},$$

where

$$\|\vec{v}\|_\sigma := \max_{t\in[0,\sigma]} |w(t)| + \max_{t\in[0,\sigma]} |W(t)|. \tag{16}$$

We define the map $B : C_{R,\sigma} \to C_{R,\sigma}$, such that

$$\vec{\tilde{v}}(t) = B(\vec{v}(t)) = \begin{pmatrix} B_1(w(t), W(t)) \\ B_2(w(t), W(t)) \end{pmatrix},$$

where

$$\begin{aligned} B_1(w(t), W(t)) = {} & 2\int_0^b h'(\xi) G(s(t), t, \xi, 0) d\xi + 2\int_0^t g(\tau)[W(\tau) - f(\tau)] N_x(s(t), t, 0, \tau) d\tau \\ & + 2\int_0^t w(\tau) N_x(s(t), t, s(\tau), \tau) d\tau + 2\int_0^t G(s(t), t, s(\tau), \tau) F(W(\tau)) d\tau \end{aligned} \tag{17}$$

and

$$\begin{aligned} B_2(w(t), W(t)) = {} & \int_0^b h(\xi) N(0, t, \xi, 0) d\xi + \int_0^t g(\tau)[W(\tau) - f(\tau)] N(0, t, 0, \tau) d\tau + \int_0^t w(\tau) N(0, t, s(\tau), \tau) d\tau \\ & - \int\int_{D(t)} N(0, t, \xi, \tau) F(W(\tau)) d\tau d\xi. \end{aligned} \tag{18}$$

Firstly, we have some preliminary Lemmas.

Lemma 2. *Let $w \in C^0[0,\sigma]$, $\max_{t\in[0,\sigma]} |w(t)| \leqslant R$ and $2R\sigma \leqslant b$ then $s(t)$ defined by (6) satisfies*

$$|s(t) - s(\tau)| \leqslant R|t - \tau|, \quad \forall \tau,\ t \in [0,\sigma], \tag{19}$$

$$|s(t) - b| \leqslant \frac{b}{2}, \quad \forall t \in [0,\sigma]. \tag{20}$$

To prove the following Lemmas we need the classical inequality

$$\frac{\exp\left(\frac{-x^2}{\alpha(t-\tau)}\right)}{(t-\tau)^{\frac{n}{2}}} \leqslant \left(\frac{n\alpha}{2ex^2}\right)^{\frac{n}{2}}, \quad \alpha,\ x > 0, \quad t > \tau, \quad n \in \mathbb{N} \tag{21}$$

and we define

$$\|f\|_t := \max_{\tau\in[0,t]} |f(\tau)|. \tag{22}$$

Lemma 3. *Let $\sigma \leqslant 1$, $R \geqslant 1$, $g \in C^0(\mathbb{R}_0^+)$, $h \in C^1[0,b]$, $h(b) = 0$, $h'(0) = g(0)[h(0) - f(0)]$, F a Lipschitz function over $C^0(\mathbb{R}_0^+)$ with a Lipschitz constant $L > 0$. Under the hypothesis of Lemma 2 we have the following properties*

$$\int_0^b |h'(\xi)||G(s(t), t, \xi, 0)| d\xi \leqslant \|h'\|; \tag{23}$$

$$\int_0^t |g(\tau)[W(\tau) - f(\tau)]||N_x(s(t), t, 0, \tau)| d\tau \leqslant A_1\left(\|g\|_t, R, \|f\|_t, \frac{1}{b}\right) t; \tag{24}$$

$$\int_0^t |w(\tau)||N_x(s(t), t, s(\tau), \tau)| d\tau \leqslant A_2\left(R, \frac{1}{b}\right)\sqrt{t}; \tag{25}$$

$$\int_0^t |G(s(t), t, s(\tau), \tau)||F(W(\tau))| d\tau \leqslant A_3(L, R)\sqrt{t}; \tag{26}$$

$$\int_0^b |h(\xi)||N(0, t, \xi, 0)| d\xi \leqslant \|h\|; \tag{27}$$

$$\int_0^t |g(\tau)[W(\tau)-f(\tau)]||N(0,t,0,\tau)|d\tau \leqslant A_4(\|g\|_t, R, \|f\|_t)\sqrt{t}; \tag{28}$$

$$\int_0^t |w(\tau)||N(0,t,s(\tau),\tau)|d\tau \leqslant A_5(R)\sqrt{t}; \tag{29}$$

$$\int\int_{D(t)} |N(0,t,\xi,\tau)||F(W(\tau))|d\xi d\tau \leqslant A_6(b,L,R)\sqrt{t}, \tag{30}$$

where the constants A_i (i = 1,…,6) are increasing functions on their arguments.

Proof. Inequality (23) holds because

$$\int_0^\infty |G(s(t),t,\xi,0)|d\xi \leqslant \int_0^\infty |N(s(t),t,\xi,0)|d\xi \leqslant 1.$$

To prove (24) we have

$$|N_x(s(t),t,0,\tau)| = |K_x(s(t),t,0,\tau) - K_x(-s(t),t,0,\tau)| \leqslant \frac{|s(t)|\exp\left(\frac{-(s(t))^2}{4(t-\tau)}\right)}{2\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} \leqslant \frac{|s(t)|\exp\left(\frac{-b^2}{16(t-\tau)}\right)}{2\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} \leqslant \alpha_1\left(\frac{1}{b}\right).$$

Then (24) holds. To prove (25) we have

$$|N_x(s(t),t,s(\tau),\tau)| = |-2K_x(-s(t),t,s(\tau),\tau) + G_x(s(t),t,s(\tau),\tau)|$$

$$|-2K_x(-s(t),t,s(\tau),\tau)| = \frac{|s(t)+s(\tau)|}{\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\exp\left(\frac{-(s(t)+s(\tau))^2}{4(t-\tau)}\right) \leqslant \alpha_2\left(\frac{1}{b}\right)$$

and

$$\begin{aligned}|G_x(s(t),t,s(\tau),\tau)| &= |K_x(s(t),t,s(\tau),\tau) + K_x(-s(t),t,s(\tau),\tau)| \\ &= \frac{(t-\tau)^{-\frac{3}{2}}}{4\sqrt{\pi}}\left|(s(t)-s(\tau))\exp\left(\frac{-(s(t)-s(\tau))^2}{4(t-\tau)}\right) - (s(t)+s(\tau))\exp\left(\frac{-(s(t)+s(\tau))^2}{4(t-\tau)}\right)\right| \\ &\leqslant \frac{1}{4\sqrt{\pi}}\left(R(t-\tau)^{-\frac{1}{2}} + 3b\left(\frac{2}{3eb^2}\right)^{\frac{3}{2}}\right),\end{aligned}$$

then (25) holds.

We obtain (26) by using $|G(s(t),t,s(\tau),\tau)| \leqslant \frac{1}{\sqrt{\pi(t-\tau)}}$ and F, a Lipschitz function. The inequality (27) can be proved in the same way as (23). To prove (28), we have

$$\int_0^t |N(0,t,0,\tau)||g(\tau)[W(\tau)-f(\tau)]|d\tau \leqslant \|g\|_t(R+\|f\|_t)\int_0^t |N(0,t,0,\tau)|d\tau = \frac{\|g\|_t(R+\|f\|_t)}{\sqrt{\pi}}2\sqrt{t}$$

and inequality (29) holds because $|N(0,t,s(\tau),\tau)| \leqslant \frac{1}{\sqrt{\pi(t-\tau)}}$ and

$$\int_0^t |w(\tau)||N(0,t,s(\tau),\tau)|d\tau \leqslant \frac{2R\sqrt{t}}{\sqrt{\pi}}.$$

In order to prove (30) we have

$$\int\int_{D(t)} |N(0,t,\xi,\tau)||F(W(\tau))|d\xi d\tau = \int_0^t \left|\int_0^{s(\tau)} N(0,t,\xi,\tau)||F(W(\tau))|d\xi\right| d\tau \leqslant LR\int_0^t \frac{|s(\tau)|}{\sqrt{\pi(t-\tau)}}d\tau \leqslant \frac{3bLR\sqrt{t}}{\sqrt{\pi}}$$

and therefore the thesis holds. □

Lemma 4. *Let s_1 and s_2 be the functions corresponding to w_1 and w_2 in $C^0[0,\sigma]$ respectively with $\max_{t\in[0,\sigma]}|w_i(t)| \leqslant R$, i = 1, 2. Then we have*

$$\begin{cases} |s_2(t)-s_1(t)| \leqslant t\|w_2-w_1\|_t, \\ |s_i(t)-s_i(\tau)| \leqslant R|t-\tau|, \quad i=1,2, \\ \frac{b}{2} \leqslant s_i(t) \leqslant \frac{3b}{2}, \quad \forall t\in[0,\sigma], i=1,2. \end{cases} \tag{31}$$

Lemma 5. *Let be $g \in C^0(\mathbb{R}_0^+)$, $h \in C^1[0,b]$, F a Lipschitz function over $C^0(\mathbb{R}_0^+)$. Under the hypothesis of Lemma 4 we have*

$$\int_0^t |w_1(\tau)N(0,t,s_1(\tau),\tau) - w_2(\tau)N(0,t,s_2(\tau),\tau)|d\tau \leqslant A_7\left(\frac{1}{b},R\right)t\|w_1 - w_2\|_t; \tag{32}$$

$$\left|\int\int_{D_1(t)} N(0,t,\xi,\tau)F(W_1(\tau))d\xi d\tau - \int\int_{D_2(t)} N(0,t,\xi,\tau)F(W_2(\tau))d\xi d\tau\right| \leqslant [A_8(L,R)\|w_1 - w_2\|_t + A_9(b,L)\|W_1 - W_2\|_t]\sqrt{t}, \tag{33}$$

where $D_i(t) = \{(\xi,\tau)/0 < \xi < s_i(\tau), 0 < \tau < t\}, \quad i = 1,2;$

$$\int_0^b |h'(\xi)||G(s_1(t),t,\xi,0) - G(s_2(t),t,\xi,0)|d\xi \leqslant A_{10}(\|h'\|)\sqrt{t}\|w_1 - w_2\|_t; \tag{34}$$

$$\int_0^t |N_x(s_1(t),t,0,\tau)g(\tau)[W_1(\tau) - f(\tau)] - N_x(s_2(t),t,0,\tau)g(\tau)[W_2(\tau) - f(\tau)]|d\tau, \leqslant A_{11}(\|g\|_t,\frac{1}{b})t\|W_1 - W_2\|_t + A_{12}(\|g\|_t,\frac{1}{b},R,\|f\|_t)t\|w_1 - w_2\|_t; \tag{35}$$

$$\int_0^t |N(0,t,0,\tau)g(\tau)[W_1(\tau) - f(\tau)] - N(0,t,0,\tau)g(\tau)[W_2(\tau) - f(\tau)]|d\tau \leqslant A_{13}(\|g\|_t)\sqrt{t}\|W_1 - W_2\|_t; \tag{36}$$

$$\int_0^t |G(s_1(t),t,s_1(\tau),\tau)F(W_1(\tau)) - G(s_2(t),t,s_2(\tau),\tau)F(W_2(\tau))|d\tau \leqslant \left[A_{14}(L)\|W_1 - W_2\|_t + A_{15}(R,L,\frac{1}{b^2})\|w_1 - w_2\|_t\right]\sqrt{t} \tag{37}$$

and

$$\int_0^t |w_1(\tau)N_x(s_1(t),t,s_1(\tau),\tau) - w_2(\tau)N_x(s_2(t),t,s_2(\tau),\tau)|d\tau \leqslant A_{16}\left(R,\frac{1}{b}\right)\sqrt{t}\|w_1 - w_2\|_t. \tag{38}$$

Proof. To prove (32) we have

$$|w_1(\tau)N(0,t,s_1(\tau),\tau) - w_2(\tau)N(0,t,s_2(\tau),\tau)| \leqslant |w_1(\tau) - w_2(\tau)||N(0,t,s_1(\tau),\tau)| + |w_2(\tau)||N(0,t,s_1(\tau),\tau) - N(0,t,s_2(\tau),\tau)|.$$

Taking into account that

$$|N(0,t,s_1(\tau),\tau)| \leqslant \frac{\exp\left(\frac{-b^2}{16(t-\tau)}\right)}{\sqrt{\pi}(t-\tau)^{\frac{1}{2}}} \leqslant \left(\frac{8}{eb^2}\right)^{\frac{1}{4}}\frac{1}{\sqrt{\pi}}$$

and

$$|N(0,t,s_1(\tau),\tau) - N(0,t,s_2(\tau),\tau)| \leqslant \frac{3b}{4\sqrt{\pi}}\left(\frac{24}{eb^2}\right)^{\frac{3}{2}}t\|w_1 - w_2\|_t$$

then

$$\int_0^t |w_1(\tau)N(0,t,s_1(\tau),\tau) - w_2(\tau)N(0,t,s_2(\tau),\tau)|d\tau \leqslant A_7\left(\frac{1}{b},R\right)t\|w_1 - w_2\|_t.$$

To prove (33) we have

$$\int\int_{D_1(t)} N(0,t,\xi,\tau)F(W_1(\tau))d\xi d\tau - \int\int_{D_2(t)} N(0,t,\xi,\tau)F(W_2(\tau))d\xi d\tau = \int\int_{D_1(t)} N(0,t,\xi,\tau)(F(W_1(\tau)) - F(W_2(\tau)))d\xi d\tau + \int\int_{D_1(t)} N(0,t,\xi,\tau)F(W_2(\tau))d\xi d\tau - \int\int_{D_2(t)} N(0,t,\xi,\tau)F(W_2(\tau))d\xi d\tau.$$

Because

$$\left|\int\int_{D_1(t)} N(0,t,\xi,\tau)(F(W_1(\tau)) - F(W_2(\tau)))d\xi d\tau\right| \leqslant \frac{3b}{2\sqrt{\pi}}L\sqrt{t}\|W_1 - W_2\|_t$$

and

$$\left|\int\int_{D_1(t)} N(0,t,\xi,\tau)F(W_2(\tau))d\xi d\tau - \int\int_{D_2(t)} N(0,t,\xi,\tau)F(W_2(\tau))d\xi d\tau\right| \leqslant \int_0^t |F(W_2(\tau))|\left|\int_{s_1(\tau)}^{s_2(\tau)} N(0,t,\xi,\tau)d\xi\right|d\tau$$
$$\leqslant \frac{LR}{\sqrt{\pi}}t^{\frac{3}{2}}\|w_1-w_2\|_t$$

then (33) holds. To prove (34) we have

$$|G(s_1(t),t,\xi,0)-G(s_2(t),t,\xi,0)| \leqslant |K(s_1(t),t,\xi,0)-K(s_2(t),t,\xi,0)| + |K(-s_1(t),t,\xi,0)-K(-s_2(t),t,\xi,0)|$$

and by the mean value theorem there exists $d = d(t)$ between $s_1(t)$ and $s_2(t)$ such that

$$|K(s_1(t),t,\xi,0)-K(s_2(t),t,\xi,0)| = |s_1(t)-s_2(t)|K(d(t),t,\xi,0)\frac{|d(t)-\xi|}{2t}$$

then

$$\int_0^b |s_1(t)-s_2(t)|K(d(t),t,\xi,0)\frac{|d(t)-\xi|}{2t}d\xi \leqslant t\|w_1-w_2\|_t\int_0^b \frac{|d(t)-\xi|}{\exp\left(\frac{(d(t)-\xi)^2}{4t}\right)4\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}d\xi \leqslant \frac{\sqrt{t}\|w_1-w_2\|_t}{\sqrt{\pi}}.$$

In the same way we have

$$\int_0^b |K(-s_1(t),t,\xi,0)-K(-s_2(t),t,\xi,0)|d\xi \leqslant \frac{\sqrt{t}\|w_1-w_2\|_t}{\sqrt{\pi}}.$$

Then

$$\int_0^b |h'(\xi)||G(s_1(t),t,\xi,0)-G(s_2(t),t,\xi,0)|d\xi \leqslant 2\|h'\|\frac{\sqrt{t}\|w_1-w_2\|_t}{\sqrt{\pi}}.$$

To prove (35) we consider that

$$\int_0^t |N_x(s_1(t),t,0,\tau)g(\tau)[W_1(\tau)-f(\tau)] - N_x(s_2(t),t,0,\tau)g(\tau)[W_2(\tau)-f(\tau)]|d\tau$$
$$\leqslant \int_0^t \{|N_x(s_1(t),t,0,\tau)||g(\tau)||W_1(\tau)-W_2(\tau)| - N_x(s_2(t),t,0,\tau)g(\tau)[W_2(\tau)-f(\tau)]\}d\tau$$
$$+\int_0^t |N_x(s_1(t),t,0,\tau)-N_x(s_2(t),t,0,\tau)||g(\tau)||W_2(\tau)-f(\tau)|d\tau.$$

We apply the mean value theorem and therefore there exists $c = c(t)$ between $s_1(t)$ and $s_2(t)$ such that

$$|N_x(s_1(t),t,0,\tau)-N_x(s_2(t),t,0,\tau)| = |s_1(t)-s_2(t)||N_{xx}(c(t),t,0,\tau)|,$$

$$|N_{xx}(c(t),t,0,\tau)| \leqslant \frac{\exp\left(\frac{-c^2(t)}{4(t-\tau)}\right)}{2\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} + \frac{c^2\exp\left(\frac{-c^2(t)}{4(t-\tau)}\right)}{4\sqrt{\pi}(t-\tau)^{\frac{5}{2}}} \leqslant \left(\frac{24}{eb^2}\right)^{\frac{3}{2}}\frac{1}{2\sqrt{\pi}} + \left(\frac{40}{eb^2}\right)^{\frac{5}{2}}\frac{9}{16\sqrt{\pi}}b^2.$$

Then we have

$$\int_0^t |N_x(s_1(t),t,0,\tau)-N_x(s_2(t),t,0,\tau)||g(\tau)||W_2(\tau)-f(\tau)|d\tau \leqslant A_{12}(\|g\|_t,\frac{1}{b},R,\|f\|_t)t\|w_1-w_2\|_t$$

and, using (24) we get (35). In order to prove (36) we have

$$\int_0^t |N(0,t,0,\tau)g(\tau)[W_1(\tau)-f(\tau)] - N(0,t,0,\tau)g(\tau)[W_2(\tau)-f(\tau)]|d\tau \leqslant \int_0^t |N(0,t,0,\tau)||g(\tau)||W_1(\tau)-W_2(\tau)|d\tau$$
$$\leqslant \|g\|_t\|W_1-W_2\|_t\frac{2\sqrt{t}}{\sqrt{\pi}}.$$

In order to prove (37) we have

$$|G(s_1(t),t,s_1(\tau),\tau)F(W_1(\tau)) - G(s_2(t),t,s_2(\tau),\tau)F(W_2(\tau))| \leqslant |G(s_1(t),t,s_1(\tau),\tau)||F(W_1(\tau))-F(W_2(\tau))|$$
$$+|G(s_1(t),t,s_1(\tau),\tau)-G(s_2(t),t,s_2(\tau),\tau)||F(W_2(\tau))|.$$

We obtain that

$$|G(s_1(t),t,s_1(\tau),\tau)||F(W_1(\tau))-F(W_2(\tau))| \leqslant \frac{L}{\sqrt{\pi(t-\tau)}}\|W_1-W_2\|_t$$

and, following [4] we have

$$\int_0^t |G(s_1(t),t,s_1(\tau),\tau)F(W_1(\tau)) - G(s_2(t),t,s_2(\tau),\tau)F(W_2(\tau))|d\tau$$
$$\leqslant \frac{2L\sqrt{t}}{\sqrt{\pi}}\|W_1 - W_2\|_t + \frac{R^3L\sqrt{t}}{\sqrt{\pi}}\|w_1 - w_2\|_t + \left(\frac{6}{e}\right)^{\frac{3}{2}}\frac{R^2t}{b^2\sqrt{\pi}}\|w_1 - w_2\|_t.$$

To finish the thesis, the result (38) can be found in [27]. □

Theorem 6. *Let be $g,\ f \in C^0(\mathbb{R}_0^+),\ h \in C^1[0,b]$, $h(b) = 0$, $h'(0) = g(0)[h(0) - f(0)]$, F is a Lipschitz function over $C^0(\mathbb{R}_0^+)$. The map $B: C_{R,\sigma} \to C_{R,\sigma}$ is well defined and it is a contraction map if σ satisfies the following inequalities*

$$\sigma \leqslant 1, 2R\sigma \leqslant b, \tag{39}$$

$$M(R,\frac{1}{b},\|g\|_\sigma,\|f\|_\sigma,L,\sigma) \leqslant 1, \tag{40}$$

$$H\left(\|h'\|,\frac{1}{b},\|g\|_\sigma,b,L,R,\sigma\right) < 1, \tag{41}$$

where R is given by

$$R = 1 + \|h\| + 2\|h'\| \tag{42}$$

and

$$M = 2\sum_{i=1}^{6} A_i\sqrt{\sigma},$$
$$H = 2\sum_{i=7}^{16} A_i\sqrt{\sigma},$$

where $A_i (i = 1,\ldots,16)$ given in Lemmas 3 and 5 are increasing functions on their arguments. Then there exists a unique solution on $C_{R,\sigma}$ to the system of integral Eqs. (8) and (9).

Proof. Using Lemma 3 and selecting R by (42) and σ such that (39), (40) hold that we have that B maps $C_{R,\sigma}$ into itself. Using Lemma 5, by selecting σ such that (41) holds B becomes a contracting mapping on $C_{R,\sigma}$ and therefore it has a unique fixed point.

Having proved the existence of a solution $\vec{v}(t) = (w(t), W(t))$ of the system of integral Eqs. (8) and (9) (with $s(t)$ defined by (6)), we proceed to prove that every solution to the system of integral equations (8) and (9) must coincide with $\vec{v}(t)$ in their common interval of existence following [13,14]. Suppose $\vec{v^*}(t) = (w^*(t), W^*(t)),\ 0 \leqslant t \leqslant \sigma^*$ is another solution to (8) and (9) and we assume that $\sigma^* \leqslant \sigma$; then we must prove that $\vec{v}(t) = \vec{v^*}(t)$ for all $0 \leqslant t \leqslant \sigma^*$. For this reason, we can define the constant R in the previous analysis by

$$R^* = \max(R, \|\vec{v^*}\|_{\sigma^*}),$$

where $\|\vec{v^*}\|_{\sigma^*}$ is defined as (16). Therefore we get that there exists $\bar{\sigma} \leqslant \sigma^*$ which satisfies the inequalities (39)–(41) with R^* instead of R, which guarantee that the map $B : C_{R^*,\bar{\sigma}} \to C_{R^*,\bar{\sigma}}$ is a contraction. Following the same method developed in [13], [14, pp. 222–223] the uniqueness holds. □

Remark 1. We have proved the local existence and uniqueness of the solution of a vectorial Volterra integral Eq. (17), (18), i.e. of the non-classical free boundary problem (1). It is an open problem to prove the global existence and uniqueness of a vectorial Volterra integral equation by using the method given in [13], [14, pp. 223–225] for a Volterra integral equation corresponding to a classical free boundary problem..

3. Comparison with null source term

We define

$$T = \sup\{\sigma > 0/\sigma \text{ verifies inequalities}(39){-}(41)\}.$$

First, by using the maximum principle we obtain the following results:

Theorem 7. *Let be $g, f \in C^0([0,T])$, $h \in C^1[0,b]$, $h(b) = 0$, $h'(0) = g(0)[h(0) - f(0)]$, F is a Lipschitz function over $C^0([0,T])$ such that satisfies (2) Under the hypothesis of Theorem 6 we have that the solution u to the free boundary problem (1) satisfies that $u(x,t) \geqslant 0$ for all $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant T$ where $s(t)$ is given by (6). Moreover $s(t)$ is an increasing function of the time t.*

Proof. We suppose that there exists $0 < t_0 \leqslant T$ such that $u(0,t_0) = 0$ and $u(0,t) > 0$ for all $0 \leqslant t < t_0$. Taking into account (2) we have $F(u(0,t)) < 0$ for all $0 \leqslant t < t_0$.

Then, if we consider the domain $D = \{(x,t)/0 < x < s(t),\ 0 < t < t_0\}$ by the minimum principle we have that $\min\{u(x,t) : (x,t) \in \overline{D}\}$ is on the parabolic boundary of D. Since $u(s(t),t) = 0$ and $u(x,0) = h(x) \geqslant 0$ we have that the minimum is obtained by $u(0,t_0) = 0$ and by Hopf's principle we get $u_x(0,t_0) > 0$ which is in contradiction with condition (1(*ii*)) which gives $u_x(0,t_0) = -g(t_0)f(t_0) < 0$.

Therefore we have $u(0,t) > 0$, for all $0 \leqslant t \leqslant T$ and then $u(x,t) \geqslant 0$ for all $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant T$. Similarly, we obtain that $s(t)$ is an increasing function because $\dot{s}(t) = -u_x(s(t),t) > 0$ for all $0 \leqslant t \leqslant T$. □

Next, we compare the solution u, s of the non-classical Stefan problem (1) with the solution u_0, s_0 of the analogous classical Stefan problem without control function (i.e. $F = 0$).

Theorem 8. *Let be $g, f \in C^0([0,T])$, $h \in C^1[0,b]$, $h(b) = 0$, $h'(0) = g(0)[h(0) - f(0)]$, F is a Lipschitz function over $C^0([0,T])$ such that satisfies (2). We assume that the hypothesis of Theorem 6 hold. Then, the solution u, s of the problem (1) satisfies that it $s(t) \geqslant s_0(t)$ for all $0 \leqslant t \leqslant T$ and $u(x,t) \geqslant u_0(x,t)$ for all $0 \leqslant x \leqslant s_0(t)$, $0 \leqslant t \leqslant T$ where u_0, s_0 is the solution of the following Stefan problem with null source term*

$$\begin{cases} (i)u_t - u_{xx} = 0, & 0 < x < s_0(t), 0 < t < T \\ (ii)u_x(0,t) = g(t)[u(0,t) - f(t)], f(t) \geqslant 0, & 0 < t < T, \\ (iii)u(s(t),t) = 0, & 0 < t < T, \\ (iv)u_x(s(t),t) = -\dot{s}(t), & 0 < t < T, \\ (v)u(x,0) = h(x) \geqslant 0, & 0 \leqslant x \leqslant b, \\ (vi)s(0) = b(b > 0). \end{cases} \tag{43}$$

Proof. We suppose that there exists $0 < t_0 \leqslant T$ such that $s(t) > s_0(t)$, for all $0 < t < t_0$ and $s(t_0) = s_0(t_0)$. We define $v(x,t) = u(x,t) - u_0(x,t)$, $0 \leqslant x \leqslant s_0(t)$, $0 \leqslant t < t_0$. Function v satisfies the following Stefan problem

$$\begin{cases} (i)\,v_t - v_{xx} = -F(u(0,t)) > 0, & 0 < x < s_0(t), 0 < t < t_0, \\ (ii)\,v_x(0,t) = g(t)\,v(0,t), & 0 < t < t_0, \\ (iii)\,v(s_0(t),t) = u(s_0(t),t) > 0, & 0 < t < t_0, \\ (iv)\,v_x(s_0(t),t) = -u_x(s_0(t),t) + \dot{s}_0(t), & 0 < t < t_0, \\ (v)\,v(x,0) = 0, & 0 \leqslant x \leqslant b, \\ (vi)s_0(0) = b(b > 0). \end{cases} \tag{44}$$

From (44)(*iv*) we have

$$v_x(s_0(t_0),t_0) = -u_x(s(t_0),t_0) + \dot{s}_0(t_0) = -\dot{s}(t_0) + \dot{s}_0(t_0) \geqslant 0$$

and

$$v(s_0(t_0),t_0) = 0,$$

which is a contradiction because by Hopf's' principle we have $v_x(s_0(t_0),t_0) < 0$. Therefore we have $s(t) \geqslant s_0(t)$ for all $0 \leqslant t \leqslant T$.

Now we will prove that $u(x,t) \geqslant u_0(x,t)$ for all $0 \leqslant x \leqslant s_0(t)$, $0 \leqslant t \leqslant T$. We suppose that there exists $0 < t_1 \leqslant T$ such that $u(0,t) > u_0(0,t)$, for all $0 < t < t_1$ and $u(0,t_1) = u_0(0,t_1)$. Then the function $v(x,t) = u(x,t) - u_0(x,t)$, $0 \leqslant x \leqslant s_0(t)$, $0 \leqslant t \leqslant t_1$ satisfies $v(0,t_1) = 0$ and $v(0,t) > 0$ for all $0 < t < t_1$. Moreover, we have $v_x(0,t_1) = g(t_1)v(0,t_1) = 0$ which is a contradiction because v have a minimum on $(0,t_1)$. Then $v(0,t) > 0$ for all $0 < t \leqslant T$ and v have a minimum on the boundary. Therefore $u(x,t) \geqslant u_0(x,t)$ for all $0 \leqslant x \leqslant s_0(t)$, $0 \leqslant t \leqslant T$. □

Remark 2. From Theorem 8 we can deduce that when we have a source F (which is a heater for condition (2)) then the fusion process is more rapid and effective than in the case in which the source is null. A similar result only for the heat transfer process has been obtained in [3].

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